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Berwald-type inequalities for Sugeno integral with respect to $(\alpha, m, r)_g$-concave functions

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Abstract
In this paper, we introduce the concept of an $(\alpha, m, r)_g$-concave function as a generalization of a concave function. Then we establish Berwald-type inequalities for the Sugeno integral based on this kind of functions. Our work generalizes the previous results in the literature. Finally, we give some conclusions and problems for further investigations.

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Keywords: Berwald-type inequality; Sugeno integral; $(\alpha, m, r)_g$-concave function

1 Introduction

As a tool for modeling nondeterministic problems, fuzzy measures and a fuzzy integral introduced by Sugeno in [1] have been successfully applied to various fields. The fuzzy integral provides a practical tool for many problems in engineering and social choice, where the aggregation of data is required. However, the practicality of fuzzy integral is restricted for the special operators used in the construction. Thus, many scholars have generalized the Sugeno integral by using some other operators to replace the special operator(s) $\vee$ and/or $\wedge$. They proposed the Choquet-like integral [2], the Shilkret integral [3], $\perp$-integral [4], the generalized fuzzy integral [5], the Sugeno-like integral [6], the λ-generalized Sugeno integral [7], the pseudo-integral [8], the interval-valued generalized fuzzy integral [9], and the set-valued pseudo-integral [10]. Suárez García and Gil Álvarez [11] presented two families of fuzzy integrals, the so-called seminormed fuzzy integral and semiconormed fuzzy integral. Klement *et al.* [12] investigated a concept of universal integrals generalizing both the Choquet integral and the Sugeno integral. Wang and Klir [13] provided a general overview on fuzzy measurement and fuzzy integration.

The integral inequalities are significant mathematical tools both in theory and applications. Different integral inequalities including Chebyshev, Jensen, Hölder, and Minkowski inequalities are widely used in various fields of mathematics, such as probability theory, differential equations, decision-making under risk, forecasting of time-series, and information sciences.

The convexity for a given function is one of the most powerful tools in establishing analytic inequalities. Especially, there are many important applications in the theory of higher transcendental functions. However, for many problems encountered in economics and engineering, the notion of convexity is unsuitable. Hence, it is necessary to extend the notion of convexity, and various generalizations of convexity have appeared in the literature. Hanson [14] gave the notion of invexity as a significant generalization of classical convexity. Ben-Israel and Mond [15] studied the preinvex functions, a special case of invex functions. Breckner [16] introduced the s-convex functions, and Varošanec [17] presented the h-convex functions as a generalization of convex functions. Mihesan [18] proposed the definition of (α, m)-convex functions. For recent results and generalizations concerning m-convex and (α, m)-convex functions, see [19, 20]. Latif and Shoaib [21] discussed the concept of m-preinvex functions and (α, m)-preinvex functions. Gill *et al.* [22] provided the concept of r-mean convex functions.

On the other hand, recently, some researchers have showed that several integral inequalities hold not only in the classical context but also for the fuzzy context. Román-Flores *et al.* investigated several kinds of classical integral inequalities for fuzzy integral including a Chebyshev-type inequality [23], a Young-type inequality [24], a Jensen-type inequality [25], a Hardy-type inequality [26], a convolution-type inequality [27], a Stolarsky-type inequality [28], and a Markov-type inequality [29]. Agahi *et al.* proved a general Chebyshev-type inequality [30], a Hölder-type inequality [31], a Berwald-type inequality [32], a general Minkowski-type inequality [33], and a general Barnes-Godun-Levin-type inequality [34] for the Sugeno integral. Caballero and Sadarangani presented Cauchy-Schwarz [35], Chebyshev [36], Fritz Carlson [37], and Sandor [38] inequalities for the Sugeno integral. Mesiar and Ouyang proposed Chebyshev [39], Yong [40], general Chebyshev [41], and Minkowski [42] inequalities for Sugeno integral.

Agahi *et al.* [32] illustrated a Berwald-type inequality for the Sugeno integral of a convex function. Agahi *et al.* [43] also obtained a Berwald-type inequality for a universal integral based on a convex function. Song *et al.* [44] proved Berwald-type inequalities for an extreme universal integral from the situation of convex functions to (α, m)-convex functions. Particularly, for pseudo-multiplication $\otimes = \wedge$, a Berwald-type inequality for the Sugeno integral based on (α, m)-concave functions is obtained. The purpose of this paper is to prove Berwald-type inequalities for the Sugeno integral related to $(\alpha, m, r)_g$-concavity. Some examples are given to illustrate the results.

After some preliminaries of some known results on the Sugeno integral and the notion of an $(\alpha, m, r)_g$-concave function in Section 2, Section 3 deals with Berwald inequalities for the Sugeno integral based on $(\alpha, m, r)_g$-concave functions and reverse Berwald-type inequalities for the Sugeno integral based on $(\alpha, m, r)_g$-convex functions. Finally, some examples are given to illustrate the results and some remarks are obtained as special cases.

