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# Lyapunov inequalities for a class of nonlinear dynamic systems on time scales

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**Abstract**

The purpose of this work is to obtain several Lyapunov inequalities for the nonlinear dynamic systems

$$\begin{cases} x^{\Delta}(t) = -A(t)x(\sigma(t)) - B(t)y(t)|\sqrt{B(t)}y(t)|^{p-2}, \\ y^{\Delta}(t) = C(t)x(\sigma(t))|x(\sigma(t))|^{q-2} + A^{T}(t)y(t), \end{cases}$$

on a given time scale interval $[a,b]_{\mathbb{T}}$ ($a,b \in \mathbb{T}$ with $\sigma(a) < b$), where $p,q \in (1,+\infty)$ satisfy $1/p + 1/q = 1$, $A(t)$ is a real $n \times n$ matrix-valued function on $[a,b]_{\mathbb{T}}$ such that $I + \mu(t)A(t)$ is invertible, $B(t)$ and $C(t)$ are two real $n \times n$ symmetric matrix-valued functions on $[a,b]_{\mathbb{T}}$, $B(t)$ is positive definite, and $x(t)$, $y(t)$ are two real $n$-dimensional vector-valued functions on $[a,b]_{\mathbb{T}}$.

**MSC:** 34K11; 39A10; 39A99

**Keywords:** Lyapunov inequality; nonlinear dynamic system; time scale

## 1 Introduction

The theory of dynamic equations on time scales, which follows Hilger's landmark paper [1], is a new study area of mathematics that has received a lot of attention. For example, we refer the reader to monographs [2, 3] and the references therein. During the last few years, some Lyapunov inequalities for dynamic equations on time scales have been obtained by many authors [4–7].

In 2002, Bohner *et al.* [8] investigated the second-order Sturm-Liouville dynamic equation

$$x^{\Delta^2}(t) + q(t)x^{\sigma}(t) = 0 \tag{1.1}$$

on time scale $\mathbb{T}$ under the conditions $x(a) = x(b) = 0$ ($a,b \in \mathbb{T}$ with $a < b$) and $q \in C_{\mathrm{rd}}(\mathbb{T},(0,\infty))$ and showed that if $x(t)$ is a solution of (1.1) with $\max_{t\in[a,b]_{\mathbb{T}}} |x(t)| > 0$, then

$$\int_a^b q(t)\Delta t \geq \frac{b-a}{C},$$

where $[a,b]_{\mathbb{T}} \equiv \{t \in \mathbb{T} : a \leq t \leq b\}$ and $C = \max\{(t-a)(b-t) : t \in [a,b]_{\mathbb{T}}\}$.

When $\mathbb{T} = \mathbb{R}$, (1.1) reduces to the Hills equation

$$x''(t) + u(t)x(t) = 0. \tag{1.2}$$

In 1907, Lyapunov [9] showed that if $u \in C([a,b],\mathbb{R})$ and $x(t)$ is a solution of (1.2) satisfying $x(a) = x(b) = 0$ and $\max_{t\in[a,b]} |x(t)| > 0$, then the following classical Lyapunov inequality holds:

$$\int_a^b |u(t)|\,dt > \frac{4}{b-a}.$$

This was later strengthened with $|u(t)|$ replaced by $u^+(t) = \max\{u(t), 0\}$ by Wintner [10] and thereafter by some other authors:

$$\int_a^b u^+(t)\,dt > \frac{4}{b-a}.$$

Moreover, the last inequality is optimal.

When $\mathbb{T}$ is the set $\mathbb{Z}$ of the integers, (1.1) reduces to the linear difference equation

$$\Delta^2 x(n) + u(n)x(n+1) = 0. \tag{1.3}$$

In 1983, Cheng [11] showed that if $a, b \in \mathbb{Z}$ with $0 < a < b$ and $x(n)$ is a solution of (1.3) satisfying $x(a) = x(b) = 0$ and $\max_{n\in\{a,a+1,\dots,b\}} |x(n)| > 0$, then

$$\sum_{n=a}^{b-2} |u(n)| \geq \begin{cases} \frac{4(b-a)}{(b-a)^2-1} & \text{if } b-a-1 \text{ is even,} \\ \frac{4}{b-a} & \text{if } b-a-1 \text{ is odd.} \end{cases}$$

The purpose of this paper is to establish several Lyapunov inequalities for the nonlinear dynamic system

$$\begin{cases} x^\Delta(t) = -A(t)x(\sigma(t)) - B(t)y(t)|\sqrt{B(t)}y(t)|^{p-2}, \\ y^\Delta(t) = C(t)x(\sigma(t))|x(\sigma(t))|^{q-2} + A^T(t)y(t), \end{cases} \tag{1.4}$$

on a given time scale interval $[a,b]_\mathbb{T}$ ($a, b \in \mathbb{T}$ with $\sigma(a) < b$), where $p, q \in (1, +\infty)$ satisfy $1/p + 1/q = 1$, $A(t)$ is a real $n \times n$ matrix-valued function on $[a,b]_\mathbb{T}$ such that $I + \mu(t)A(t)$ is invertible, $B(t)$ and $C(t)$ are two real $n \times n$ symmetric matrix-valued functions on $[a,b]_\mathbb{T}$, $B(t)$ being positive definite, $A^T(t)$ is the transpose of $A(t)$, and $x(t)$, $y(t)$ are two real $n$-dimensional vector-valued functions on $[a,b]_\mathbb{T}$.

