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# Some identities related to Riemann zeta-function

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**Abstract**

It is well known that the Riemann zeta-function $\zeta(s)$ plays a very important role in the study of analytic number theory. In this paper, we use the elementary method and some new inequalities to study the computational problem of one kind of reciprocal sums related to the Riemann zeta-function at the integer point $s \geq 2$, and for the special values $s = 2, 3$, we give two exact identities for the integer part of the reciprocal sums of the Riemann zeta-function. For general integer $s \geq 4$, we also propose an interesting open problem.

**MSC:** 11B83; 11M06

**Keywords:** Riemann zeta-function; inequality; function [x]; identity; elementary method

## 1 Introduction

Let complex number $s = \sigma + it$, if $\sigma > 1$, then the famous Riemann zeta-function $\zeta(s)$ is defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and it is analytic everywhere except for a simple pole at $s = 1$ with residue 1.

As regards the various properties of $\zeta(s)$, many mathematicians have studied them and obtained abundant research results. Some related work can be found in [1–3], and [4]. However, many research results as regards the Riemann zeta-function basically can be summarized in three aspects: (A) the estimation of the order for the Riemann zeta-function; (B) the mean value theorem for the Riemann zeta-function; (C) the zeros density estimation for the Riemann zeta-function. Particularly with regard to a most important problem related to the zeros density estimation of the Riemann zeta-function one has the most famous Riemann hypothesis.

This paper is inspired by [5, 6], and [7], we will study the properties of the Riemann zeta-function from another angle.

For convenience, we first introduce the concept of the Fibonacci sequence $\{F_n\}$: For all integers $n \geq 1$, the famous Fibonacci sequence $F_n$ is defined by

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

It is clear that $F_n$ is a second-order linear recurrence sequence. That is, $F_{n+1} = F_n + F_{n-1}$ with $F_0 = 0$ and $F_1 = 1$. This sequence has many important positions in the research of pure mathematics and applied mathematics, and it has attracted of many scholars attention and interest. For example, Ohtsuka and Nakamura [5] first studied the properties of the reciprocal sums of $F_n$, and they proved two identities:

$$\left[\left(\sum_{k=n}^{\infty}\frac{1}{F_k}\right)^{-1}\right] = \begin{cases} F_{n-2}, & \text{if } n \geq 2 \text{ is even;} \\ F_{n-2} - 1, & \text{if } n \geq 1 \text{ is odd,} \end{cases}$$

$$\left[\left(\sum_{k=n}^{\infty}\frac{1}{F_k^2}\right)^{-1}\right] = \begin{cases} F_{n-1}F_n - 1, & \text{if } n \geq 2 \text{ is even;} \\ F_nF_{n-1}, & \text{if } n \geq 1 \text{ is odd,} \end{cases}$$

where the function $[x]$ denotes the greatest integer $\leq x$.

Xu and Wang [6] studied a similar problem. They considered the infinite sum of the cubes of the reciprocal $F_n$ and obtained a complex computational formula:

$$\left[\left(\sum_{k=n}^{\infty}\frac{1}{F_k^3}\right)^{-1}\right].$$

Zhang and Wang [7] considered the computational problem of Pell numbers and proved the identity

$$\left[\left(\sum_{k=n}^{\infty}\frac{1}{P_k}\right)^{-1}\right] = \begin{cases} P_{n-1} + P_{n-2}, & \text{if } n \geq 2 \text{ is an even number;} \\ P_{n-1} + P_{n-2} - 1, & \text{if } n \geq 1 \text{ is an odd number,} \end{cases}$$

where the Pell numbers $P_n$ are defined by $P_0 = 0$, $P_1 = 1$, and $P_{n+1} = 2P_n + P_{n-1}$ for all integers $n \geq 1$.

Some other results related to recursive sequence, recursive polynomial and their promotion forms can also be found in [8–15], here no longer list them one by one.

Inspired by the above, we may naturally ask: for the part sums of reciprocal Riemann zeta-function, does there exist a beautiful computational formula? That is to say, for any integers $s \geq 2$ and $n > 1$, does there exist an interesting computational formula for

$$\left[\left(\sum_{k=n}^{\infty}\frac{1}{k^s}\right)^{-1}\right]. \tag{1}$$

About this problem, as far as we know, it seems that none had studied it yet, we also have not seen any related result before. But we think this problem is very interesting and important, because it depicts other important properties of the Riemann zeta-function, especially the asymptotic properties of its part sums.

The main purpose of this paper is to study this problem, and use the elementary method and some new inequalities to give two interesting identities for (1) with $s = 2$ and 3. That is, we shall prove the following two conclusions.