2 Preliminaries

In this section, we recall some basic definitions and properties of the fuzzy integral and introduce the $(\alpha, m, r)_g$-convex functions. For details, we refer the reader to Refs. [1, 13].

Suppose that $\wp$ is a σ-algebra of subsets of X and let $\mu : \wp \to [0, \infty)$ be a nonnegative, extended real-valued set function. We say that μ is a fuzzy measure if it satisfies:

(1) $\mu(\emptyset) = 0$;

(2) $E, F \in \wp$ and $E \subset F$ imply $\mu(E) \leq \mu(F)$;

(3) $\{E_n\} \subset \wp$, $E_1 \subset E_2 \subset \cdots$, imply $\lim_{n\to\infty} \mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n)$;

(4) $\{E_n\} \subset \wp$, $E_1 \supset E_2 \supset \cdots$, $\mu(E_1) < \infty$, imply $\lim_{n\to\infty} \mu(E_n) = \mu(\bigcap_{n=1}^{\infty} E_n)$.

If f is a nonnegative real-valued function defined on X, we denote the set $\{x \in X : f(x) \geq \alpha\} = \{x \in X : f \geq \alpha\}$ by F_α for $\alpha \geq 0$. Note that if $\alpha \leq \beta$, then $F_\beta \subset F_\alpha$.

Let $(X, \wp, \mu)$ be a fuzzy measure space. We denote by M^+ the set of all nonnegative measurable functions with respect to $\wp$.

Definition 2.1 (Sugeno [1]) Let $(X, \wp, \mu)$ be a fuzzy measure space, $f \in M^+$, and $A \in \wp$. The Sugeno integral (or the fuzzy integral) of f on A with respect to the fuzzy measure μ is defined as

$$(\mathrm{S}) \int_A f \, d\mu = \bigvee_{\alpha \geq 0} \left[\alpha \wedge \mu(A \cap F_\alpha)\right];$$

when $A = X$,

$$(\mathrm{S}) \int_X f \, d\mu = \bigvee_{\alpha \geq 0} \left[\alpha \wedge \mu(F_\alpha)\right],$$

where $\vee$ and $\wedge$ denote the operations sup and inf on $[0, \infty)$, respectively.

The properties of the fuzzy integral are well known and can be found in [13].

Proposition 2.2 *Let $(X, \wp, \mu)$ be a fuzzy measure space, $A, B \in \wp$, and $f, g \in M^+$. Then*:

(1) $(\mathrm{S}) \int_A f \, d\mu \leq \mu(A)$;
(2) $(\mathrm{S}) \int_A k \, d\mu = k \wedge \mu(A)$ *for a nonnegative constant* k;
(3) $(\mathrm{S}) \int_A f \, d\mu \leq (\mathrm{S}) \int_A g \, d\mu$ *iff* $f \leq g$;
(4) $\mu(A \cap \{f \geq \alpha\}) \geq \alpha \Rightarrow (\mathrm{S}) \int_A f \, d\mu \geq \alpha$;
(5) $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \Rightarrow (\mathrm{S}) \int_A f \, d\mu \leq \alpha$;
(6) $(\mathrm{S}) \int_A f \, d\mu > \alpha \Leftrightarrow$ *there exists* $\gamma > \alpha$ *such that* $\mu(A \cap \{f \geq \gamma\}) > \alpha$;
(7) $(\mathrm{S}) \int_A f \, d\mu < \alpha \Leftrightarrow$ *there exists* $\gamma < \alpha$ *such that* $\mu(A \cap \{f \geq \gamma\}) < \alpha$.

Remark 2.3 Consider the distribution function F associated to f on A, that is, $F(\alpha) = \mu(A \cap \{f \geq \alpha\})$. Then, due to (4) and (5) of Proposition 2.2, we have $F(\alpha) = \alpha \Rightarrow (\mathrm{S}) \int_A f \, d\mu = \alpha$. Thus, from a numerical point of view, the fuzzy integral can be calculated by solving the equation $F(\alpha) = \alpha$.

Definition 2.4 Let $I \subseteq \mathbb{R}$ be an interval, $\lambda, \alpha, m \in [0,1]$, $r \in \mathbb{R}$, and g be a continuous and monotonous function on $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is said to be $(\alpha, m, r)_g$-concave on I if, for all $x, y \in I$,

$$f\left(\left[\lambda x^r + m(1-\lambda) y^r\right]^{1/r}\right) \geq g^{-1}\left(\left[\lambda^\alpha (g \circ f)^r(x) + m\left(1 - \lambda^\alpha\right)(g \circ f)^r(y)\right]^{1/r}\right), \quad r \neq 0$$

or

$$f\left(x^\lambda y^{m(1-\lambda)}\right) \geq g^{-1}\left((g \circ f)^{\lambda^\alpha}(x)(g \circ f)^{m(1-\lambda^\alpha)}(y)\right), \quad r = 0.$$

By reversing the inequalities we obtain the definition of an $(\alpha, m, r)_g$-convex function f on I.