When $n = 1$ and $p = q = 2$, (1.4) reduces to

$$\begin{cases} x^\Delta(t) = u(t)x(\sigma(t)) + v(t)y(t), \\ y^\Delta(t) = -w(t)x(\sigma(t)) - u(t)y(t), \end{cases} \tag{1.5}$$

where $u(t)$, $v(t)$, and $w(t)$ are real-valued rd-continuous functions on $\mathbb{T}$ satisfying $v(t) \geq 0$ for any $t \in \mathbb{T}$.

In 2011, He *et al.* [12] obtained the following result.

**Theorem 1.1** ([12]) *Let* $1 - \mu(t)u(t) > 0$ *for any* $t \in \mathbb{T}$ *and* $a, b \in \mathbb{T}^k$ *with* $\sigma(a) \leq b$. *If* (1.5) *has a real solution* $(x(t), y(t))$ *such that*

$$x(a) = 0 \quad or \quad x(a)x\bigl(\sigma(a)\bigr) < 0;$$
$$x(b) = 0 \quad or \quad x(b)x\bigl(\sigma(b)\bigr) < 0; \qquad \max_{t\in[a,b]_{\mathbb{T}}} \bigl|x(t)\bigr| > 0,$$

*then we have the following inequality*:

$$\int_a^b \bigl|u(t)\bigr|\Delta(t) + \left[\int_a^{\sigma(b)} v(t)\Delta(t) \int_a^b w^+(t)\Delta(t)\right]^{1/2} \geq 2,$$

*where* $w^+(t) = \max\{w(t), 0\}$.

In 2016, Liu *et al.* [13] obtained the following theorem.

**Theorem 1.2** *Let* $p = q = 2$ *and* $a, b \in \mathbb{T}$ *with* $\sigma(a) < b$. *If* (1.4) *has a solution* $(x(t), y(t))$ *such that*

$$x(a) = x(b) = 0 \quad and \quad \max_{t\in[a,b]_{\mathbb{T}}} x^T(t)x(t) > 0, \tag{1.6}$$

*then for any* $n \times n$ *symmetric matrix-valued function* $C_1(t)$ *with* $C_1(t) - C(t) \geq 0$, *we have the following inequalities*:

(1)

$$\int_a^b \frac{[\int_a^{\sigma(t)} |B(s)||e_{\ominus A}(\sigma(t), s)|^2\Delta s][\int_{\sigma(t)}^b |B(s)||e_{\ominus A}(\sigma(t), s)|^2\Delta s]}{\int_a^b |B(s)||e_{\ominus A}(\sigma(t), s)|^2\Delta s}\bigl|C_1(t)\bigr|\Delta t \geq 1,$$

(2)

$$\int_a^b \bigl|C_1(t)\bigr|\left\{\int_a^b \bigl|B(s)\bigr|\bigl|e_{\ominus A}\bigl(\sigma(t), s\bigr)\bigr|^2\Delta s\right\}\Delta t \geq 4,$$

(3)

$$\int_a^b \bigl|A(t)\bigr|\Delta t + \left(\int_a^b \bigl|\sqrt{B(t)}\bigr|^2\Delta t\right)^{1/2}\left(\int_a^b \bigl|C_1(t)\bigr|\Delta t\right)^{1/2} \geq 2.$$

For some other related results on Lyapunov-type inequalities, see, for example, [14–23].

## 2 Preliminaries and some lemmas

Throughout this paper, we adopt basic definitions and notation of monograph [2]. A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. On a time scale $\mathbb{T}$, the forward jump operator, the backward jump operator, and the graininess function are defined as

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad \text{and} \quad \mu(t) = \sigma(t) - t,$$

respectively.

The point $t \in \mathbb{T}$ is said to be left-dense (resp. left-scattered) if $\rho(t) = t$ (resp. $\rho(t) < t$). The point $t \in \mathbb{T}$ is said to be right-dense (resp. right-scattered) if $\sigma(t) = t$ (resp. $\sigma(t) > t$). If $\mathbb{T}$ has a left-scattered maximum $M$, then we define $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if $f$ is continuous at right-dense points and has finite left-sided limits at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions from $\mathbb{T}$ to $\mathbb{R}$ is denoted by $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. For a function $f : \mathbb{T} \to \mathbb{R}$, the notation $f^\sigma$ means the composition $f \circ \sigma$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the (delta) derivative $f^\Delta(t)$ at $t \in \mathbb{T}$ is defined as the number (if it exists) such that for given any $\varepsilon > 0$, there is a neighborhood $U$ of $t$ with

$$\left|f\left(\sigma(t)\right) - f(s) - f^\Delta(t)\left(\sigma(t) - s\right)\right| \leq \varepsilon\left|\sigma(t) - s\right|$$

for all $s \in U$. If the (delta) derivative $f^\Delta(t)$ exists for every $t \in \mathbb{T}^k$, then we say that $f$ is $\Delta$-differentiable on $\mathbb{T}$.

