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Some generalizations of operator inequalities for positive linear maps

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Abstract
In this paper, we generalize some operator inequalities for positive linear maps due to Lin (Stud. Math. 215:187-194, 2013) and Zhang (Banach J. Math. Anal. 9:166-172, 2015).

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1 Introduction

Throughout this paper, let M, M', m, m' be scalars, I be the identity operator, and $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. The operator norm is denoted by $\|\cdot\|$. We write $A \geq 0$ if the operator A is positive. If $A - B \geq 0$, then we say that $A \geq B$. For $A, B > 0$, we use the following notation:

$A \nabla_\mu B = (1-\mu)A + \mu B$, $A \sharp_\mu B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\mu A^{\frac{1}{2}}$, where $0 \leq \mu \leq 1$.

When $\mu = \frac{1}{2}$ we write $A \nabla B$ and $A \sharp B$ for brevity for $A \nabla_{\frac{1}{2}} B$ and $A \sharp_{\frac{1}{2}} B$, respectively; see Kubo and Ando [3].

A linear map Φ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be unital if $\Phi(I) = I$. We say that Φ is 2-positive if whenever the 2×2 operator matrix $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ is positive, then so is $\begin{bmatrix} \Phi(A) & \Phi(B) \\ \Phi(B^*) & \Phi(C) \end{bmatrix}$.

Let $0 < m \leq A, B \leq M$ and Φ be positive unital linear map. Lin [1], Theorem 2.1, proved the following reversed operator AM-GM inequalities:

$$\Phi^2\left(\frac{A+B}{2}\right) \leq K^2(h)\Phi^2(A \sharp B), \tag{1.1}$$

$$\Phi^2\left(\frac{A+B}{2}\right) \leq K^2(h)\bigl(\Phi(A) \sharp \Phi(B)\bigr)^2, \tag{1.2}$$

where $K(h) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$ is the Kantorovich constant.

Can the inequalities (1.1) and (1.2) be improved? Lin [1], Conjecture 4.2, conjectured that the constant $K(h)$ can be replaced by the Specht ratio $S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}$ in (1.1) and (1.2), which remains as an open question.

Zhang [2], Theorem 2.6, generalized (1.1) and (1.2) when $p \geq 2$:

$$\Phi^{2p}\left(\frac{A+B}{2}\right) \leq \frac{(K(M^2+m^2))^{2p}}{16M^{2p}m^{2p}}\Phi^{2p}(A \sharp B), \tag{1.3}$$

$$\Phi^{2p}\left(\frac{A+B}{2}\right) \leq \frac{(K(M^2+m^2))^{2p}}{16M^{2p}m^{2p}}\left(\Phi(A)\sharp\Phi(B)\right)^{2p}. \tag{1.4}$$

We will present some operator inequalities which are generalizations of (1.1), (1.2), (1.3), and (1.4) in the next section.

Bhatia and Davis [4] proved that if $0 < m \leq A \leq M$ and X and Y are two partial isometries on $\mathcal{H}$ whose final spaces are orthogonal to each other. Then for every 2-positive unital linear map Φ,

$$\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX) \leq \left(\frac{M-m}{M+m}\right)^2\Phi(X^*AX). \tag{1.5}$$

Lin [5], Conjecture 3.4, conjectured that the following inequality could be true:

$$\left\|\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX)\Phi(X^*AX)^{-1}\right\| \leq \left(\frac{M-m}{M+m}\right)^2.$$

Recently, Fu and He [6], Theorem 5, proved

$$\left\|\Phi(X^*AY)\Phi(Y^*AY)^{-1}\Phi(Y^*AX)\Phi(X^*AX)^{-1}\right\| \leq \frac{1}{4}\left(\left(\frac{M-m}{M+m}\right)^2 M + \frac{1}{m}\right)^2. \tag{1.6}$$

We will get a stronger result than (1.6).

2 Main results

We begin this section with the following lemmas.

Lemma 1 [7] *Let $A, B > 0$. Then the following norm inequality holds*:

$$\|AB\| \leq \frac{1}{4}\|A+B\|^2. \tag{2.1}$$

Lemma 2 [8] *Let $A > 0$. Then for every positive unital linear map Φ*,

$$\Phi(A^{-1}) \geq \Phi^{-1}(A). \tag{2.2}$$

Lemma 3 [9] *Let $A, B > 0$. Then, for $1 \leq r < \infty$*,

$$\left\|A^r + B^r\right\| \leq \left\|(A+B)^r\right\|. \tag{2.3}$$

Lemma 4 ([10], Theorem 7) *Suppose that two operators A, B and positive real numbers m, m', M, M' satisfy either of the following conditions*:

(1) $0 < m \leq A \leq m' < M' \leq B \leq M$,

(2) $0 < m \leq B \leq m' < M' \leq A \leq M$.

Then

$$A \nabla_\mu B \geq K^r(h')A \sharp_\mu B$$

for all $\mu \in [0,1]$, where $r = \min(\mu, 1-\mu)$ and $h' = \frac{M'}{m'}$.

