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Some novel inequalities for fuzzy variables on the variance and its rational upper bound

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Abstract

Variance is of great significance in measuring the degree of deviation, which has gained extensive usage in many fields in practical scenarios. The definition of the variance on the basis of the credibility measure was first put forward in 2002. Following this idea, the calculation of the accurate value of the variance for some special fuzzy variables, like the symmetric and asymmetric triangular fuzzy numbers and the Gaussian fuzzy numbers, is presented in this paper, which turns out to be far more complicated. Thus, in order to better implement variance in real-life projects like risk control and quality management, we suggest a rational upper bound of the variance based on an inequality, together with its calculation formula, which can largely simplify the calculation process within a reasonable range. Meanwhile, some discussions between the variance and its rational upper bound are presented to show the rationality of the latter. Furthermore, two inequalities regarding the rational upper bound of variance and standard deviation of the sum of two fuzzy variables and their individual variances and standard deviations are proved. Subsequently, some numerical examples are illustrated to show the effectiveness and the feasibility of the proposed inequalities.

Keywords: fuzzy variable; credibility distribution; variance; rational upper bound; inequality

1 Introduction

In real life, many projects are usually implemented in an uncertain environment, in which there exist many indeterminate elements. In 1978, Zadeh [1] initiated the possibility theory, where the possibility measure is thought to be suitable to deal with the uncertainty, especially under the circumstance of sparse data. Meanwhile, Dubois and Prade [2] suggested a fuzzification principle to extend the usual algebraic operations to fuzzified operations on fuzzy numbers. Nevertheless, they reminded that frequent using of this principle may lead to wrong results. Wang and Kerre [3] proposed nine axioms serving as several reasonable properties for the sake of finding the rationalness during an ordering procedure among fuzzy quantities. Then, Dubois and Prade [4] further defined the notion of expectation for fuzzy variables of intervals, viewing them as a constant random set. Moreover, Bàn [5] set forth the notion of fuzzy-valued measure and the concept of conditional expectation for fuzzy-valued variables. Since then, the applications of the expected value of fuzzy variables have been already applied to some certain domains. For example, fuzzy expected value was adopted in a computer program by Kuramoto *et al.* [6] for ventilatory control during clinical anesthesia, and Jaccard [7] applied it into snow avalanches to measure their semantic damage.

Apart from the concept and calculation of the expected value, Carlsson and Fullér [8] brought up the notions of crisp possibilistic expected value and crisp possibilistic variance for fuzzy variables with continuous possibility distributions. They also presented the process about the calculation of the variance on linear combination of fuzzy variables, which turned out to be figured in an analogous way as in probability theory. Chen and Tan [9] further studied the definitions of variance and covariance in multiplication of fuzzy variables, which were applied in portfolio to build possibilistic models for better selection under an uncertain situation.

However, due to the absence of self-duality, using a possibility measure would sometimes lead to the exaggeration of the reality. Since the self-duality is of great significance in both theory and practice, in 2002, Liu and Liu [10] further investigated the fuzzy set theory and defined a credibility measure, which equals the mean value of the possibility measure and the necessity measure. Following that, they introduced a novel expected value operator for fuzzy variables and designed a fuzzy simulation technique for calculation. Then, in 2006, the credibility measure was systematically studied by Li and Liu [11], proving it as a set function satisfying four axioms, that is, normality, monotonicity, duality, and maximality. After that, the credibility measure was widely acknowledged and used in the fields of fuzzy decision [12, 13], fuzzy process [14, 15], fuzzy inference [16], and so on. For instance, Huang [17] put forward a mean-variance model for optimal allocation of capital, and a genetic algorithm on the basis of fuzzy simulation was generated to address this optimization problem. Recently, the expected value based on the credibility measure was applied in industry by Virivinti and Mitra [18] to solving the multiobjective optimization of an industrial grinding problem. Zhou *et al.* [19] suggested an equivalent form of the expected value operator in terms of the inverse credibility distribution of strictly monotone functions.

As an important measurement index of deviation degree in the credibility theory, the variance is vital in practice as well. However, the existing researches regarding the variance based on the credibility measure are hardly seen due to the complexity of the calculation process. Thus, in this paper, we further investigate the variance based on the credibility measure from the point of view of inequalities for the sake of better applications. To begin with, this paper presents the calculation process of accurate variances using the original definition for three kinds of fuzzy variables with regular credibility distributions, that is, the symmetric triangular fuzzy number, the asymmetric triangular fuzzy number, and the Gaussian fuzzy number. Since the whole calculation processes are too intricate, an inequality about the variance is proved, which is used for defining a rational upper bound of the variance (RUBV). Following this idea, a formula to calculate this RUBV is suggested, together with some discussions and comparisons with the accurate value of the variance to demonstrate the feasibility and the practical meaning of the upper bound. After that, in order to calculate the RUBV for a bunch of fuzzy variables, a new theorem in regard to strictly monotone functions that contains several independent fuzzy variables with regular credibility distributions is deduced. Subsequently, two inequalities on the RUBV are proved, one of which shows the relationship between the RUBV of the sum of two fuzzy variables and the sum of their individual RUBVs, and the other focuses on the standard deviation.

The rest of this paper is arranged as follows. Section 2 reviews some fundamental notions and theories of fuzzy variables including the credibility distribution, the inverse credibility distribution, and the expected value. Section 3 presents the calculation process of the accurate variance for fuzzy variables. Section 4 introduces the definition of the RUBV based on an inequality of the variance. Correspondingly, a calculation formula in terms of this RUBV is suggested, together with some examples and comparisons with the accurate values of variance. Finally, two inequalities with respect to the RUBV of the sum of two fuzzy variables are recommended in Section 5.

2 Distribution and expected value

For the purpose of measuring a fuzzy event, Zadeh [1] introduced the concept of the possibility measure. After that, Liu and Liu [10] further pioneered the credibility measure, based on which the credibility theory was constructed by Liu [20] as a branch of mathematics for studying the behavior of fuzzy phenomena. In this section, some basic knowledge of the credibility theory is reviewed involving fuzzy variable, credibility measure, credibility distribution, inverse credibility distribution, and expected value of fuzzy variables.

2.1 Fuzzy variable

Suppose that Θ is a nonempty set, $\mathcal{P}(\Theta)$ is the power set of Θ, and Pos is a possibility measure. Then the triplet $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$ is called a possibility space. A fuzzy variable is defined as a function from a possibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$ to the set of real numbers.