Let $F, f \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ satisfy $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^k$. Then, for any $c, d \in \mathbb{T}$, the Cauchy integral of $f$ is defined as

$$\int_c^d f(t)\Delta t = F(d) - F(c).$$

For any $z \in \mathbb{R}^n$ and any $S \in \mathbb{R}^{n\times n}$ (the space of real $n \times n$ matrices), write

$$|z| = \sqrt{z^T z} \quad \text{and} \quad |S| = \max_{z\in\mathbb{R}^n, z\neq 0} \frac{|Sz|}{|z|},$$

which are called the Euclidean norm of $z$ and the matrix norm of $S$, respectively. It is obvious that, for any $z \in \mathbb{R}^n$ and $U, V \in \mathbb{R}^{n\times n}$,

$$|Uz| \leq |U||z| \quad \text{and} \quad |UV| \leq |U||V|.$$

Let $\mathbb{R}_s^{n\times n}$ be the set of all symmetric real $n \times n$ matrices. We can show that, for any $U \in \mathbb{R}_s^{n\times n}$,

$$|U| = \max_{|\lambda I - U| = 0} |\lambda| \quad \text{and} \quad |U^2| = |U|^2.$$

A matrix $S \in \mathbb{R}_s^{n\times n}$ is said to be positive definite (resp. semipositive definite), written as $S > 0$ (resp. $S \geq 0$), if $y^T Sy > 0$ (resp. $y^T Sy \geq 0$) for any $y \in \mathbb{R}^n$ with $y \neq 0$. If $S$ is positive definite (resp. semipositive definite), then there exists a unique positive definite matrix (resp. semipositive definite matrix), written as $\sqrt{S}$, satisfying $[\sqrt{S}]^2 = S$.

In this paper, we establish Lyapunov inequalities for (1.4) that has a solution $(x(t), y(t))$ satisfying

$$x(a) = x(b) = 0 \quad \text{and} \quad \max_{t\in[a,b]_{\mathbb{T}}} x^T(t)x(t) > 0. \tag{2.1}$$

We first introduce the following lemmas.

**Lemma 2.1** ([2]) *Let $1/p + 1/q = 1$ ($p,q \in (1,+\infty)$) and $a,b \in \mathbb{T}$ ($a < b$). Then, for any $f,g \in C_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R})$,*

$$\int_a^b \left|f(t)g(t)\right|\Delta t \leq \left(\int_a^b \left|f(t)\right|^p \Delta t\right)^{\frac{1}{p}} \left(\int_a^b \left|g(t)\right|^q \Delta t\right)^{\frac{1}{q}}.$$

**Lemma 2.2** *Let $a,b \in \mathbb{T}$ with $a < b$. Suppose that $\alpha,\beta,\gamma,\delta \in \mathbb{R}$ and $p,q \in (1,+\infty)$ with $\alpha/p + \beta/q = \gamma/p + \delta/q = 1/p + 1/q = 1$. Then, for any $f,g \in C_{\mathrm{rd}}([a,b]_{\mathbb{T}},(-\infty,0) \cup (0,\infty))$,*

$$\int_a^b \left|f(t)g(t)\right|\Delta t \leq \left(\int_a^b \left|f(t)\right|^\alpha \left|g(t)\right|^\gamma \Delta t\right)^{\frac{1}{p}} \left(\int_a^b \left|f(t)\right|^\beta \left|g(t)\right|^\delta \Delta t\right)^{\frac{1}{q}}.$$

*Proof* Let $M(t) = (|f(t)|^\alpha |g(t)|^\gamma)^{\frac{1}{p}}$ and $N(t) = (|f(t)|^\beta |g(t)|^\delta)^{\frac{1}{q}}$. Then by Lemma 2.1 we have

$$\begin{aligned}
\int_a^b \left|f(t)g(t)\right|\Delta t &= \int_a^b M(t)N(t)\Delta t \\
&\leq \left(\int_a^b M^p(t)\Delta t\right)^{\frac{1}{p}} \left(\int_a^b N^q(t)\Delta t\right)^{\frac{1}{q}} \\
&= \left(\int_a^b \left|f(t)\right|^\alpha \left|g(t)\right|^\gamma \Delta t\right)^{\frac{1}{p}} \left(\int_a^b \left|f(t)\right|^\beta \left|g(t)\right|^\delta \Delta t\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof of Lemma 2.2. □

**Remark 2.3** Let $\gamma = 0$ in Lemma 2.2. Then we obtain that, for any $f,g \in C_{\mathrm{rd}}([a,b]_{\mathbb{T}},(-\infty,0) \cup (0,\infty))$,

$$\int_a^b \left|f(t)g(t)\right|\Delta t \leq \left\{\max_{t\in[a,b]_{\mathbb{T}}} \left|f(t)\right|^\beta\right\}^{\frac{1}{q}} \left(\int_a^b \left|f(t)\right|^\alpha \Delta t\right)^{\frac{1}{p}} \left(\int_a^b \left|g(t)\right|^q \Delta t\right)^{\frac{1}{q}}.$$

**Lemma 2.4** ([2]) *If $A \in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^{n\times n})$ with invertible $I + \mu(t)A(t)$, $f \in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^n)$, $t_0 \in \mathbb{T}$, and $a \in \mathbb{R}^n$, then*

$$x(t) = e_{\ominus A}(t,t_0)a + \int_{t_0}^t e_{\ominus A}(t,\tau)f(\tau)\Delta\tau$$