Theorem 1 *Let* $0 < m \le A \le m' < M' \le B \le M$. *Then*

$$\frac{A+B}{2} + MmK^{\frac{1}{2}}(h')(A \sharp B)^{-1} \le M + m, \tag{2.4}$$

where $K(h') = \frac{(h'+1)^2}{4h'}$ *with* $h' = \frac{M'}{m'}$.

Proof It is easy to see that

$$\frac{1}{2}(M - A)(m - A)A^{-1} \le 0,$$

then

$$Mm\frac{A^{-1}}{2} + \frac{A}{2} \le \frac{M+m}{2}.$$

Similarly,

$$Mm\frac{B^{-1}}{2} + \frac{B}{2} \le \frac{M+m}{2}.$$

Summing up the above two inequalities, we get

$$\frac{A+B}{2} + Mm\frac{A^{-1}+B^{-1}}{2} \le M + m.$$

By $(A \sharp B)^{-1} = A^{-1} \sharp B^{-1}$ and Lemma 4, we have

$$\begin{aligned} \frac{A+B}{2} + MmK^{\frac{1}{2}}(h')(A \sharp B)^{-1} &= \frac{A+B}{2} + MmK^{\frac{1}{2}}(h')\left(A^{-1} \sharp B^{-1}\right) \\ &\le \frac{A+B}{2} + Mm\frac{A^{-1}+B^{-1}}{2} \\ &\le M + m. \end{aligned}$$

This completes the proof. □

Theorem 2 *Let* $0 < m \le A \le m' < M' \le B \le M$. *Then for every positive unital linear map* Φ,

$$\Phi^2\left(\frac{A+B}{2}\right) \le \frac{K^2(h)}{K(h')}\Phi^2(A \sharp B) \tag{2.5}$$

and

$$\Phi^2\left(\frac{A+B}{2}\right) \le \frac{K^2(h)}{K(h')}\left(\Phi(A) \sharp \Phi(B)\right)^2, \tag{2.6}$$

where $K(h) = \frac{(h+1)^2}{4h}$, $K(h') = \frac{(h'+1)^2}{4h'}$, $h = \frac{M}{m}$, *and* $h' = \frac{M'}{m'}$.

Proof The inequality (2.5) is equivalent to

$$\left\| \Phi\left(\frac{A+B}{2}\right)\Phi^{-1}(A \sharp B)\right\| \leq \frac{K(h)}{K^{\frac{1}{2}}(h')}. \tag{2.7}$$

Compute

$$\begin{aligned}
&\left\| \Phi\left(\frac{A+B}{2}\right) MmK^{\frac{1}{2}}(h')\Phi^{-1}(A \sharp B)\right\| \\
&\leq \frac{1}{4}\left\| \Phi\left(\frac{A+B}{2}\right) + MmK^{\frac{1}{2}}(h')\Phi^{-1}(A \sharp B)\right\|^2 \quad \text{(by (2.1))} \\
&\leq \frac{1}{4}\left\| \Phi\left(\frac{A+B}{2}\right) + MmK^{\frac{1}{2}}(h')\Phi\big((A \sharp B)^{-1}\big)\right\|^2 \quad \text{(by (2.2))} \\
&= \frac{1}{4}\left\| \Phi\left(\frac{A+B}{2} + MmK^{\frac{1}{2}}(h')(A \sharp B)^{-1}\right)\right\|^2 \\
&\leq \frac{1}{4}\left\| \Phi(M+m)\right\|^2 \quad \text{(by (2.4))} \\
&= \frac{1}{4}(M+m)^2.
\end{aligned}$$

That is,

$$\left\| \Phi\left(\frac{A+B}{2}\right)\Phi^{-1}(A \sharp B)\right\| \leq \frac{(M+m)^2}{4MmK^{\frac{1}{2}}(h')} = \frac{K(h)}{K^{\frac{1}{2}}(h')}.$$

Thus, (2.7) holds. The proof of (2.6) is similar, we omit the details.

This completes the proof. □

Remark 1 Because of $\frac{K^2(h)}{K(h')} < K^2(h)$, inequalities (2.5) and (2.6) are refinements of (1.1) and (1.2), respectively.

Theorem 3 *Let $0 < m \leq A \leq m' < M' \leq B \leq M$ and $2 \leq p < \infty$. Then for every positive unital linear map Φ,*

$$\Phi^{2p}\left(\frac{A+B}{2}\right) \leq \frac{1}{16}\left(\frac{K^2(h)(M^2+m^2)^2}{K(h')M^2m^2}\right)^p \Phi^{2p}(A \sharp B) \tag{2.8}$$

and

$$\Phi^{2p}\left(\frac{A+B}{2}\right) \leq \frac{1}{16}\left(\frac{K^2(h)(M^2+m^2)^2}{K(h')M^2m^2}\right)^p \big(\Phi(A) \sharp \Phi(B)\big)^{2p}, \tag{2.9}$$

where $K(h) = \frac{(h+1)^2}{4h}$, $K(h') = \frac{(h'+1)^2}{4h'}$, $h = \frac{M}{m}$, and $h' = \frac{M'}{m'}$.