Definition 1 (Liu [21]) Let ξ be a fuzzy variable defined on the possibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$. Then its membership function is derived from the possibility measure by

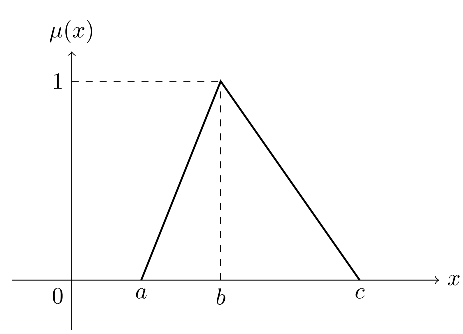
$$\mu(x) = \mathrm{Pos}\{\theta \in \Theta \mid \xi(\theta) = x\}, \quad x \in \Re. \tag{1}$$

Example 1 A fuzzy variable ξ is called a triangular fuzzy number and denoted by $\xi \sim \mathcal{T}(a, b, c)$ if its membership function is

$$\mu(x) = \begin{cases} \frac{x-a}{b-a} & \text{if } a \le x \le b, \\ \frac{c-x}{c-b} & \text{if } b < x \le c, \\ 0 & \text{otherwise}, \end{cases} \tag{2}$$

as depicted in Figure 1, where a, b, and c are real numbers satisfying $a < b < c$.

Figure 1 The membership function in Example 1. The piecewise function $\mu(x)$ is the membership function of a triangular fuzzy number $\xi \sim \mathcal{T}(a, b, c)$.



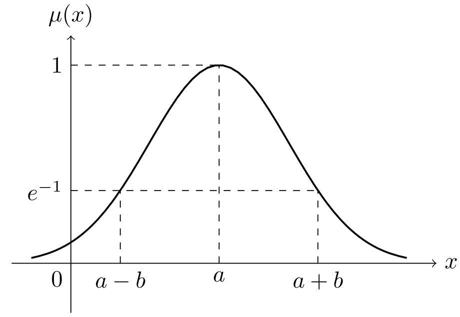


Figure 2 The membership function in Example 2. $\mu(x)$ is the membership function of a Gaussian fuzzy number $\xi \sim \mathcal{N}(a,b)$, and e^{-1} is 0.368.

Example 2 A fuzzy variable ξ is called a Gaussian fuzzy number and denoted by $\xi \sim \mathcal{N}(a,b)$ if its membership function is

$$\mu(x) = e^{-(\frac{x-a}{b})^2} \tag{3}$$

as depicted in Figure 2, where a and b are real numbers satisfying $b > 0$.

2.2 Credibility measure

Suppose that ξ is a fuzzy variable, μ is the membership function of ξ, and r is a real number. The fuzzy event $\{\xi \le r\}$ has the possibility [1] and the necessity [22] as follows:

$$\mathrm{Pos}\{\xi \le r\} = \sup_{x \le r} \mu(x), \qquad \mathrm{Nec}\{\xi \le r\} = 1 - \sup_{x > r} \mu(x). \tag{4}$$

Theorem 1 (Liu [21]) *Let* $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$ *be a possibility space, and A a set in* $\mathcal{P}(\Theta)$. *Then the credibility of A is defined by*

$$\mathrm{Cr}\{A\} = \frac{1}{2}\big(\mathrm{Pos}\{A\} + \mathrm{Nec}\{A\}\big). \tag{5}$$

Note that from Eqs. (4) and (5) it is easy to deduce

$$\mathrm{Cr}\{\xi \le r\} = \frac{1}{2}\Big(\sup_{x \le r} \mu(x) + 1 - \sup_{x > r} \mu(x)\Big). \tag{6}$$

Moreover, Liu and Liu [10] proved that the credibility measure Cr is self-dual, that is,

$$\mathrm{Cr}\{A\} + \mathrm{Cr}\big\{A^c\big\} = 1 \tag{7}$$

for any set $A \in \mathcal{P}(\Theta)$. Since a self-dual measure is thoroughly indispensable both theoretically and practically, the credibility measure is taken into consideration in this paper.

Theorem 2 (Liu [21]) *The credibility measure is subadditive, that is,*

$$\mathrm{Cr}\{A \cup B\} \le \mathrm{Cr}\{A\} + \mathrm{Cr}\{B\} \tag{8}$$

for any events A and B.

Figure 3 The credibility distribution of $\xi \sim \mathcal{T}(a, b, c)$ in Example 3. The piecewise function $\Phi(x)$ is the credibility distribution of $\xi \sim \mathcal{T}(a, b, c)$ whose values are 0, 0.5, and 1 when x are a, b, and c, respectively.

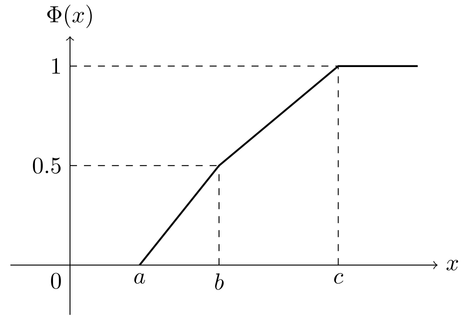
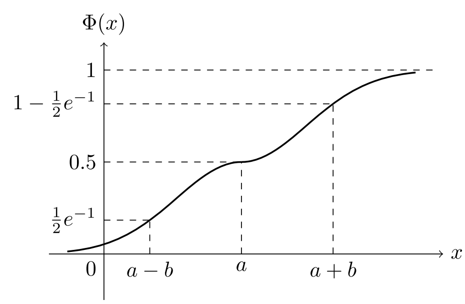


Figure 4 The credibility distribution of $\xi \sim \mathcal{N}(a, b)$ in Example 4. The piecewise function $\Phi(x)$ is the credibility distribution of $\xi \sim \mathcal{N}(a, b)$ whose values are $\frac{1}{2}e^{-1}$, 0.5, and $1 - \frac{1}{2}e^{-1}$ when x are $a - b$, a, and $a + b$, respectively. In addition, $\Phi(x)$ approaches 0 as $x \to -\infty$ and approaches 1 as $x \to +\infty$.



2.3 Credibility distribution

Definition 2 (Liu [21]) The credibility distribution $\Phi : \Re \to [0, 1]$ of a fuzzy variable ξ is defined by

$$\Phi(x) = \mathrm{Cr}\{\theta \in \Theta \mid \xi(\theta) \le x\}. \tag{9}$$

Here, $\Phi(x)$ means the credibility when ξ takes a value less than or equal to x, which was proved to be a nondecreasing function on $\Re$ with $\Phi(-\infty) = 0$ and $\Phi(+\infty) = 1$ by Liu [20].

Example 3 Suppose that $\xi \sim \mathcal{T}(a, b, c)$ is a triangular fuzzy number with the membership function in Eq. (2). Then the credibility distribution of ξ is

$$\Phi(x) = \begin{cases} 0 & \text{if } x \le a, \\ (x - a)/2(b - a) & \text{if } a < x \le b, \\ (x + c - 2b)/2(c - b) & \text{if } b < x \le c, \\ 1 & \text{if } x > c; \end{cases} \tag{10}$$

see Figure 3.