*is the unique solution of the initial value problem*

$$\begin{cases} x^\Delta(t) = -A(t)x(\sigma(t)) + f(t), \\ x(t_0) = a, \end{cases}$$

*where $(\ominus A)(t) = -[I + \mu(t)A(t)]^{-1}A(t)$ for any $t \in \mathbb{T}^k$, and $e_{\ominus A}(t,t_0)$ is the unique matrix-valued solution of the initial value problem*

$$\begin{cases} Y^\Delta(t) = (\ominus A)(t)Y(t), \\ Y(t_0) = I. \end{cases}$$

**Lemma 2.5** ([2]) *Let $A,B \in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^{n\times n})$ be $\Delta$-differentiable. Then*

$$\big(A(t)B(t)\big)^\Delta = A^\sigma(t)B^\Delta(t) + A^\Delta(t)B(t) = A^\Delta(t)B^\sigma(t) + A(t)B^\Delta(t).$$

**Lemma 2.6** ([13]) *If $f_1(t), f_2(t), \dots, f_n(t)$ are $\Delta$-integrable on $[a,b]_{\mathbb{T}}$ and $x(t) = (f_1(t), f_2(t), \dots, f_n(t))$, then*

$$\left|\int_a^b x(t)\Delta t\right| = \left\{\sum_{i=1}^n \left(\int_a^b f_i(t)\Delta t\right)^2\right\}^{\frac{1}{2}} \le \int_a^b \left\{\sum_{i=1}^n f_i^2(t)\right\}^{\frac{1}{2}} \Delta t = \int_a^b \left|x(t)\right|\Delta t.$$

**Lemma 2.7** ([13]) *If $A_1, A_2 \in \mathbb{R}_s^{n\times n}$ and $A_1 - A_2 \ge 0$, then, for any $x \in \mathbb{R}^n$,*

$$\left(x^\sigma\right)^T A_2 x^\sigma \le |A_1|\left|x^\sigma\right|^2.$$

## 3 Main results and proofs

In this section, we assume that $\alpha, \beta \in \mathbb{R}$ and $p, q \in (1, +\infty)$ satisfy

$$\alpha/p + \beta/q = 1/p + 1/q = 1.$$

For any $t, \tau \in [a,b]_{\mathbb{T}}$, write

$$\begin{aligned}
&F(t,\tau) = \left|e_{\ominus A}\left(\sigma(t), \tau\right)\right|\left|\sqrt{B(\tau)}\right|,\\
&G(t) = \left|\sqrt{B(t)}y(t)\right|^{p-2} y^T(t)B(t)y(t) = \left|\sqrt{B(t)}y(t)\right|^p,\\
&\Phi\left(\sigma(t)\right) = \left(\int_a^{\sigma(t)} F^\alpha(t,s)\Delta s\right)^{\frac{q}{p}},\\
&\Psi\left(\sigma(t)\right) = \left(\int_{\sigma(t)}^b F^\alpha(t,s)\Delta s\right)^{\frac{q}{p}},\\
&P(t) = \Phi\left(\sigma(t)\right)\Psi\left(\sigma(t)\right) \max_{a\le\tau\le\sigma(t)} F^\beta(t,\tau) \max_{\sigma(t)\le\tau\le b} F^\beta(t,\tau),\\
&Q(t) = \Phi\left(\sigma(t)\right) \max_{a\le\tau\le\sigma(t)} F^\beta(t,\tau) + \Psi\left(\sigma(t)\right) \max_{\sigma(t)\le\tau\le b} F^\beta(t,\tau).
\end{aligned}$$

**Theorem 3.1** *Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$ and $C_1 \in \mathbb{R}_s^{n\times n}$ with $C_1(t) - C(t) \ge 0$. If (1.4) has a solution $(x(t), y(t))$ with $x(t), y(t) \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$ satisfying (2.1) on the interval $[a,b]_{\mathbb{T}}$, then*

$$\int_a^b \frac{P(t)}{Q(t)}\left|C_1(t)\right|\Delta t \ge 1. \tag{3.1}$$

*Proof* Since $(x(t), y(t))$ is a solution of (1.4), we have

$$\left(y^T(t)x(t)\right)^\Delta = \left(x^\sigma(t)\right)^T C(t)x^\sigma(t)\left|x^\sigma(t)\right|^{q-2} - G(t). \tag{3.2}$$

Integrating (3.2) from $a$ to $b$ and noting that $x(a) = x(b) = 0$, we obtain

$$\int_a^b G(t)\Delta t = \int_a^b \left|x^\sigma(t)\right|^{q-2}\left(x^\sigma(t)\right)^T C(t)x^\sigma(t)\Delta t.$$

Noting that $B(t) > 0$, we know that $y^T(t)B(t)y(t) \ge 0$, $t \in [a,b]_{\mathbb{T}}$.