Remark 2.5 If in Definition 2.4, $g = \mathrm{id}$ (*i.e.*, $g(x) = x$ for any $x \in I$), then we obtain the definition of (α, m, r)-concavity.

If in Definition 2.4, $\alpha, m = 1$, then we obtain the definition of r_g-mean concavity.

If in Definition 2.4, $\alpha, m = 1$ and $g = \mathrm{id}$, then we obtain the definition of r-mean concavity [45].

If in Definition 2.4, $r = 1$, then we obtain the definition of $(\alpha, m)_g$-concavity.

If in Definition 2.4, $r = 1$ and $g = \mathrm{id}$, then we obtain the definition of (α, m)-concavity [18].

If $(\alpha, m, r) \in \{(0,0,1), (1,m,1), (1,1,1), (\alpha,1,1)\}$ and $g = \mathrm{id}$ in Definition 2.4, we obtain the following classes of functions: decreasing, m-concave, concave, and α-concave.

3 Berwald-type inequalities for Sugeno integral based on $(\alpha, m, r)_g$-concave function

The following Berwald inequality is well known [46].

Let f be a nonnegative concave function on $[a, b]$. Then, for all u, v such that $0 < u < v < \infty$,

$$\frac{(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}} \left(\frac{\int_a^b f^v(x)\, dx}{b-a} \right)^{\frac{1}{v}} \le \left(\frac{\int_a^b f^u(x)\, dx}{b-a} \right)^{\frac{1}{u}}. \tag{3.1}$$

Unfortunately, the following example shows that the Berwald inequality for the Sugeno integral based on $(\alpha, m, r)_g$-concave functions is not valid.

Example Consider $X = [0,1]$ and μ be the Lebesgue measure on X. Take the function $f(x) = g(x) = \sqrt{x}$; then $f(x)$ is a $(\frac{2}{3}, \frac{1}{3}, 2)_{\frac{1}{2}}$-concave function. In fact,

$$\begin{aligned} \sqrt{x} &= f\left(\left(x^2 \cdot 1^2 + \frac{1}{3}(1-x^2)0^2 \right)^{\frac{1}{2}} \right) \\ &\ge \left(\left(\sqrt[3]{x^4} \cdot 1 + \frac{1}{3}\left(1 - \sqrt[3]{x^4}\right) \cdot 0 \right)^{\frac{1}{2}} \right)^2 = \sqrt[3]{x^4} \end{aligned}$$

for $x \in [0,1]$.

Let $u = \frac{1}{3}$ and $v = \frac{1}{2}$. A straightforward calculus shows that

$$\begin{aligned} (S)\int_0^1 f^{\frac{1}{2}}(x)\, d\mu &= \bigvee_{\beta \in [0,1]} \beta \wedge \mu\left([0,1] \cap \{x \ge \beta^4\}\right) = 0.7245, \\ (S)\int_0^1 f^{\frac{1}{3}}(x)\, d\mu &= \bigvee_{\beta \in [0,1]} \beta \wedge \mu\left([0,1] \cap \{x \ge \beta^6\}\right) = 0.7781. \end{aligned}$$

Therefore,

$$0.4982 = \left(\frac{(1+\frac{1}{2})^2}{(1+\frac{1}{3})^3} \right) \left((S)\int_0^1 f^{\frac{1}{2}}(x)\, d\mu \right)^2 \ge \left((S)\int_0^1 f^{\frac{1}{3}}(x)\, d\mu \right)^3 = 0.4711.$$

This proves that the Berwald inequality is not satisfied for the Sugeno integral based on $(\alpha, m, r)_g$-concave functions.

Now we present Berwald inequalities for the Sugeno integral based on $(\alpha, m, r)_g$-concave functions.

Theorem 3.1 *Let $(\alpha,m)\in(0,1]^2$, $r\in\mathbb{R}$, $r\neq 0$, g be a continuous and monotonous function, $f:[0,1]\to[0,\infty)$ be an $(\alpha,m,r)_g$-concave function, and μ be the Lebesgue measure on $\mathbb{R}$. Then*:

Case (i). *If $(g\circ f)^r(1)-m(g\circ f)^r(0)>0$, then*

$$\left((\mathrm{S})\int_0^1 f^u(x)\,d\mu\right)^{\frac{1}{u}}\ge\beta\wedge\left(1-\left(\frac{g^r(\beta)-m(g\circ f)^r(0)}{(g\circ f)^r(1)-m(g\circ f)^r(0)}\right)^{\frac{1}{\alpha r}}\right)^{\frac{1}{u}},$$

where $\beta=\frac{(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}((\mathrm{S})\int_0^1 f^v(x)\,d\mu)^{\frac{1}{v}}$.