Example 4 Suppose that $\xi \sim \mathcal{N}(a, b)$ is a Gaussian fuzzy number with the membership function in Eq. (3). Then the credibility distribution of ξ is

$$\Phi(x) = \begin{cases} \frac{1}{2}e^{-(\frac{x-a}{b})^2} & \text{if } x \le a, \\ 1 - \frac{1}{2}e^{-(\frac{x-a}{b})^2} & \text{if } x > a; \end{cases} \tag{11}$$

see Figure 4.

Figure 5 The inverse credibility distribution of $\xi \sim \mathcal{T}(a,b,c)$ in Example 5. The piecewise function $\Phi^{-1}(\alpha)$ is the inverse credibility distribution of $\xi \sim \mathcal{T}(a,b,c)$ whose values are a, b, and c when α are 0, 0.5, and 1, respectively.

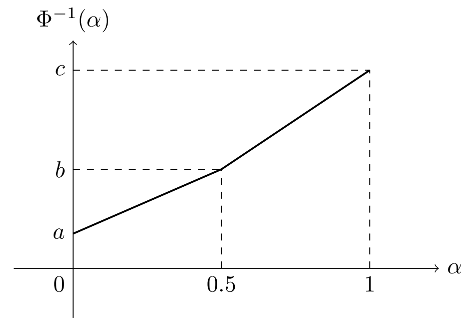
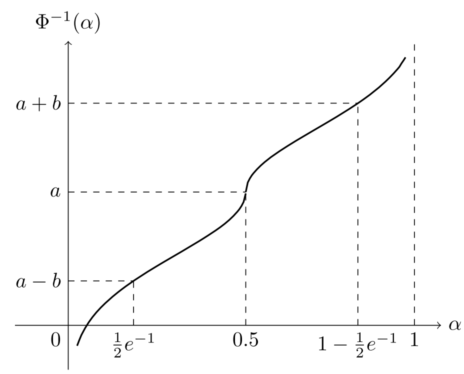


Figure 6 The inverse credibility distribution of $\xi \sim \mathcal{N}(a,b)$ in Example 6. The piecewise function $\Phi^{-1}(\alpha)$ is the inverse credibility function of $\xi \sim \mathcal{N}(a,b)$ whose values are $a-b$, a, and $a+b$ when α are $\frac{1}{2}e^{-1}$, 0.5, and $1-\frac{1}{2}e^{-1}$, respectively.



2.4 Regular credibility distribution

Definition 3 (Zhou *et al.* [19]) A credibility distribution Φ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$ and if

$$\lim_{x \to -\infty} \Phi(x) = 0, \qquad \lim_{x \to +\infty} \Phi(x) = 1.$$

Definition 4 (Zhou *et al.* [19]) Let ξ be a fuzzy variable with a regular credibility distribution Φ. Then the inverse function Φ^{-1} is called the inverse credibility distribution of ξ.

Example 5 The inverse credibility distribution of a triangular fuzzy number $\xi \sim \mathcal{T}(a,b,c)$ is

$$\Phi^{-1}(\alpha) = \begin{cases} (2b-2a)\alpha + a & \text{if } \alpha < 0.5, \\ (2c-2b)\alpha + 2b - c, & \text{if } \alpha \geq 0.5, \end{cases} \tag{12}$$

as shown in Figure 5.

Example 6 The inverse credibility distribution of a Gaussian fuzzy number $\xi \sim \mathcal{N}(a,b)$ is

$$\Phi^{-1}(\alpha) = \begin{cases} a - b\sqrt{-\ln(2\alpha)} & \text{if } \alpha \leq 0.5, \\ a + b\sqrt{-\ln(2-2\alpha)} & \text{if } \alpha > 0.5, \end{cases} \tag{13}$$

as shown in Figure 6.

Theorem 3 (Zhou *et al.* [19]) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent fuzzy variables with regular credibility distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then $\xi = f(\xi_1, \ldots, \xi_m, \xi_{m+1}, \ldots, \xi_n)$ is a fuzzy variable with an inverse credibility distribution*

$$\Phi^{-1}(\alpha) = f\bigl(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\bigr).$$

2.5 Expected value

In 2002, the following general definition of expected value for fuzzy variables was provided by Liu and Liu [10] via the credibility distribution.

Definition 5 (Liu and Liu [10]) Let ξ be a fuzzy variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi \geq r\}\,\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{\xi \leq r\}\,\mathrm{d}r, \tag{14}$$

provided that at least one of the two integrals is finite.

Theorem 4 (Zhou *et al.* [19]) *Let ξ be a fuzzy variable with the inverse credibility distribution Φ^{-1}. If its expected value exists, then*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\,\mathrm{d}\alpha. \tag{15}$$

Example 7 According to Eqs. (12) and (15), the expected value of a triangular fuzzy number $\xi \sim \mathcal{T}(a,b,c)$ can be calculated as

$$\begin{aligned} E[\xi] &= \int_0^{0.5} \bigl((1-2\alpha)a + 2\alpha b\bigr)\,\mathrm{d}\alpha + \int_{0.5}^{1} \bigl((2-2\alpha)b + (2\alpha-1)c\bigr)\,\mathrm{d}\alpha \\ &= \frac{a+2b+c}{4}. \end{aligned}$$

Provided that $b - a = c - b$, which means that ξ is a symmetric triangular fuzzy number, it is clear that the expected value is $E[\xi] = b$.

Example 8 According to Eqs. (13) and (15), the expected value of a Gaussian fuzzy number $\xi \sim \mathcal{N}(a,b)$ can be calculated as

$$\begin{aligned} E[\xi] &= \int_0^{0.5} \bigl(a - b\sqrt{-\ln(2\alpha)}\bigr)\,\mathrm{d}\alpha + \int_{0.5}^{1} \bigl(a + b\sqrt{-\ln(2-2\alpha)}\bigr)\,\mathrm{d}\alpha \\ &= 0.5a - b\int_0^{0.5} \sqrt{-\ln(2\alpha)}\,\mathrm{d}\alpha + 0.5a + b\int_{0.5}^{1} \sqrt{-\ln(2-2\alpha)}\,\mathrm{d}\alpha \\ &= a - 0.5b\int_0^1 \sqrt{-\ln t}\,\mathrm{d}t - 0.5b\int_1^0 \sqrt{-\ln t}\,\mathrm{d}t \\ &= a. \end{aligned}$$

Theorem 5 (Zhou *et al.* [19]) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent fuzzy variables with regular credibility distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then the expected value of the fuzzy variable $\xi = f(\xi_1, \ldots, \xi_m, \xi_{m+1}, \ldots, \xi_n)$ is*

$$E[\xi] = \int_0^1 f\big(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\big)\, \mathrm{d}\alpha.$$

Theorem 6 (Liu and Liu [10]) *Let ξ and η be independent fuzzy variables with finite expected values. Then for any real variables a and b, we have*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]. \tag{16}$$

3 Variance

This paper focuses on the variance of fuzzy variables with regular credibility distributions. According to the definition of the variance for fuzzy variables, in this section, we present a calculation of the variance for three kinds of commonly used fuzzy variables, that is, the symmetric triangular fuzzy number, the asymmetric triangular fuzzy number, and the Gaussian fuzzy number.