We claim that $y^T(t)B(t)y(t) \not\equiv 0$ $(t \in [a,b]_{\mathbb{T}})$. Indeed, if $y^T(t)B(t)y(t) \equiv 0$ $(t \in [a,b]_{\mathbb{T}})$, then

$$\left|\sqrt{B(t)}y(t)\right|^2 = y^T(t)B(t)y(t) \equiv 0,$$

which implies $B(t)y(t) \equiv 0$ ($t \in [a,b]_{\mathbb{T}}$). Thus, the first equation of (1.4) reduces to

$$x^{\Delta}(t) = -A(t)x\bigl(\sigma(t)\bigr), \qquad x(a) = 0.$$

By Lemma 2.4 it follows

$$x(t) = e_{\ominus A}(t,a) \cdot 0 = 0,$$

which is a contradiction to (2.1). Hence, we obtain that

$$\int_a^b \bigl|x^{\sigma}(t)\bigr|^{q-2}\bigl(x^{\sigma}(t)\bigr)^T C(t)x^{\sigma}(t)\Delta t = \int_a^b G(t)\Delta t > 0, \tag{3.3}$$

and it follows from Lemma 2.4 that, for $t \in [a,b]_{\mathbb{T}}$,

$$\begin{aligned} x(t) &= -\int_a^t e_{\ominus A}(t,\tau)B(\tau)y(\tau)\bigl|\sqrt{B(\tau)}y(\tau)\bigr|^{p-2}\Delta\tau \\ &= -\int_b^t e_{\ominus A}(t,\tau)B(\tau)y(\tau)\bigl|\sqrt{B(\tau)}y(\tau)\bigr|^{p-2}\Delta\tau, \end{aligned}$$

which implies that, for $t \in [a,b)_{\mathbb{T}}$,

$$\begin{aligned} x^{\sigma}(t) &= -\int_a^{\sigma(t)} e_{\ominus A}\bigl(\sigma(t),\tau\bigr)B(\tau)y(\tau)\bigl|\sqrt{B(\tau)}y(\tau)\bigr|^{p-2}\Delta\tau \\ &= +\int_{\sigma(t)}^b e_{\ominus A}\bigl(\sigma(t),\tau\bigr)B(\tau)y(\tau)\bigl|\sqrt{B(\tau)}y(\tau)\bigr|^{p-2}\Delta\tau. \end{aligned}$$

Note that, for $a \le \sigma(t) \le b$,

$$\begin{aligned} &\bigl|e_{\ominus A}\bigl(\sigma(t),\tau\bigr)B(\tau)y(\tau)\bigl|\sqrt{B(\tau)}y(\tau)\bigr|^{p-2}\bigr| \\ &\quad\le \bigl|e_{\ominus A}\bigl(\sigma(t),\tau\bigr)\bigr|\bigl|B(\tau)y(\tau)\bigr|\bigl|\sqrt{B(\tau)}y(\tau)\bigr|^{p-2} \\ &\quad\le F(t,\tau)\bigl|\sqrt{B(\tau)}y(\tau)\bigr|\bigl|\sqrt{B(\tau)}y(\tau)\bigr|^{p-2} \\ &\quad= F(t,\tau)G^{\frac{1}{q}}(\tau). \end{aligned}$$

Then by Remark 2.3 and Lemma 2.6 we obtain

$$\begin{aligned} \bigl|x^{\sigma}(t)\bigr|^q &= \left|\int_a^{\sigma(t)} e_{\ominus A}\bigl(\sigma(t),\tau\bigr)B(\tau)y(\tau)\bigl|\sqrt{B(\tau)}y(\tau)\bigr|^{p-2}\Delta\tau\right|^q \\ &\le \left[\int_a^{\sigma(t)} \bigl|e_{\ominus A}\bigl(\sigma(t),\tau\bigr)B(\tau)y(\tau)\bigl|\sqrt{B(\tau)}y(\tau)\bigr|^{p-2}\bigr|\Delta\tau\right]^q \\ &\le \left[\int_a^{\sigma(t)} F(t,\tau)G^{\frac{1}{q}}(\tau)\Delta\tau\right]^q \\ &\le \left(\int_a^{\sigma(t)} F^{\alpha}(t,\tau)\Delta\tau\right)^{\frac{q}{p}} \int_a^{\sigma(t)} F^{\beta}(t,\tau)G(\tau)\Delta\tau \\ &\le \max_{a\le\tau\le\sigma(t)} F^{\beta}(t,\tau)\left(\int_a^{\sigma(t)} F^{\alpha}(t,\tau)\Delta\tau\right)^{\frac{q}{p}} \int_a^{\sigma(t)} G(\tau)\Delta\tau, \end{aligned}$$

that is,

$$\left|x^{\sigma}(t)\right|^{q} \leq \max_{a\leq\tau\leq\sigma(t)} F^{\beta}(t,\tau)\Phi\bigl(\sigma(t)\bigr)\int_{a}^{\sigma(t)} G(\tau)\Delta\tau. \tag{3.4}$$

Similarly, for $a \leq \sigma(t) \leq b$, we have

$$\left|x^{\sigma}(t)\right|^{q} \leq \max_{\sigma(t)\leq\tau\leq b} F^{\beta}(t,\tau)\Psi\bigl(\sigma(t)\bigr)\int_{\sigma(t)}^{b} G(\tau)\Delta\tau. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\left|x^{\sigma}(t)\right|^{q} \leq \frac{P(t)}{Q(t)}\int_{a}^{b} G(\tau)\Delta\tau.$$