Case (ii). *If $(g\circ f)^r(1)-m(g\circ f)^r(0)=0$, then*

$$\left((\mathrm{S})\int_0^1 f^u(x)\,d\mu\right)^{\frac{1}{u}}\ge f(0)\sqrt[r]{m}\wedge 1.$$

Case (iii). *If $(g\circ f)^r(1)-m(g\circ f)^r(0)<0$, then*

$$\left((\mathrm{S})\int_0^1 f^u(x)\,d\mu\right)^{\frac{1}{u}}\ge\beta\wedge\left(\frac{g^r(\beta)-m(g\circ f)^r(0)}{(g\circ f)^r(1)-m(g\circ f)^r(0)}\right)^{\frac{1}{u\alpha r}},$$

where $\beta=\frac{(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}((\mathrm{S})\int_0^1 f^v(x)\,d\mu)^{\frac{1}{v}}$.

Proof Let $0<u<v<\infty$ and $(\mathrm{S})\int_0^1 f^v(x)\,d\mu=t$. Since f is an $(\alpha,m,r)_g$-concave function for $x\in[0,1]$, we have

$$\begin{aligned}f(x)&=f\left(\left[x^r\cdot 1^r+m\left(1-x^r\right)\cdot 0^r\right]^{1/r}\right)\\&\ge g^{-1}\left(\left[x^{\alpha r}(g\circ f)^r(1)+m\left(1-x^{\alpha r}\right)(g\circ f)^r(0)\right]^{1/r}\right)=h(x).\end{aligned}$$

By Proposition 2.2(3) we have

$$\begin{aligned}&\left((\mathrm{S})\int_0^1 f^u(x)\,d\mu\right)^{\frac{1}{u}}\\&\quad\ge\left((\mathrm{S})\int_0^1 h^u(x)\,d\mu\right)^{\frac{1}{u}}=\left(\bigvee_{\gamma\in[0,1]}\left(\gamma\wedge\mu\left([0,1]\cap\left\{h^u\ge\gamma\right\}\right)\right)\right)^{\frac{1}{u}}\\&\quad=\left(\bigvee_{\gamma\in[0,1]}\left(\gamma\wedge\mu\left([0,1]\cap\left\{h\ge\gamma^{\frac{1}{u}}\right\}\right)\right)\right)^{\frac{1}{u}}\\&\quad=\left(\bigvee_{\gamma\in[0,1]}\left(\gamma\wedge\mu\left([0,1]\cap\left\{x\,\middle|\,\begin{array}{c}((g\circ f)^r(1)-m(g\circ f)^r(0))x^{\alpha r}\\\ge g^r(\gamma^{\frac{1}{u}})-m(g\circ f)^r(0)\end{array}\right\}\right)\right)\right)^{\frac{1}{u}}\\&\quad\ge\left(\left(\frac{(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}t^{\frac{1}{v}}\right)^u\wedge\mu\left([0,1]\cap\left\{x\,\middle|\,\begin{array}{c}((g\circ f)^r(1)-m(g\circ f)^r(0))x^{\alpha r}\\\ge g^r(\frac{(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}t^{\frac{1}{v}})-m(g\circ f)^r(0)\end{array}\right\}\right)\right)^{\frac{1}{u}}.\end{aligned}$$

By Proposition 2.2(1) and Remark 2.3 we get:

Case (i). If $(g\circ f)^r(1) - m(g\circ f)^r(0) > 0$, then

$$\left((\mathrm{S})\int_0^1 f^u(x)\,d\mu\right)^{\frac{1}{u}} \geq \beta \wedge \left(1-\left(\frac{g^r(\beta)-m(g\circ f)^r(0)}{(g\circ f)^r(1)-m(g\circ f)^r(0)}\right)^{\frac{1}{\alpha r}}\right)^{\frac{1}{u}},$$

where $\beta = \frac{(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}((\mathrm{S})\int_0^1 f^v(x)\,d\mu)^{\frac{1}{v}}$.

Case (ii). If $(g\circ f)^r(1) - m(g\circ f)^r(0) = 0$, then

$$\left((\mathrm{S})\int_0^1 f^u(x)\,d\mu\right)^{\frac{1}{u}} \geq f(0)\sqrt[r]{m} \wedge 1.$$

Case (iii). If $(g\circ f)^r(1) - m(g\circ f)^r(0) < 0$, then

$$\left((\mathrm{S})\int_0^1 f^u(x)\,d\mu\right)^{\frac{1}{u}} \geq \beta \wedge \left(\frac{g^r(\beta)-m(g\circ f)^r(0)}{(g\circ f)^r(1)-m(g\circ f)^r(0)}\right)^{\frac{1}{u\alpha r}},$$

where $\beta = \frac{(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}((\mathrm{S})\int_0^1 f^v(x)\,d\mu)^{\frac{1}{v}}$.