Definition 6 (Liu [23]) Let ξ be a fuzzy variable with finite expected value e. Then the variance of ξ is defined by

$$V[\xi] = E\big[(\xi - e)^2\big]. \tag{17}$$

If we know the credibility distribution of a fuzzy variable ξ and the finite expected value $E[\xi] = e$, then the variance satisfies

$$V[\xi] = E\big[(\xi - e)^2\big] = \int_0^{+\infty} \mathrm{Cr}\big\{(\xi - e)^2 \geq r\big\}\, \mathrm{d}r. \tag{18}$$

Now let us show a calculation process of the variance for some fuzzy variables via Eq. (18) and also the corresponding credibility distributions.

Example 9 Let $\xi \sim \mathcal{T}(a, b, c)$ be a symmetric triangular fuzzy number, that is, $b - a = c - b$ and $E(\xi) = e = b$ from Example 7. Based on Definition 1 and the membership function in Example 1, we can obtain the membership function of $\eta = (\xi - e)^2$:

$$\mu_\eta(x) = \begin{cases} 1 - \frac{\sqrt{x}}{c-b} & \text{if } 0 \leq x \leq (c-b)^2, \\ 0 & \text{otherwise,} \end{cases}$$

as shown in Figure 7.

Since $\mathrm{Cr}\{\eta < r\} = \frac{1}{2}(\sup_{x<r} \mu_\eta(x) + 1 - \sup_{x \geq r} \mu_\eta(x))$ by Eq. (6), it follows immediately that

$$\mathrm{Cr}\{\eta < r\} = \frac{1}{2}\left(1 + 1 - \left(1 - \frac{\sqrt{r}}{c-b}\right)\right) = \frac{1}{2}\left(1 + \frac{\sqrt{r}}{c-b}\right), \quad 0 \leq r \leq (c-b)^2.$$

Figure 7 The membership function of $\eta = (\xi - e)^2$ in Example 9. $\mu_\eta(x)$ is the membership function of the fuzzy number $\eta = (\xi - e)^2$ where $\xi \sim \mathcal{T}(a, b, c)$ is a symmetric triangular fuzzy number, that is, $b - a = c - b$. Values of $\mu_\eta(x)$ are 1 and 0 when x are 0 and $(c - b)^2$, respectively.

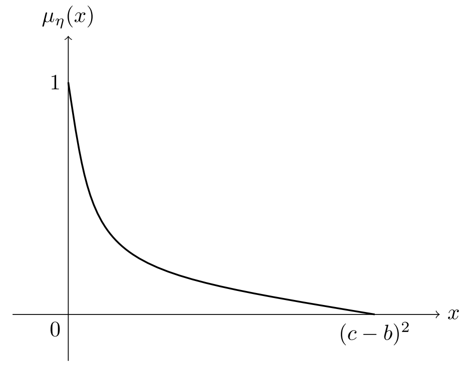
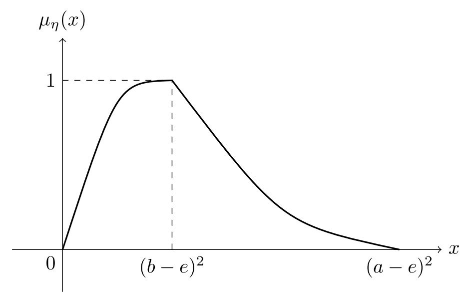


Figure 8 The membership function of $\eta = (\xi - e)^2$ in Example 10. $\mu_\eta(x)$ is the membership function of the fuzzy number $\eta = (\xi - e)^2$ where $\xi \sim \mathcal{T}(a, b, c)$ is an asymmetric triangular fuzzy number, that is, $a - b \neq b - c$. Values of $\mu_\eta(x)$ are 1 and 0 when x are $(b - e)^2$ and 0 or $(a - e)^2$, respectively.



As for the case $r > (c - b)^2$, it can be derived similarly that $\text{Cr}\{\eta < r\} = 1$. Subsequently, in view of the self-duality of the credibility measure, we have

$$\text{Cr}\{\eta \geq r\} = 1 - \text{Cr}\{\eta < r\} = \begin{cases} \frac{1}{2}(1 - \frac{\sqrt{r}}{c-b}), & 0 \leq r \leq (c - b)^2, \\ 0, & r > (c - b)^2. \end{cases}$$

Thus, the symmetric triangular fuzzy number $\xi \sim \mathcal{T}(a, b, c)$ has the variance

$$\begin{aligned} V[\xi] &= \int_0^{+\infty} \text{Cr}\{(\xi - e)^2 \geq r\}\, \mathrm{d}r \\ &= \int_0^{(c-b)^2} \frac{1}{2}\left(1 - \frac{\sqrt{r}}{c - b}\right) \mathrm{d}r = \frac{1}{6}(c - b)^2. \end{aligned} \tag{19}$$

Provided that $\xi \sim \mathcal{T}(1, 2, 3)$, we obtain that $E[\xi] = 2$ and $V[\xi] = 1/6$.

Example 10 Let $\xi \sim \mathcal{T}(a, b, c)$ be an asymmetric triangular fuzzy number. From Example 7 we know that $E(\xi) = e = (a + 2b + c)/4$. Without loss of generality, suppose that $b - a < c - b$. Based on Definition 1 and Example 1, it is not hard to get the membership function of $\eta = (\xi - e)^2$:

$$\mu_\eta(x) = \begin{cases} \frac{c-e+\sqrt{x}}{c-b} & \text{if } 0 \leq x \leq (b - e)^2, \\ \frac{e-a-\sqrt{x}}{b-a} & \text{if } (b - e)^2 < x \leq (a - e)^2, \\ 0 & \text{otherwise,} \end{cases}$$

as shown in Figure 8.