Then by (3.3) and Lemma 2.7 we have

$$\begin{aligned}
&\int_{a}^{b}\left|C_1(t)\right|\left|x^{\sigma}(t)\right|^{q}\Delta t \\
&\quad\leq \int_{a}^{b}\left|C_1(t)\right|\frac{P(t)}{Q(t)}\Delta t\int_{a}^{b} G(t)\Delta t \\
&\quad= \int_{a}^{b}\left|C_1(t)\right|\frac{P(t)}{Q(t)}\Delta t\int_{a}^{b}\left|x^{\sigma}(t)\right|^{q-2}\bigl(x^{\sigma}(t)\bigr)^{T}C(t)x^{\sigma}(t)\Delta t \\
&\quad\leq \int_{a}^{b}\left|C_1(t)\right|\frac{P(t)}{Q(t)}\Delta t\int_{a}^{b}\left|C_1(t)\right|\left|x^{\sigma}(t)\right|^{q}\Delta t.
\end{aligned}$$

Since

$$\int_{a}^{b}\left|C_1(t)\right|\left|x^{\sigma}(t)\right|^{q}\Delta t \geq \int_{a}^{b}\left|x^{\sigma}(t)\right|^{q-2}\bigl(x^{\sigma}(t)\bigr)^{T}C(t)x^{\sigma}(t)\Delta t = \int_{a}^{b} G(t)\Delta t > 0,$$

we get

$$\int_{a}^{b}\frac{P(t)}{Q(t)}\left|C_1(t)\right|\Delta t \geq 1.$$

This completes the proof of Theorem 3.1. □

**Corollary 3.2** *Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$ and $C_1 \in \mathbb{R}_s^{n\times n}$ with $C_1(t) - C(t) \geq 0$. If* (1.4) *has a solution $(x(t), y(t))$ with $x(t), y(t) \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$ satisfying* (2.1) *on the interval $[a, b]_{\mathbb{T}}$, then*

$$\int_{a}^{b} Q(t)\left|C_1(t)\right|\Delta t \geq 4. \tag{3.6}$$

*Proof* Note that

$$\frac{P(t)}{Q(t)} \leq \frac{Q(t)}{4}.$$

It follows from (3.1) that

$$\int_{a}^{b}\frac{Q(t)}{4}\left|C_1(t)\right|\Delta t \geq 1,$$

that is,

$$\int_a^b Q(t)\big|C_1(t)\big|\Delta t \geq 4.$$

This completes the proof of Corollary 3.2. □

**Corollary 3.3** *Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$ and $C_1 \in \mathbb{R}_s^{n\times n}$ with $C_1(t) - C(t) \geq 0$. If* (1.4) *has a solution $(x(t), y(t))$ with $x(t), y(t) \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$ satisfying* (2.1) *on the interval $[a, b]_{\mathbb{T}}$, then*

$$\int_a^b \sqrt{P(t)}\big|C_1(t)\big|\Delta t \geq 2. \tag{3.7}$$

*Proof* Note that

$$Q(t) \geq 2\sqrt{P(t)}.$$

It follows from (3.1) that

$$\int_a^b \sqrt{P(t)}\big|C_1(t)\big|\Delta t \geq \int_a^b 2\frac{P(t)}{Q(t)}\big|C_1(t)\big|\Delta t \geq 2.$$

This completes the proof of Corollary 3.3. □

**Theorem 3.4** *Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$ and $C_1 \in \mathbb{R}_s^{n\times n}$ with $C_1(t) - C(t) \geq 0$. If* (1.4) *has a solution $(x(t), y(t))$ with $x(t), y(t) \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$ satisfying* (2.1) *on the interval $[a, b]_{\mathbb{T}}$, then there exists $c \in (a, b)$ such that*

$$\begin{cases} \int_a^{\sigma(c)} \Phi(\sigma(t)) \max_{a\leq\tau\leq\sigma(t)} F^\beta(t,\tau)|C_1(t)|\Delta t \geq 1, \\ \int_c^b \Psi(\sigma(t)) \max_{\sigma(t)\leq\tau\leq b} F^\beta(t,\tau)|C_1(t)|\Delta t \geq 1. \end{cases} \tag{3.8}$$

*Proof* Set $U(t) = \Phi(\sigma(t)) \max_{a\leq\tau\leq\sigma(t)} F^\beta(t,\tau)$ and $V(t) = \Psi(\sigma(t)) \max_{\sigma(t)\leq\tau\leq b} F^\beta(t,\tau)$. Let

$$f(t) = \int_a^t U(s)\big|C_1(s)\big|\Delta s - \int_t^b V(s)\big|C_1(s)\big|\Delta s.$$

Then we have $f(a) < 0$ and $f(b) > 0$. Hence, we can choose $c \in (a, b)$ such that $f(c) \leq 0$ and $f(\sigma(c)) \geq 0$, that is,

$$\int_a^c U(s)\big|C_1(s)\big|\Delta s \leq \int_c^b V(s)\big|C_1(s)\big|\Delta s \tag{3.9}$$

and

$$\int_a^{\sigma(c)} U(s)\big|C_1(s)\big|\Delta s \geq \int_{\sigma(c)}^b V(s)\big|C_1(s)\big|\Delta s. \tag{3.10}$$