This completes the proof. □

Remark 3.2 If $\alpha = 0$ in Theorem 3.1, then

$$\left((\mathrm{S})\int_0^1 f^u(x)\,d\mu\right)^{\frac{1}{u}} \geq \min\bigl\{f(1),1\bigr\}.$$

Example Consider $X = [0,1]$ with the Lebesgue measure μ on it. Take the function $f(x) = g(x) = \sqrt{x}$; then $f(x)$ is a $(\frac{2}{3},\frac{1}{3},2)_{\frac{1}{2}}$-concave function. In fact,

$$\sqrt{x} = f\left(\left(x^2\cdot 1^2 + \frac{1}{3}\left(1-x^2\right)0^2\right)^{\frac{1}{2}}\right) \geq \left(\left(\sqrt[3]{x^4}\cdot 1 + \frac{1}{3}\left(1-\sqrt[3]{x^4}\right)\cdot 0\right)^{\frac{1}{2}}\right)^2 = \sqrt[3]{x^4}$$

for $x \in [0,1]$.

Let $u = \frac{1}{3}$ and $v = \frac{1}{2}$. A straightforward calculus shows that

$$(\mathrm{S})\int_0^1 f^{\frac{1}{2}}(x)\,d\mu = \bigvee_{\beta\in[0,1]} \beta \wedge \mu\bigl([0,1]\cap\bigl\{x \geq \beta^4\bigr\}\bigr) = 0.7245,$$

$$(\mathrm{S})\int_0^1 f^{\frac{1}{3}}(x)\,d\mu = \bigvee_{\beta\in[0,1]} \beta \wedge \mu\bigl([0,1]\cap\bigl\{x \geq \beta^6\bigr\}\bigr) = 0.7781,$$

$$\left(\frac{(1+\frac{1}{2})^2}{(1+\frac{1}{3})^3}\right)\left((\mathrm{S})\int_0^1 f^{\frac{1}{2}}(x)\,d\mu\right)^2 = 0.4982.$$

By Theorem 3.1 we have

$$\begin{aligned} 0.4711 &= \left((\mathrm{S})\int_0^1 f^{\frac{1}{3}}(x)\,d\mu\right)^3 \\ &\geq \left(\frac{(1+\frac{1}{2})^2}{(1+\frac{1}{3})^3}\right)\left((\mathrm{S})\int_0^1 f^{\frac{1}{2}}(x)\,d\mu\right)^2 \end{aligned}$$

$$\wedge\left(\left(1-\left(\frac{(\frac{(1+\frac{1}{2})^2}{(1+\frac{1}{3})^3})((\mathrm{S})\int_0^1 f^{\frac{1}{2}}(x)\,d\mu)^2-\frac{1}{3}\sqrt{0}}{\sqrt{1}-\frac{1}{3}\sqrt{0}}\right)\right)^{\frac{3}{4}}\right)^3$$
$$= 0.4982 \wedge 0.0674 = 0.0674.$$

Now, we will prove the general cases of Theorem 3.1.

Theorem 3.3 *Let* $(\alpha, m) \in (0,1]^2$, $r \in \mathbb{R}$, $r \neq 0$, g *be a continuous and monotonous function,* $f : [a,b] \to [0,\infty)$ *be an* $(\alpha, m, r)_g$*-concave function, and* μ *be the Lebesgue measure on* $\mathbb{R}$*. Then*:

Case (i). *If* $(g\circ f)^r(b) - m(g\circ f)^r(a) > 0$, *then*

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \geq \beta \wedge \left(b-\left(\left(\frac{g^r(\beta)-m(g\circ f)^r(a)}{(g\circ f)^r(b)-m(g\circ f)^r(a)}\right)^{1/\alpha}\left(b^r-ma^r\right)+ma^r\right)^{1/r}\right)^{\frac{1}{u}},$$

where $\beta = \frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}\left(\frac{(\mathrm{S})\int_a^b f^v(x)\,d\mu}{b-a}\right)^{\frac{1}{v}}$.

Case (ii). *If* $(g\circ f)^r(b) - m(g\circ f)^r(a) = 0$, *then*

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \geq \min\left\{f(a)\sqrt[r]{m}, (b-a)^{\frac{1}{u}}\right\}.$$

Case (iii). *If* $(g\circ f)^r(b) - m(g\circ f)^r(a) < 0$, *then*

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \geq \beta \wedge \left(\left(\left(\frac{g^r(\beta)-m(g\circ f)^r(a)}{(g\circ f)^r(b)-m(g\circ f)^r(a)}\right)^{1/\alpha}\left(b^r-ma^r\right)+ma^r\right)^{1/r}-a\right)^{\frac{1}{u}},$$

where $\beta = \frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}\left(\frac{(\mathrm{S})\int_a^b f^v(x)\,d\mu}{b-a}\right)^{\frac{1}{v}}$.