**Theorem 1** *For any $n \geq 1$, we have the identity*

$$\left[\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right)^{-1}\right] = n - 1.$$

**Theorem 2** *For any positive integer $n$, we have*

$$\left[\left(\sum_{k=n}^{\infty}\frac{1}{k^3}\right)^{-1}\right] = 2n(n-1).$$

For $s = 4$, through inspection of the data, we found a very strange problem: there is no such an integral coefficient polynomial $f(x)$ with degree 3 that the following identity holds:

$$\left[\left(\sum_{k=n}^{\infty}\frac{1}{k^4}\right)^{-1}\right] = f(n).$$

Therefore, how to give a precise calculation formula for (1) with $s = 4$ is a very complicated problem. So we propose the following.

**Open problem** For integer $s = 4$, does there exist an exact computational formula for (1)?

We hope people who are interested in this problem can study it together with us, and solve this problem finally.

## 2 Several lemmas

In this section, we shall give some simple lemmas, which are necessary in the proofs of our theorems. First we have the following inequality.

**Lemma 1** *For any integer $n > 1$, we have the inequality*

$$\sum_{k=n}^{\infty}\frac{1}{k^3} < \frac{1}{2n(n-1)}.$$

*Proof* In fact, for any integer $n \geq 2$, note that we have the decomposition

$$\frac{1}{(n-1)(n+1)} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) \quad \text{and} \quad \frac{1}{(n-1)n} = \frac{1}{n-1} - \frac{1}{n}$$

and we have

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{1}{k^3} &< \sum_{k=n}^{\infty}\frac{1}{k(k^2-1)} = \sum_{k=n}^{\infty}\frac{1}{k(k-1)(k+1)} = \sum_{k=n}^{\infty}\frac{1}{2k}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) \\ &= \frac{1}{2}\sum_{k=n}^{\infty}\left(\frac{1}{k(k-1)} - \frac{1}{k(k+1)}\right) = \frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{k(k-1)} - \frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{k(k+1)} \\ &= \frac{1}{2}\sum_{k=n}^{\infty}\left(\frac{1}{k-1} - \frac{1}{k}\right) - \frac{1}{2}\sum_{k=n}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n}\right) = \frac{1}{2n(n-1)}.\end{aligned}$$

This proves Lemma 1. □

**Lemma 2** *For any integer $n \geq 2$, we also have the inequality*

$$\frac{1}{2(2n-2)(2n-1)+1} - \frac{1}{2(2n)(2n+1)+1} < \frac{1}{(2n-1)^3} + \frac{1}{(2n)^3}.$$

*Proof* It is clear that for all integers $n \geq 2$, the inequality in Lemma 2 is equivalent to the following inequality:

$$\frac{1}{2} \cdot \frac{(2n)(2n+1) - (2n-2)(2n-1)}{((2n-2)(2n-1) + \frac{1}{2}) \cdot ((2n)(2n+1) + \frac{1}{2})} < \frac{16n^3 - 12n^2 + 6n - 1}{8n^3(2n-1)^3}$$

or

$$\frac{4n-1}{(4n^2 - 6n + \frac{5}{2}) \cdot (4n^2 + 2n + \frac{1}{2})} < \frac{16n^3 - 12n^2 + 6n - 1}{8n^3(8n^3 - 12n^2 + 6n - 1)}, \tag{2}$$

but the inequality (2) is equivalent to

$$\begin{aligned} &8\bigl(4n^4 - n^3\bigr)\bigl(8n^3 - 12n^2 + 6n - 1\bigr) \\ &\quad < \bigl(16n^3 - 12n^2 + 6n - 1\bigr)\left(16n^4 - 16n^3 + 2n + \frac{5}{4}\right) \end{aligned}$$

or

$$\begin{aligned} &256n^7 - 448n^6 + 288n^5 - 80n^4 + 8n^3 \\ &\quad < 256n^7 - 448n^6 + 288n^5 - 80n^4 + 12n^3 - 3n^2 + \frac{11}{2}n - \frac{5}{4}. \end{aligned} \tag{3}$$

It is clear that (3) is obvious, if $n \geq 2$. So inequality (2) is also correct. This proves Lemma 2. □

**Lemma 3** *For any integer $n \geq 2$, we have the inequality*

$$\frac{1}{2(2n)(2n-1)+1} - \frac{1}{2(2n+2)(2n+1)+1} < \frac{1}{(2n)^3} + \frac{1}{(2n+1)^3}.$$

*Proof* It is clear that, to prove Lemma 3, we only need to prove the inequality

$$\frac{4n+1}{(4n^2 + 6n + \frac{5}{2}) \cdot (4n^2 - 2n + \frac{1}{2})} < \frac{16n^3 + 12n^2 + 6n + 1}{8n^3(8n^3 + 12n^2 + 6n + 1)}. \tag{4}$$