Thereby,

$$\mathrm{Cr}\{\eta < r\} = \begin{cases} \frac{c-e+\sqrt{r}}{2(c-b)} & \text{if } 0 \le r \le (b-e)^2, \\ 1 - \frac{e-a-\sqrt{r}}{2(b-a)} & \text{if } (b-e)^2 < r \le (a-e)^2, \\ 1 & \text{if } r > (a-e)^2. \end{cases}$$

Since $\mathrm{Cr}\{\eta < r\} + \mathrm{Cr}\{\eta \ge r\} = 1$, we can get that

$$\mathrm{Cr}\{\eta \ge r\} = \begin{cases} 1 - \frac{c-e+\sqrt{r}}{2(c-b)} & \text{if } 0 \le r \le (b-e)^2, \\ \frac{e-a-\sqrt{r}}{2(b-a)} & \text{if } (b-e)^2 < r \le (a-e)^2, \\ 0 & \text{if } r > (a-e)^2. \end{cases}$$

Thus, the variance of $\xi \sim \mathcal{T}(a,b,c)$ is

$$\begin{aligned} V[\xi] &= \int_0^{+\infty} \mathrm{Cr}\{(\xi - e)^2 \ge r\}\,\mathrm{d}r \\ &= \int_0^{(b-e)^2} \frac{e-a-\sqrt{r}}{2(b-a)}\,\mathrm{d}r + \int_{(b-e)^2}^{(a-e)^2} \left(1 - \frac{c-e+\sqrt{r}}{2(c-b)}\right)\mathrm{d}r. \end{aligned}$$

Under the assistance of some well-developed software packages like Matlab, we finally get

$$\begin{aligned} V[\xi] = {} & \frac{1}{384(c-b)}\bigl(84ab^2 - 2(a-2b+c)^3 - 30a^2b + 33ac^2 \\ & + 21a^2c - 78bc^2 + 132b^2c + 3a^3 - 72b^3 + 15c^3 - 108abc\bigr) \\ & + \frac{1}{48(b-a)}\left(\frac{1}{4}(a-2b+c)^3 - \frac{1}{4}(2b-3a+c)^3 - 6ab^2 \right. \\ & \left. + 15a^2b + 12a^2c - 3ac^2 + 3bc^2 + 6b^2c - 9a^3 - 18abc\right). \end{aligned} \tag{20}$$

Provided that $\xi \sim \mathcal{T}(2,4,8)$, we have $E[\xi] = 4.5$ and $V[\xi] = 83/64$.

Example 11 Let $\xi \sim \mathcal{N}(a,b)$ be a Gaussian fuzzy number. From Example 8 we know $E[\xi] = e = a$. As a result, the membership function of $\eta = (\xi - e)^2$ is deduced according to Definition 1 and Example 2 as

$$\mu_\eta(x) = \begin{cases} e^{-\frac{x}{b^2}}, & x \ge 0, \\ 0, & x < 0; \end{cases}$$

see Figure 9.

Hence,

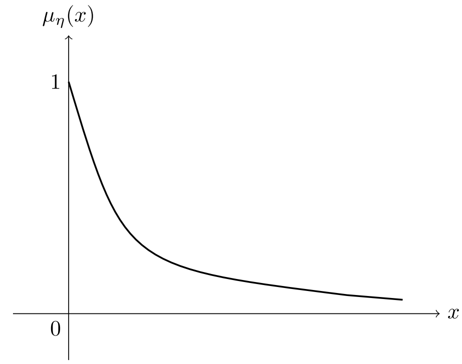
$$\mathrm{Cr}\{\eta \ge r\} = 1 - \mathrm{Cr}\{\eta < r\} = 1 - \frac{1}{2}\bigl(1 + 1 - e^{-\frac{r}{b^2}}\bigr) = \frac{1}{2}e^{-\frac{r}{b^2}}, \quad r \ge 0.$$

Consequently, we can obtain that

$$V[\xi] = \int_0^{+\infty} \mathrm{Cr}\{(\xi - e)^2 \ge r\}\,\mathrm{d}r = \frac{1}{2}\int_0^{+\infty} e^{-\frac{r}{b^2}}\,\mathrm{d}r = \frac{1}{2}b^2. \tag{21}$$

Provided that $\xi \sim \mathcal{N}(2,4)$, we have $E[\xi] = 2$ and $V[\xi] = 8$.

Figure 9 The membership function of $\eta = (\xi - e)^2$ in Example 11. $\mu_\eta(x)$ is the membership function of the fuzzy number $\eta = (\xi - e)^2$ where $\xi \sim \mathcal{N}(a, b)$ is a Gaussian fuzzy number, which takes value 1 when x is 0 and approaches 0 as $x \to +\infty$.



4 A rational upper bound of the variance

We can see from Section 3 that the whole calculation process of the variance for a single fuzzy variable is not easy even for the simplest symmetric triangular fuzzy number. However, in real-world scenarios, many decision-making problems involve a bunch of variables like $\xi_1, \xi_2, \dots, \xi_n$, which usually accompany the measurement of potential risk like $V[f(\xi_1, \xi_2, \dots, \xi_n)]$. Under this circumstance, it would be far more complicated to measure the risk according to the original definition of the variance when $f(\xi_1, \xi_2, \dots, \xi_n)$ is a nonlinear or complex function. In order to tackle this problem, in this section, we suggest a rational upper bound of the variance and its calculation formula for some special fuzzy variables on account of the following theorem.

Theorem 7 *Let ξ be a fuzzy variable with credibility distribution Φ. If the expected value e exists, then the variance of ξ satisfies*

$$V[\xi] \le \int_0^{+\infty} \bigl(1 - \Phi(e + \sqrt{x}) + \Phi(e - \sqrt{x})\bigr)\,\mathrm{d}x. \tag{22}$$

Proof According to the definition of the variance and Eq. (18), we have

$$\begin{aligned} V[\xi] &= \int_0^{+\infty} \mathrm{Cr}\bigl\{(\xi - e)^2 \ge x\bigr\}\,\mathrm{d}x \\ &= \int_0^{+\infty} \mathrm{Cr}\bigl\{(\xi \ge e + \sqrt{x}) \cup (\xi \le e - \sqrt{x})\bigr\}\,\mathrm{d}x. \end{aligned}$$

Since the credibility measure is subadditive in accordance with Theorem 2, we have

$$\begin{aligned} V[\xi] &\le \int_0^{+\infty} \bigl(\mathrm{Cr}\bigl\{(\xi \ge e + \sqrt{x})\bigr\} + \mathrm{Cr}\bigl\{(\xi \le e - \sqrt{x})\bigr\}\bigr)\,\mathrm{d}x \\ &= \int_0^{+\infty} \bigl(1 - \Phi(e + \sqrt{x}) + \Phi(e - \sqrt{x})\bigr)\,\mathrm{d}x. \end{aligned}$$ □

Based on Theorem 7, we suggest a new concept with respect to the variance for the sake of better applications.

Definition 7 Let ξ be a fuzzy variable with credibility distribution Φ and finite expected value e. Then the rational upper bound of the variance (RUBV) is defined by

$$\overline{V}[\xi] = \int_0^{+\infty} \big(1 - \Phi(e + \sqrt{x}) + \Phi(e - \sqrt{x})\big)\,\mathrm{d}x. \tag{23}$$

Next, a formula is given to calculate the RUBV of a fuzzy variable using the inverse credibility distribution.