Proof Let $0 < u < v < \infty$ and $(\mathrm{S})\int_a^b f^v(x)\,d\mu = t$. Since f is an $(\alpha, m, r)_g$-concave function for $x \in [a,b]$, we have

$$f(x) = f\left(\left[m\left(1-\frac{x^r-ma^r}{b^r-ma^r}\right)a^r + \frac{x^r-ma^r}{b^r-ma^r}b^r\right]^{1/r}\right)$$
$$\geq g^{-1}\left(\left[m\left(1-\left(\frac{x^r-ma^r}{b^r-ma^r}\right)^{\alpha}\right)(g\circ f)^r(a) + \left(\frac{x^r-ma^r}{b^r-ma^r}\right)^{\alpha}(g\circ f)^r(b)\right]^{1/r}\right) = h(x).$$

By Proposition 2.2(3) we have

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \geq \left((\mathrm{S})\int_a^b h^u(x)\,d\mu\right)^{\frac{1}{u}} = \left(\bigvee_{\gamma\in[0,b-a]}\left(\gamma \wedge \mu\left([a,b]\cap\left\{x|h \geq \gamma^{\frac{1}{u}}\right\}\right)\right)\right)^{\frac{1}{u}}$$

$$= \left(\bigvee_{\gamma \in [0,b-a]} \left(\gamma \wedge \mu \left([a,b] \cap \left\{ x \middle| \begin{array}{l} ((g\circ f)^r(b) - m(g\circ f)^r(a))(\frac{x^r - ma^r}{b^r - ma^r})^\alpha \\ \geq g^r(\gamma^{\frac{1}{u}}) - m(g\circ f)^r(a) \end{array} \right\} \right) \right) \right)^{\frac{1}{u}}$$

$$\geq \frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}} \left(\frac{t}{b-a} \right)^{\frac{1}{v}}$$

$$\wedge \left(\mu \left([a,b] \cap \left\{ x \middle| \begin{array}{l} ((g\circ f)^r(b) - m(g\circ f)^r(a))(\frac{x^r - ma^r}{b^r - ma^r})^\alpha \\ \geq g^r(\frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}(\frac{t}{b-a})^{\frac{1}{v}}) - m(g\circ f)^r(a) \end{array} \right\} \right) \right)^{\frac{1}{u}}.$$

By Proposition 2.2(1) and Remark 2.3 we get:

Case (i). If $(g\circ f)^r(b) - m(g\circ f)^r(a) > 0$, then

$$\left((\mathrm{S}) \int_a^b f^u(x)\,d\mu \right)^{\frac{1}{u}}$$
$$\geq \beta \wedge \left(b - \left(\left(\frac{g^r(\beta) - m(g\circ f)^r(a)}{(g\circ f)^r(b) - m(g\circ f)^r(a)} \right)^{1/\alpha} (b^r - ma^r) + ma^r \right)^{1/r} \right)^{\frac{1}{u}},$$

where $\beta = \frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}(\frac{(\mathrm{S})\int_a^b f^v(x)\,d\mu}{b-a})^{\frac{1}{v}}$.

Case (ii). If $(g\circ f)^r(b) - m(g\circ f)^r(a) = 0$, then

$$\left((\mathrm{S}) \int_a^b f^u(x)\,d\mu \right)^{\frac{1}{u}} \geq \frac{(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}} t^{\frac{1}{v}} \wedge f(a) \sqrt[ur]{m}.$$

Case (iii). If $(g\circ f)^r(b) - m(g\circ f)^r(a) < 0$, then

$$\left((\mathrm{S}) \int_a^b f^u(x)\,d\mu \right)^{\frac{1}{u}}$$
$$\geq \beta \wedge \left(\left(\left(\frac{g^r(\beta) - m(g\circ f)^r(a)}{(g\circ f)^r(b) - m(g\circ f)^r(a)} \right)^{1/\alpha} (b^r - ma^r) + ma^r \right)^{1/r} - a \right)^{\frac{1}{u}},$$

where $\beta = \frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}(\frac{(\mathrm{S})\int_a^b f^v(x)\,d\mu}{b-a})^{\frac{1}{v}}$.

This completes the proof. □

Remark 3.4 If $\alpha = 0$ in Theorem 3.3, then

$$\left((\mathrm{S}) \int_a^b f^u(x)\,d\mu \right)^{\frac{1}{u}} \geq \min\{f(b), (b-a)^{\frac{1}{u}}\}.$$

Example Consider $X = [1,2]$ with the Lebesgue measure μ on it. Take the functions $f(x) = \ln(x+2)$ and $g(x) = \mathrm{id}$; $f(x)$ is a $(1,0,3)$-concave function. In fact,

$$\ln(x+2) = f\left(\left[\left(\frac{x^3 - 0\cdot 1^3}{2^3 - 0\cdot 1^3} \right) \cdot 2^3 + 0 \left(1 - \left(\frac{x^3 - 0\cdot 1^3}{2^3 - 0\cdot 1^3} \right) \right) \cdot 1^3 \right]^{1/3} \right)$$
$$\geq \left[\left(\frac{x^3 - 0\cdot 1^3}{2^3 - 0\cdot 1^3} \right) \cdot \ln^3(4) + 0 \left(1 - \left(\frac{x^3 - 0\cdot 1^3}{2^3 - 0\cdot 1^3} \right)^{1/2} \right) \cdot \ln^3(3) \right]^{1/3} = \frac{\ln(4)}{2} x$$

for $x \in [1,2]$.