The inequality (4) is equivalent to the inequality

$$\begin{aligned} &8\bigl(4n^4 + n^3\bigr)\bigl(8n^3 + 12n^2 + 6n + 1\bigr) \\ &\quad < \bigl(16n^3 + 12n^2 + 6n + 1\bigr)\left(16n^4 + 16n^3 - 2n + \frac{5}{4}\right) \end{aligned}$$

or

$$\begin{aligned} &256n^7 + 448n^6 + 288n^5 + 80n^4 + 8n^3 \\ &\quad < 256n^7 + 448n^6 + 288n^5 + 80n^4 + 12n^3 + 3n^2 + \frac{11}{2n} + \frac{5}{4}. \end{aligned} \tag{5}$$

It is clear that (5) holds, so inequality (4) is correct. That means that Lemma 3 is correct. □

## 3 Proof of the theorems

In this section, we shall complete the proofs of our theorems. First for any integer $n \geq 2$, note that we have the inequalities

$$\frac{1}{n} = \sum_{k=n}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \sum_{k=n}^{\infty}\frac{1}{k(k+1)} < \sum_{k=n}^{\infty}\frac{1}{k^2}$$

and

$$\sum_{k=n}^{\infty}\frac{1}{k^2} < \sum_{k=n}^{\infty}\frac{1}{k(k-1)} = \sum_{k=n}^{\infty}\left(\frac{1}{k-1} - \frac{1}{k}\right) = \frac{1}{n-1}.$$

So we have the inequality

$$n-1 < \left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right)^{-1} < n.$$

It is clear that this inequality also holds if $n = 1$. So for all integers $n \geq 1$, we have the identity

$$\left[\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right)^{-1}\right] = n-1.$$

This proves Theorem 1.

Now we prove Theorem 2. First we prove the inequality

$$\frac{1}{2n(n-1)+1} < \sum_{k=n}^{\infty}\frac{1}{k^3}. \tag{6}$$

In fact if $n = 2m-1 > 1$ is an odd number, then from Lemma 2 we have

$$\begin{aligned}\sum_{k=2m-1}^{\infty}\frac{1}{k^3} &= \sum_{k=m}^{\infty}\frac{1}{(2k-1)^3} + \sum_{k=m}^{\infty}\frac{1}{(2k)^3} = \sum_{k=m}^{\infty}\left(\frac{1}{(2k-1)^3} + \frac{1}{(2k)^3}\right) \\ &> \frac{1}{2}\sum_{k=m}^{\infty}\left(\frac{1}{(2k-2)(2k-1)+\frac{1}{2}} - \frac{1}{(2k)(2k+1)+\frac{1}{2}}\right) \\ &= \frac{1}{2(2m-2)(2m+1)+1}.\end{aligned} \tag{7}$$

Similarly, if $n = 2m$ is an even number, then from Lemma 3 we have

$$\begin{aligned}\sum_{k\geq 2m}^{\infty}\frac{1}{k^3} &= \sum_{k=m}^{\infty}\frac{1}{(2k)^3} + \sum_{k=m}^{\infty}\frac{1}{(2k+1)^3} = \sum_{k=m}^{\infty}\left(\frac{1}{(2k)^3} + \frac{1}{(2k+1)^3}\right) \\ &> \frac{1}{2}\sum_{k=m}^{\infty}\left(\frac{1}{(2k)(2k-1)+\frac{1}{2}} - \frac{1}{(2k+2)(2k+1)+\frac{1}{2}}\right) \\ &= \frac{1}{2(2m-1)(2m)+1}.\end{aligned} \tag{8}$$

Combining inequalities (7), (8), and Lemma 1 we may deduce the inequality

$$\frac{1}{2n(n-1)+1} < \sum_{k=n}^{\infty} \frac{1}{k^3} < \frac{1}{2n(n-1)},$$

which implies

$$2n(n-1) < \left( \sum_{k=n}^{\infty} \frac{1}{k^3} \right)^{-1} < 2n(n-1)+1. \tag{9}$$

For any integer $n \geq 2$, from the definition of the function $[x]$ and (9) we may immediately deduce the identity

$$\left[ \left( \sum_{k=n}^{\infty} \frac{1}{k^3} \right)^{-1} \right] = 2n(n-1). \tag{10}$$

If $n = 1$, then $\sum_{k=1}^{\infty} \frac{1}{k^3} > 1$, so

$$0 < \left( \sum_{k=1}^{\infty} \frac{1}{k^3} \right)^{-1} < 1 \quad \text{or} \quad \left[ \left( \sum_{k=1}^{\infty} \frac{1}{k^3} \right)^{-1} \right] = 0.$$

This completes the proof of Theorem 2.

**Competing interests**
The author declares that they have no competing interests.

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