Theorem 8 *Let ξ be a fuzzy variable with regular credibility distribution Φ. If the expected value e exists, then the RUBV can be calculated by*

$$\overline{V}[\xi] = \int_0^1 \big(\Phi^{-1}(\alpha) - e\big)^2\,\mathrm{d}\alpha. \tag{24}$$

Proof According to the definition of the RUBV, we obtain

$$\begin{aligned}\overline{V}[\xi] &= \int_0^{+\infty} \big(1 - \Phi(e + \sqrt{x}) + \Phi(e - \sqrt{x})\big)\,\mathrm{d}x \\ &= \int_0^{+\infty} \big(1 - \Phi(e + \sqrt{x})\big)\,\mathrm{d}x + \int_0^{+\infty} \Phi(e - \sqrt{x})\,\mathrm{d}x.\end{aligned}$$

For the first part, replacing $\Phi(e + \sqrt{x})$ with α and x with $(\Phi^{-1}(\alpha) - e)^2$, respectively, we get

$$\begin{aligned}\int_0^{+\infty} \big(1 - \Phi(e + \sqrt{x})\big)\,\mathrm{d}x &= \int_{\Phi(e)}^1 (1-\alpha)\,\mathrm{d}\big(\Phi^{-1}(\alpha) - e\big)^2 \\ &= (1-\alpha)\big(\Phi^{-1}(\alpha) - e\big)^2)|_{\Phi(e)}^1 - \int_{\Phi(e)}^1 \big(\Phi^{-1}(\alpha) - e\big)^2\,\mathrm{d}(1-\alpha) \\ &= \int_{\Phi(e)}^1 \big(\Phi^{-1}(\alpha) - e\big)^2\,\mathrm{d}\alpha.\end{aligned}$$

As for the second part, replacing $\Phi(e - \sqrt{x})$ with α and x with $(e - \Phi^{-1}(\alpha))^2$, respectively, we can obtain that

$$\begin{aligned}\int_0^{+\infty} \Phi(e - \sqrt{x})\,\mathrm{d}x &= \int_{\Phi(x)}^0 \alpha\,\mathrm{d}\big(e - \Phi^{-1}(\alpha)\big)^2 \\ &= \alpha\big(e - \Phi^{-1}(\alpha)\big)^2|_{\Phi(e)}^0 - \int_{\Phi(e)}^0 \big(e - \Phi^{-1}(\alpha)\big)^2\,\mathrm{d}\alpha \\ &= \int_0^{\Phi(e)} \big(\Phi^{-1}(\alpha) - e\big)^2\,\mathrm{d}\alpha.\end{aligned}$$

Consequently, the RUBV of ξ is

$$\overline{V}[\xi] = \int_{\Phi(e)}^1 \big(\Phi^{-1}(\alpha) - e\big)^2\,\mathrm{d}\alpha + \int_0^{\Phi(e)} \big(\Phi^{-1}(\alpha) - e\big)^2\,\mathrm{d}\alpha = \int_0^1 \big(\Phi^{-1}(\alpha) - e\big)^2\,\mathrm{d}\alpha. \qquad \square$$

Example 12 Consider the RUBV of a symmetric triangular fuzzy number $\xi \sim \mathcal{T}(a, b, c)$, which means that $b - a = c - b$. Since the inverse credibility distribution of ξ has been deduced in Eq. (12), and the expected value is b from Example 7, we get

$$\begin{aligned}\overline{V}[\xi] &= \int_0^{0.5} \big((2b-2a)\alpha + a - b\big)^2 \,\mathrm{d}\alpha + \int_{0.5}^{1} \big((2c-2b)\alpha + 2b - c - b\big)^2 \,\mathrm{d}\alpha \\ &= \frac{1}{3}(c-b)^2. \end{aligned} \tag{25}$$

Compared with the accurate result $V[\xi] = (c-b)^2/6$ in Eq. (19), although the RUBV $\overline{V}[\xi]$ of the symmetric triangular fuzzy number ξ is twice as large as its variance, it seems more convenient and simpler in terms of the deduction process.

Example 13 Consider the RUBV of an asymmetric triangular fuzzy number $\xi \sim \mathcal{T}(a,b,c)$. Since the expected value is $(a+2b+c)/4$ by Example 7, we have

$$\begin{aligned}\overline{V}[\xi] &= \int_0^{0.5} \left((2b-2a)\alpha + a - \frac{a+2b+c}{4}\right)^2 \mathrm{d}\alpha \\ &\quad + \int_{0.5}^{1} \left((2c-2b)\alpha + 2b - c - \frac{a+2b+c}{4}\right)^2 \mathrm{d}\alpha \\ &= \frac{5}{48}a^2 - \frac{1}{12}ab - \frac{1}{8}ac + \frac{1}{12}b^2 - \frac{1}{12}bc + \frac{5}{48}c^2. \end{aligned} \tag{26}$$

Provided that $\xi \sim \mathcal{T}(2,4,8)$, the RUBV of ξ is $\overline{V}[\xi] = 37/12 \approx 3.08$. Compared with Example 10, the upper bound of the variance calculated by Eq. (26) is approximately 1.78 larger than the accurate result $V[\xi] = 83/64 \approx 1.30$ calculated by Eq. (20), but obviously the deduction process and the calculation process are much easier.

Example 14 Consider the RUBV of a Gaussian fuzzy number $\xi \sim \mathcal{N}(a,b)$. According to the inverse credibility distribution of ξ in Eq. (13) and the expected value $E[\xi] = a$ from Example 8, we get

$$\overline{V}[\xi] = \int_0^{0.5} \big(a - b\sqrt{-\ln(2\alpha)} - a\big)^2 \,\mathrm{d}\alpha + \int_{0.5}^{1} \big(a + b\sqrt{-\ln(2-2\alpha)} - a\big)^2 \,\mathrm{d}\alpha = b^2. \tag{27}$$

Provided that $\xi \sim \mathcal{N}(2,4)$, we have $E[\xi] = 2$ and $\overline{V}[\xi] = 16$, which is just as twice as the result in Example 11, that is, $V[\xi] = 8$.

As mentioned before, many decision-making problems usually contain a pile of fuzzy variables $\xi_1, \xi_2, \ldots, \xi_n$, and it is thus required to calculate the variance of a function like $f(\xi_1, \xi_2, \ldots, \xi_n)$. Since the calculation of the variance $V[f(\xi_1, \xi_2, \ldots, \xi_n)]$ is too complicated, the following theorem is presented instead for dealing with this kind of problems.

Theorem 9 *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent fuzzy variables with regular credibility distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then the RUBV of fuzzy variable $\xi = f(\xi_1, \ldots, \xi_m, \xi_{m+1}, \ldots, \xi_n)$ is*

$$\overline{V}[\xi] = \int_0^1 \Big(f\big(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\big) - e\Big)^2 \,\mathrm{d}\alpha, \tag{28}$$

where e is the expected value of ξ calculated via Theorem 5.