Proof By the operator reverse monotonicity of inequality (2.5), we have

$$\Phi^{-2}(A \sharp B) \leq L^2\Phi^{-2}\left(\frac{A+B}{2}\right), \tag{2.10}$$

where $L = \frac{K(h)}{K^{\frac{1}{2}}(h')}$.

Compute

$$\begin{aligned}
&\left\|\Phi^p\left(\frac{A+B}{2}\right)M^p m^p \Phi^{-p}(A\sharp B)\right\| \\
&\leq \frac{1}{4}\left\|L^{\frac{p}{2}}\Phi^p\left(\frac{A+B}{2}\right)+\left(\frac{M^2m^2}{L}\right)^{\frac{p}{2}}\Phi^{-p}(A\sharp B)\right\|^2 \quad \text{(by (2.1))} \\
&\leq \frac{1}{4}\left\|L\Phi^2\left(\frac{A+B}{2}\right)+\frac{M^2m^2}{L}\Phi^{-2}(A\sharp B)\right\|^p \quad \text{(by (2.3))} \\
&\leq \frac{1}{4}\left\|L\Phi^2\left(\frac{A+B}{2}\right)+LM^2m^2\Phi^{-2}\left(\frac{A+B}{2}\right)\right\|^p \quad \text{(by (2.10))} \\
&\leq \frac{1}{4}\big(L\big(M^2+m^2\big)\big)^p \quad \text{(by [1], (4.7))}.
\end{aligned}$$

That is,

$$\left\|\Phi^p\left(\frac{A+B}{2}\right)\Phi^{-p}(A\sharp B)\right\| \leq \frac{1}{4}\left(\frac{L(M^2+m^2)}{Mm}\right)^p = \frac{1}{4}\left(\frac{K^2(h)(M^2+m^2)^2}{K(h')M^2m^2}\right)^{\frac{p}{2}}.$$

Thus, (2.8) holds. By inequality (2.6), the proof of (2.9) is similar, we omit the details.

This completes the proof. □

Remark 2 Since $K(h') > 1$, inequalities (2.8) and (2.9) are sharper than (1.3) and (1.4), respectively.

Theorem 4 *Let $0 < m \leq A \leq M$ and let X, Y be two isometries on $\mathcal{H}$ whose final spaces are orthogonal to each other. Then for every 2-positive unital linear map Φ,*

$$\big\|\Phi\big(X^*AY\big)\Phi\big(Y^*AY\big)^{-1}\Phi\big(Y^*AX\big)\Phi\big(X^*AX\big)^{-1}\big\| \leq \frac{(M-m)^2}{4Mm}. \tag{2.11}$$

Proof Since X is isometric and $0 < m \leq A \leq M$, $m \leq \Phi(X^*AX) \leq M$ and $\frac{1}{M} \leq \Phi(X^*AX)^{-1} \leq \frac{1}{m}$.

Compute

$$\begin{aligned}
&\left(\frac{M-m}{M+m}\right)^2 Mm\big\|\Phi\big(X^*AY\big)\Phi\big(Y^*AY\big)^{-1}\Phi\big(Y^*AX\big)\Phi\big(X^*AX\big)^{-1}\big\| \\
&\leq \frac{1}{4}\left\|\Phi\big(X^*AY\big)\Phi\big(Y^*AY\big)^{-1}\Phi\big(Y^*AX\big)+\left(\frac{M-m}{M+m}\right)^2 Mm\Phi\big(X^*AX\big)^{-1}\right\|^2 \quad \text{(by (2.1))} \\
&\leq \frac{1}{4}\left\|\left(\frac{M-m}{M+m}\right)^2\Phi\big(X^*AX\big)+\left(\frac{M-m}{M+m}\right)^2 Mm\Phi\big(X^*AX\big)^{-1}\right\|^2 \quad \text{(by (1.5))} \\
&\leq \frac{1}{4}\left(\frac{M-m}{M+m}\right)^4 (M+m)^2.
\end{aligned}$$

Hence,

$$\big\|\Phi\big(X^*AY\big)\Phi\big(Y^*AY\big)^{-1}\Phi\big(Y^*AX\big)\Phi\big(X^*AX\big)^{-1}\big\| \leq \frac{(M-m)^2}{4Mm}.$$

This completes the proof. □

Remark 3 Since $0 < m \le M$,

$$\frac{1}{4}\left(\left(\frac{M-m}{M+m}\right)^2 M + \frac{1}{m}\right)^2 \ge \left(\frac{M-m}{M+m}\right)^2 \frac{M}{m} \ge \left(\frac{M-m}{M+m}\right)^2 \frac{(M+m)^2}{4Mm} = \frac{(M-m)^2}{4Mm}.$$

Thus, (2.11) is tighter than (1.6).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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