By (3.4) we have that

$$\big|C_1(t)\big|\big|x^\sigma(t)\big|^q \leq U(t)\big|C_1(t)\big|\int_a^{\sigma(t)} G(\tau)\Delta\tau. \tag{3.11}$$

Integrating (3.11) from $a$ to $\sigma(c)$, we obtain

$$\begin{aligned}\int_a^{\sigma(c)} |C_1(t)||x^\sigma(t)|^q \Delta t &\le \int_a^{\sigma(c)} U(t)|C_1(t)|\left(\int_a^{\sigma(t)} G(\tau)\Delta\tau\right)\Delta t \\ &\le \int_a^c U(t)|C_1(t)|\Delta t \int_a^{\sigma(c)} G(\tau)\Delta\tau \\ &\quad + U(c)|C_1(c)|(\sigma(c)-c)\int_a^{\sigma(c)} G(\tau)\Delta\tau \\ &= \int_a^{\sigma(c)} U(t)|C_1(t)|\Delta t \int_a^{\sigma(c)} G(\tau)\Delta\tau.\end{aligned}$$

Similarly, we obtain from (3.4) and (3.10) that

$$\begin{aligned}\int_{\sigma(c)}^b |C_1(t)||x^\sigma(t)|^q \Delta t &\le \int_{\sigma(c)}^b V(t)|C_1(t)|\Delta t \int_{\sigma(c)}^b G(\tau)\Delta\tau \\ &\le \int_a^{\sigma(c)} U(t)|C_1(t)|\Delta t \int_{\sigma(c)}^b G(\tau)\Delta\tau.\end{aligned}$$

This yields

$$\begin{aligned}\int_a^b |C_1(t)||x^\sigma(t)|^q \Delta t &\le \int_a^{\sigma(c)} U(t)|C_1(t)|\Delta t \int_a^b G(t)\Delta t \\ &= \int_a^{\sigma(c)} U(t)|C_1(t)|\Delta t \int_a^b |x^\sigma(t)|^{q-2}(x^\sigma(t))^T C(t)x^\sigma(t)\Delta t \\ &\le \int_a^{\sigma(c)} U(t)|C_1(t)|\Delta t \int_a^b |C_1(t)||x^\sigma(t)|^q \Delta t.\end{aligned}$$

Since

$$\begin{aligned}\int_a^b |C_1(t)||x^\sigma(t)|^q \Delta t &\ge \int_a^b |x^\sigma(t)|^{q-2}(x^\sigma(t))^T C(t)x^\sigma(t)\Delta t \\ &= \int_a^b G(t)\Delta t > 0,\end{aligned}$$

we have $\int_a^{\sigma(c)} U(t)|C_1(t)|\Delta t \ge 1$.

Next, we obtain from (3.5) that

$$|x^\sigma(t)|^q |C_1(t)| \le V(t)|C_1(t)| \int_{\sigma(t)}^b G(\tau)\Delta\tau. \tag{3.12}$$

Integrating (3.12) from $c$ to $b$, we have

$$\begin{aligned}\int_c^b |C_1(t)||x^\sigma(t)|^q \Delta t &\le \int_c^b V(t)|C_1(t)|\left(\int_{\sigma(t)}^b G(\tau)\Delta\tau\right)\Delta t \\ &\le \int_c^b V(t)|C_1(t)|\Delta t \int_{\sigma(c)}^b G(\tau)\Delta\tau.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\int_a^c |C_1(t)||x^\sigma(t)|^q \Delta t &\le \int_a^c U(t)|C_1(t)|\Delta t \int_a^{\sigma(c)} G(\tau)\Delta\tau \\ &\le \int_c^b V(t)|C_1(t)|\Delta t \int_a^{\sigma(c)} G(\tau)\Delta\tau.\end{aligned}$$

This yields

$$\begin{aligned}\int_a^b |C_1(t)||x^\sigma(t)|^q \Delta t &\le \int_c^b V(t)|C_1(t)|\Delta t \int_a^b G(t)\Delta t \\ &= \int_c^b V(t)|C_1(t)|\Delta t \int_a^b |x^\sigma(t)|^{q-2}\big(x^\sigma(t)\big)^T C(t)x^\sigma(t)\Delta t \\ &\le \int_c^b V(t)|C_1(t)|\Delta t \int_a^b |C_1(t)||x^\sigma(t)|^q \Delta t.\end{aligned}$$

Thus, we have $\int_c^b V(t)|C_1(t)|\Delta t \ge 1$. This completes the proof of Theorem 3.4. □

**Theorem 3.5** *Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$ and $C_1 \in \mathbb{R}_s^{n\times n}$ with $C_1(t) - C(t) \ge 0$. If (1.4) has a solution $(x(t), y(t))$ with $x(t), y(t) \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$ satisfying (2.1) on the interval $[a,b]_{\mathbb{T}}$, then*

$$\int_a^b |A(t)|\Delta t + \Big\{\max_{a\le t\le b} \big|\sqrt{B(t)}\big|^\beta\Big\}^{\frac{1}{q}} \left(\int_a^b \big|\sqrt{B(t)}\big|^\alpha \Delta t\right)^{\frac{1}{p}} \left(\int_a^b |C_1(t)|\Delta t\right)^{\frac{1}{q}} \ge 2.$$