Proof The inverse credibility distribution of ξ can be obtained via Theorem 3, that is,

$$\Phi^{-1}(\alpha) = f\big(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\big),$$

and the theorem follows immediately from Theorem 7. □

As for strictly monotone functions $f(x_1, x_2, \ldots, x_n)$, Theorem 9 provides a calculation formula for $\overline{V}[f(\xi_1, \xi_2, \ldots, \xi_n)]$, which is not difficult to perform with the aid of software packages when the individual inverse credibility distributions Φ_i^{-1} are known. In addition, if it is required to control the risk by minimizing the variance like $V[f(\xi_1, \xi_2, \ldots, \xi_n)]$ in decision-making problems, we can turn to minimizing $\overline{V}[f(\xi_1, \xi_2, \ldots, \xi_n)]$ substitutively, which can not only achieve the same effect, but also greatly reduce the calculation process simultaneously. Thus, to some extent, it is reasonable to adopt this upper bound as an acceptable measurement of deviation degree in practical problems even though there exist some differences between them.

5 Inequalities

In this section, we will focus ourselves on the RUBV of the sum of two fuzzy variables and on proving some inequalities, which show the relationship between the RUBV and the sum of their individual RUBVs. The formulation of these inequalities also has some practical applications such as quality management.

Theorem 10 *If ξ and η are two independent fuzzy variables with regular credibility distributions Φ and Ψ, respectively, then we have*

$$\overline{V}[\xi + \eta] \le 2\big(\overline{V}[\xi] + \overline{V}[\eta]\big), \tag{29}$$

and the equality holds if and only if $\Phi(x) = \Psi(x + c)$, where c is a constant.

Proof It follows from Theorem 3 that $\xi + \eta$ has the inverse credibility distribution

$$\Upsilon^{-1}(\alpha) = \Phi^{-1}(\alpha) + \Psi^{-1}(\alpha).$$

Besides, by Theorem 6 we have

$$E[\xi + \eta] = E[\xi] + E[\eta].$$

Then it follows from Theorem 9 that

$$\begin{aligned}
\overline{V}[\xi + \eta] &= \int_0^1 \big(\Upsilon^{-1}(\alpha) - E[\xi + \eta]\big)^2 \,\mathrm{d}\alpha \\
&= \int_0^1 \big(\Phi^{-1}(\alpha) + \Psi^{-1}(\alpha) - \big(E[\xi] + E[\eta]\big)\big)^2 \,\mathrm{d}\alpha \\
&= \int_0^1 \big(\big(\Phi^{-1}(\alpha) - E[\xi]\big) + \big(\Psi^{-1}(\alpha) - E[\eta]\big)\big)^2 \,\mathrm{d}\alpha \\
&\le \int_0^1 \big(2\big(\Phi^{-1}(\alpha) - E[\xi]\big)^2 + 2\big(\Psi^{-1}(\alpha) - E[\eta]\big)^2\big) \,\mathrm{d}\alpha
\end{aligned}$$

$$= 2\int_0^1 \big(\Phi^{-1}(\alpha) - E[\xi]\big)^2 \,\mathrm{d}\alpha + 2\int_0^1 \big(\Psi^{-1}(\alpha) - E[\eta]\big)^2 \,\mathrm{d}\alpha$$
$$= 2\big(\overline{V}[\xi] + \overline{V}[\eta]\big).$$

Besides, the equality holds if and only if

$$\Phi^{-1}(\alpha) - E[\xi] = \Psi^{-1}(\alpha) - E[\eta].$$

Denoting $c = E[\eta] - E[\xi]$, we can obtain that $\Phi(x) = \Psi(x + c)$. □

Example 15 Let $\xi_1 \sim \mathcal{N}(a_1, b_1)$ and $\xi_2 \sim \mathcal{N}(a_2, b_2)$ be two independent Gaussian fuzzy numbers. By Theorem 3 the fuzzy variable $\xi_1 + \xi_2$ has the inverse credibility distribution

$$\Phi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) = \begin{cases} (a_1 + a_2) - (b_1 + b_2)\sqrt{-\ln(2\alpha)} & \text{if } \alpha \le 0.5, \\ (a_1 + a_2) + (b_1 + b_2)\sqrt{-\ln(2 - 2\alpha)} & \text{if } \alpha > 0.5, \end{cases} \tag{30}$$

where Φ_1 and Φ_2 are the credibility distributions of ξ_1 and ξ_2, respectively. By Theorem 6 the fuzzy variable $\xi_1 + \xi_2$ has the expected value $E[\xi_1 + \xi_2] = E[\xi_1] + E[\xi_2] = a_1 + a_2$. It follows from Theorem 9 that

$$\begin{aligned} \overline{V}[\xi_1 + \xi_2] = {} & \int_0^{0.5} \Big((a_1 + a_2) - (b_1 + b_2)\sqrt{-\ln(2\alpha)} - (a_1 + a_2)\Big)^2 \,\mathrm{d}\alpha \\ & + \int_{0.5}^1 \Big((a_1 + a_2) + (b_1 + b_2)\sqrt{-\ln(2 - 2\alpha)} - (a_1 + a_2)\Big)^2 \,\mathrm{d}\alpha \\ = {} & (b_1 + b_2)^2. \end{aligned} \tag{31}$$

Noting that $\overline{V}[\xi_1] = {b_1}^2$ and $\overline{V}[\xi_2] = {b_2}^2$, we have

$$\overline{V}[\xi_1 + \xi_2] \le 2\big(\overline{V}[\xi_1] + \overline{V}[\xi_2]\big),$$

and the equality holds if and only if $b_1 = b_2$, which means $\Phi_1(x) = \Phi_2(x - a_1 + a_2)$.

In reality, this inequality can be applied in industrial manufacture. Due to the measurement error, suppose that the length of two standardized industrial parts are fuzzy variables that follow Gaussian distributions, for example, $\xi_1 \sim \mathcal{N}(a_1, b_1)$ and $\xi_2 \sim \mathcal{N}(a_2, b_2)$. In order to test the quality of the combination of these two parts, the calculation of the degree of deviation (*i.e.*, the variance or standard deviation) is necessary. Since the RUBV of each fuzzy variable can be easily obtained, the RUBV of $\xi_1 + \xi_2$ can be figured out directly via Eq. (31).

Theorem 11 *If ξ and η are two independent variables with regular credibility distributions Φ and Ψ, respectively, we have*

$$\sqrt{\overline{V}[\xi_1 + \xi_2]} \le \sqrt{\overline{V}[\xi_1]} + \sqrt{\overline{V}[\xi_2]}, \tag{32}$$

and the equality holds if and only if $\Phi(x) = \Psi(ax + b)$, where a and b are two constants.