Let $u = \frac{1}{2}$ and $\nu = 2$. A straightforward calculus shows that

$$(S)\int_1^2 f^2(x)\,d\mu = \bigvee_{\beta\in[1,2]} \beta \wedge \mu\big([1,2] \cap \{\ln^2(x+2) \ge \beta\}\big) = 1.1194,$$

$$(S)\int_1^2 f^{\frac{1}{2}}(x)\,d\mu = \bigvee_{\beta\in[1,2]} \beta \wedge \mu\big([1,2] \cap \{\ln(x+2) \ge \beta^2\}\big) = 1.0415,$$

$$\left(\frac{(2-1)^2\cdot(1+2)^{\frac{1}{2}}}{(1+\frac{1}{2})^2}\right)\left(\frac{(S)\int_1^2 f^2(x)\,d\mu}{2-1}\right)^{\frac{1}{2}} = 0.8144.$$

By Theorem 3.3 we have

$$\begin{aligned} 0.4260 &= 0.8144 \wedge 0.4260 \\ &= \left(\frac{(2-1)^2\cdot(1+2)^{\frac{1}{2}}}{(1+\frac{1}{2})^2}\right)\left(\frac{(S)\int_1^2 f^2(x)\,d\mu}{2-1}\right)^{\frac{1}{2}} \\ &\quad \wedge \left(2 - \left(\left(\frac{(\frac{(2-1)^2\cdot(1+2)^{\frac{1}{2}}}{(1+\frac{1}{2})^2})(\frac{(S)\int_1^2 f^2(x)\,d\mu}{2-1})^{\frac{1}{2}} - 0\cdot\ln^3(3)}{\ln^3(4) - 0\cdot\ln^3(3)}\right)\cdot 2^3\right)^{\frac{1}{3}}\right)^2 \\ &\le \left((S)\int_1^2 f^{\frac{1}{2}}(x)\,d\mu\right)^2 = 1.0847. \end{aligned}$$

Now we consider some special cases of $(\alpha,m,r)_g$-concave functions in Theorem 3.3.

Remark 3.5 Let $(\alpha,m) \in [0,1]^2$, $r \in \mathbb{R}$, $r \neq 0$, $g = \mathrm{id}$, $f : [a,b] \to [0,\infty)$ be an (α,m,r)-concave function, and μ be the Lebesgue measure on $\mathbb{R}$. Then:

Case (i). If $f^r(b) - mf^r(a) > 0$, then

$$\left((S)\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \ge \beta \wedge \left(b - \left(\left(\frac{\beta^r - mf^r(a)}{f^r(b) - mf^r(a)}\right)^{1/\alpha}\big(b^r - ma^r\big) + ma^r\right)^{1/r}\right)^{\frac{1}{u}},$$

where $\beta = \frac{(b-a)^{\frac{1}{u}}(1+\nu)^{\frac{1}{\nu}}}{(1+u)^{\frac{1}{u}}}\left(\frac{(S)\int_a^b f^\nu(x)\,d\mu}{b-a}\right)^{\frac{1}{\nu}}$.

Case (ii). If $f^r(b) - mf^r(a) = 0$, then

$$\left((S)\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \ge \min\big\{f(a)\sqrt[r]{m}, (b-a)^{\frac{1}{u}}\big\}.$$

Case (iii). If $f^r(b) - mf^r(a) < 0$, then

$$\left((S)\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \ge \beta \wedge \left(\left(\left(\frac{\beta^r - mf^r(a)}{f^r(b) - mf^r(a)}\right)^{1/\alpha}\big(b^r - ma^r\big) + ma^r\right)^{1/r} - a\right)^{\frac{1}{u}},$$

where $\beta = \frac{(b-a)^{\frac{1}{u}}(1+\nu)^{\frac{1}{\nu}}}{(1+u)^{\frac{1}{u}}}\left(\frac{(S)\int_a^b f^\nu(x)\,d\mu}{b-a}\right)^{\frac{1}{\nu}}$.

Remark 3.6 Let $\alpha = m = 1$, $r \in \mathbb{R}$, $r \neq 0$, g be a continuous and monotonous function, $f : [a,b] \to [0,\infty)$ be an r_g-mean concave function, and μ be the Lebesgue measure on $\mathbb{R}$. Then:

Case (i). If $(g\circ f)^r(b)-(g\circ f)^r(a)>0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \geq \beta\wedge\left(b-\left(\left(\frac{g^r(\beta)-(g\circ f)^r(a)}{(g\circ f)^r(b)-(g\circ f)^r(a)}\right)\left(b^r-a^r\right)+a^r\right)^{1/r}\right)^{\frac{1}{u}},$$

where $\beta=\frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}\left(\frac{(\mathrm{S})\int_a^b f^v(x)\,d\mu}{b-a}\right)^{\frac{1}{v}}$.