*Proof* Since $x(a) = x(b) = 0$, we have

$$\int_a^b G(t)\Delta t = \int_a^b |x^\sigma(t)|^{q-2}\big(x^\sigma(t)\big)^T C(t)x^\sigma(t)\Delta t.$$

It follows from the first equation of (1.4) that, for all $a \le t \le b$,

$$\begin{aligned}x(t) &= \int_a^t \big(-A(\tau)x^\sigma(\tau) - B(\tau)\big|\sqrt{B(\tau)}y(\tau)\big|^{p-2}y(\tau)\big)\Delta\tau \\ &= \int_t^b \big(A(\tau)x^\sigma(\tau) + B(\tau)\big|\sqrt{B(\tau)}y(\tau)\big|^{p-2}y(\tau)\big)\Delta\tau.\end{aligned}$$

Thus, we have

$$\begin{aligned}|x(t)| &= \left|\int_a^t \big(-A(\tau)x^\sigma(\tau) - B(\tau)y(\tau)\big)\big|\sqrt{B(\tau)}y(\tau)\big|^{p-2}\Delta\tau\right| \\ &\le \int_a^t \big|A(\tau)x^\sigma(\tau) + B(\tau)y(\tau)\big|\sqrt{B(\tau)}y(\tau)\big|^{p-2}\big|\Delta\tau \\ &\le \int_a^t |A(\tau)x^\sigma(\tau)|\Delta\tau + \int_a^t |B(\tau)y(\tau)|\big|\sqrt{B(\tau)}y(\tau)\big|^{p-2}\Delta\tau \\ &\le \int_a^t |A(\tau)||x^\sigma(\tau)|\Delta\tau + \int_a^t \big|\sqrt{B(\tau)}\big|G^{\frac{1}{q}}(\tau)\Delta\tau.\end{aligned}$$

Similarly, we have

$$\left|x(t)\right| \le \int_t^b \left|A(\tau)\right|\left|x^\sigma(\tau)\right|\Delta\tau + \int_t^b \left|\sqrt{B(\tau)}\right| G^{\frac{1}{q}}(\tau)\Delta\tau.$$

Then we obtain

$$\begin{aligned}
\left|x(t)\right| &\le \frac{1}{2}\left[\int_a^b \left|A(t)\right|\left|x^\sigma(t)\right|\Delta t + \int_a^b \left|\sqrt{B(t)}\right| G^{\frac{1}{q}}(t)\Delta t\right] \\
&\le \frac{1}{2}\left[\int_a^b \left|A(t)\right|\left|x^\sigma(t)\right|\Delta t + \left\{\max_{a\le t\le b}\left|\sqrt{B(t)}\right|^\beta\right\}^{\frac{1}{q}}\right. \\
&\quad \left.\times \left(\int_a^b \left|\sqrt{B(t)}\right|^\alpha \Delta t\right)^{\frac{1}{p}} \left(\int_a^b G(t)\Delta t\right)^{\frac{1}{q}}\right] \\
&= \frac{1}{2}\left[\int_a^b \left|A(t)\right|\left|x^\sigma(t)\right|\Delta t + \left\{\max_{a\le t\le b}\left|\sqrt{B(t)}\right|^\beta\right\}^{\frac{1}{q}} \left(\int_a^b \left|\sqrt{B(t)}\right|^\alpha \Delta t\right)^{\frac{1}{p}}\right. \\
&\quad \left.\times \left(\int_a^b \left|x^\sigma(t)\right|^{q-2}\left(x^\sigma(t)\right)^T C(t)x^\sigma(t)\Delta t\right)^{\frac{1}{q}}\right] \\
&\le \frac{1}{2}\left[\int_a^b \left|A(t)\right|\left|x^\sigma(t)\right|\Delta t + \left\{\max_{a\le t\le b}\left|\sqrt{B(t)}\right|^\beta\right\}^{\frac{1}{q}}\right. \\
&\quad \left.\times \left(\int_a^b \left|\sqrt{B(t)}\right|^\alpha \Delta t\right)^{\frac{1}{p}} \left(\int_a^b \left|C_1(t)\right|\left|x^\sigma(t)\right|^q \Delta t\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Denote $M = \max_{a\le t\le b}|x(t)| > 0$. Then

$$M \le \frac{1}{2}\left[\int_a^b \left|A(t)\right| M\Delta t + \left\{\max_{a\le t\le b}\left|\sqrt{B(t)}\right|^\beta\right\}^{\frac{1}{q}} \left(\int_a^b \left|\sqrt{B(t)}\right|^\alpha \Delta t\right)^{\frac{1}{p}} \left(\int_a^b \left|C_1(t)\right| M^q \Delta t\right)^{\frac{1}{q}}\right].$$

Thus,

$$\int_a^b \left|A(t)\right|\Delta t + \left\{\max_{a\le t\le b}\left|\sqrt{B(t)}\right|^\beta\right\}^{\frac{1}{q}} \left(\int_a^b \left|\sqrt{B(t)}\right|^\alpha \Delta t\right)^{\frac{1}{p}} \left(\int_a^b \left|C_1(t)\right|\Delta t\right)^{\frac{1}{q}} \ge 2.$$

This completes the proof of Theorem 3.5. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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