Proof Considering that the inverse credibility distribution and the expected value of $\xi + \eta$ are

$$\Upsilon^{-1}(\alpha) = \Phi^{-1}(\alpha) + \Psi^{-1}(\alpha), \qquad E[\xi + \eta] = E[\xi] + E[\eta],$$

respectively, in terms of Theorem 9, we only need to prove

$$\begin{aligned} &\sqrt{\int_0^1 \bigl(\Phi^{-1}(\alpha) - E[\xi] + \Psi^{-1}(\alpha) - E[\eta]\bigr)^2 \,\mathrm{d}\alpha} \\ &\quad \le \sqrt{\int_0^1 \bigl(\Phi^{-1}(\alpha) - E[\xi]\bigr)^2 \,\mathrm{d}\alpha} + \sqrt{\int_0^1 \bigl(\Psi^{-1}(\alpha) - E[\eta]\bigr)^2 \,\mathrm{d}\alpha}. \end{aligned} \tag{33}$$

Squaring two sides of inequality (33) and simplifying the inequality, we get

$$\begin{aligned} &\int_0^1 \bigl(\Phi^{-1}(\alpha) - E[\xi]\bigr)\bigl(\Psi^{-1}(\alpha) - E[\eta]\bigr) \,\mathrm{d}\alpha \\ &\quad \le \sqrt{\int_0^1 \bigl(\Phi^{-1}(\alpha) - E[\xi]\bigr)^2 \,\mathrm{d}\alpha}\sqrt{\int_0^1 \bigl(\Psi^{-1}(\alpha) - E[\eta]\bigr)^2 \,\mathrm{d}\alpha}, \end{aligned}$$

that is,

$$\int_0^1 \bigl(\Phi^{-1}(\alpha) - E[\xi]\bigr)\bigl(\Psi^{-1}(\alpha) - E[\eta]\bigr) \,\mathrm{d}\alpha \le \sqrt{\overline{V}[\xi]} \times \sqrt{\overline{V}[\eta]}. \tag{34}$$

Let us prove inequality (34). Since

$$\begin{aligned} &\frac{(\Phi^{-1}(\alpha) - E[\xi])(\Psi^{-1}(\alpha) - E[\eta])}{\sqrt{\overline{V}[\xi]} \times \sqrt{\overline{V}[\eta]}} \\ &\quad \le \frac{1}{2}\frac{(\Phi^{-1}(\alpha) - E[\xi])^2}{\overline{V}[\xi]} + \frac{1}{2}\frac{(\Psi^{-1}(\alpha) - E[\eta])^2}{\overline{V}[\eta]}, \end{aligned}$$

we have

$$\begin{aligned} &\int_0^1 \frac{(\Phi^{-1}(\alpha) - E[\xi])(\Psi^{-1}(\alpha) - E[\eta])}{\sqrt{\overline{V}[\xi]} \times \sqrt{\overline{V}[\eta]}} \,\mathrm{d}r \\ &\quad \le \frac{1}{2}\int_0^1 \frac{(\Phi^{-1}(\alpha) - E[\xi])^2}{\overline{V}[\xi]} \,\mathrm{d}\alpha + \frac{1}{2}\int_0^1 \frac{(\Psi^{-1}(\alpha) - E[\eta])^2}{\overline{V}[\eta]} \,\mathrm{d}\alpha \\ &\quad \le \frac{1}{2} + \frac{1}{2} = 1, \end{aligned} \tag{35}$$

which is just inequality (34). It should be noted that the equality holds if and only if

$$\frac{\Phi^{-1}(\alpha) - E[\xi]}{\sqrt{\overline{V}[\xi]}} = \frac{\Psi^{-1}(\alpha) - E[\eta]}{\sqrt{\overline{V}[\eta]}},$$

that is,

$$\Psi^{-1}(\alpha) = \sqrt{\frac{\overline{V}[\eta]}{\overline{V}[\xi]}}\Phi^{-1}(\alpha) + E[\eta] - \sqrt{\frac{\overline{V}[\eta]}{\overline{V}[\xi]}}E[\xi].$$

Writing

$$a = \sqrt{\frac{\overline{V}[\eta]}{\overline{V}[\xi]}}, \qquad b = E[\eta] - \sqrt{\frac{\overline{V}[\eta]}{\overline{V}[\xi]}}E[\xi],$$

then we obtain $\Psi^{-1}(\alpha) = a\Phi^{-1}(\alpha) + b$ or, equivalently, $\Phi(x) = \Psi(ax + b)$. □

Example 16 Consider two independent Gaussian fuzzy numbers $\xi_1 \sim \mathcal{N}(a_1, b_1)$ and $\xi_2 \sim \mathcal{N}(a_2, b_2)$. By Example 15 we have

$$\sqrt{\overline{V}[\xi_1 + \xi_2]} = \sqrt{(b_1 + b_2)^2}, \tag{36}$$

that is,

$$\sqrt{\overline{V}[\xi_1 + \xi_2]} = b_1 + b_2 = \sqrt{\overline{V}[\xi_1]} + \sqrt{\overline{V}[\xi_2]}. \tag{37}$$

Here, the equality holds because ξ_1 and ξ_2 have the same kind of distributions, which means that $\Psi^{-1}(\alpha) = a\Phi^{-1}(\alpha) + b$ always holds. Since the deduction process of two independent fuzzy variables with different types of distributions is hard to work out even by Matlab, the corresponding example is not shown in this paper.

6 Conclusions

Variance is of great significance in practice. Although the variance of fuzzy variables with respect to the possibility measure and its properties have been studied before, the variance of fuzzy variables on the basis of the credibility measure has not been formally investigated yet due to the intricate process of calculation. It is acknowledged that the credibility measure is self-dual, which is important both theoretically and practically. Thus, in this paper, we studied the variance of fuzzy variables based upon the credibility measure for better applications.

To sum up, our contributions mainly lie in the following several aspects. First of all, we calculated the variances of three different fuzzy variables on the basis of the original definition of the variance. Secondly, an inequality regarding the variance was proposed together with a calculation formula, which can greatly simplify the calculation process. Thirdly, through some discussions and comparisons, the upper bound suggested in this paper was showed to be acceptable within a reasonable range. Finally, we stated a new theorem, with the help of which two inequalities with respect to the variance and standard deviation for the sum of two fuzzy variables were deduced.

In the future, deep research is needed to take into account the case with much more fuzzy variables and the case that membership functions of fuzzy variables are more complicated. Additionally, it should be noted that although the calculation process of the RUBV has been simplified in this paper, the calculations of some fuzzy variables are not easy enough even under the help of Matlab. Some improvements on the RUBV of fuzzy variables will be done in the future work.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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