Case (ii). If $(g\circ f)^r(b)-(g\circ f)^r(a)=0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \geq \min\left\{f(a)\sqrt[r]{m},(b-a)^{\frac{1}{u}}\right\}.$$

Case (iii). If $(g\circ f)^r(b)-(g\circ f)^r(a)<0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \geq \beta\wedge\left(\left(\left(\frac{g^r(\beta)-(g\circ f)^r(a)}{(g\circ f)^r(b)-(g\circ f)^r(a)}\right)\left(b^r-a^r\right)+a^r\right)^{1/r}-a\right)^{\frac{1}{u}},$$

where $\beta=\frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}\left(\frac{(\mathrm{S})\int_a^b f^v(x)\,d\mu}{b-a}\right)^{\frac{1}{v}}$.

Remark 3.7 Let $\alpha=m=1$, $r\in\mathbb{R}$, $r\neq 0$, $g=\mathrm{id}$, $f:[a,b]\to[0,\infty)$ be an r-mean concave function, and μ be the Lebesgue measure on $\mathbb{R}$. Then:

Case (i). If $f^r(b)-f^r(a)>0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \geq \beta\wedge\left(b-\left(\left(\frac{\beta^r-f^r(a)}{f^r(b)-f^r(a)}\right)\left(b^r-a^r\right)+a^r\right)^{1/r}\right)^{\frac{1}{u}},$$

where $\beta=\frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}\left(\frac{(\mathrm{S})\int_a^b f^v(x)\,d\mu}{b-a}\right)^{\frac{1}{v}}$.

Case (ii). If $f^r(b)-f^r(a)=0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \geq \min\left\{f(a)\sqrt[r]{m},(b-a)^{\frac{1}{u}}\right\}.$$

Case (iii). If $f^r(b)-f^r(a)<0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \geq \beta\wedge\left(\left(\left(\frac{\beta^r-f^r(a)}{f^r(b)-f^r(a)}\right)\left(b^r-a^r\right)+a^r\right)^{1/r}-a\right)^{\frac{1}{u}},$$

where $\beta=\frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}\left(\frac{(\mathrm{S})\int_a^b f^v(x)\,d\mu}{b-a}\right)^{\frac{1}{v}}$.

Remark 3.8 Let $(\alpha,m)\in[0,1]^2$, $r=1$, g be a continuous and monotonous function, $f:[a,b]\to[0,\infty)$ be an $(\alpha,m)_g$-concave function, and μ be the Lebesgue measure on $\mathbb{R}$. Then:

Case (i). If $(g\circ f)(b)-m(g\circ f)(a)>0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \geq \beta\wedge\left(\left(1-\left(\frac{g(\beta)-m(g\circ f)(a)}{(g\circ f)(b)-m(g\circ f)(a)}\right)^{1/\alpha}\right)(b-ma)\right)^{\frac{1}{u}},$$

where $\beta=\frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}\left(\frac{(\mathrm{S})\int_a^b f^v(x)\,d\mu}{b-a}\right)^{\frac{1}{v}}$.

Case (ii). If $(g\circ f)(b)-m(g\circ f)(a)=0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \ge \min\left\{f(a)\sqrt[r]{m},(b-a)^{\frac{1}{u}}\right\}.$$

Case (iii). If $(g\circ f)(b)-m(g\circ f)(a)<0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \ge \beta\wedge\left(\left(\left(\frac{g(\beta)-m(g\circ f)(a)}{(g\circ f)(b)-m(g\circ f)(a)}\right)^{1/\alpha}(b-ma)+ma\right)-a\right)^{\frac{1}{u}},$$

where $\beta=\frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}(\frac{(\mathrm{S})\int_a^b f^v(x)\,d\mu}{b-a})^{\frac{1}{v}}$.

Remark 3.9 Let $(\alpha,m)\in[0,1]^2$, $r=1$, $g=\mathrm{id}$, $f:[a,b]\to[0,\infty)$ be an (α,m)-concave function, and μ be the Lebesgue measure on $\mathbb{R}$. Then:

Case (i). If $f(b)-mf(a)>0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \ge \beta\wedge\left(\left(1-\left(\frac{\beta-mf(a)}{f(b)-mf(a)}\right)^{1/\alpha}\right)(b-ma)\right)^{\frac{1}{u}},$$

where $\beta=\frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}(\frac{(\mathrm{S})\int_a^b f^v(x)\,d\mu}{b-a})^{\frac{1}{v}}$.

Case (ii). If $f(b)-mf(a)=0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \ge \min\left\{f(a)\sqrt[r]{m},(b-a)^{\frac{1}{u}}\right\}.$$

Case (iii). If $f(b)-mf(a)<0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \ge \beta\wedge\left(\left(\left(\frac{\beta-mf(a)}{f(b)-mf(a)}\right)^{1/\alpha}(b-ma)+ma\right)-a\right)^{\frac{1}{u}},$$

where $\beta=\frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}(\frac{(\mathrm{S})\int_a^b f^v(x)\,d\mu}{b-a})^{\frac{1}{v}}$.

Remark 3.10 Let $g=\mathrm{id}$ and $\alpha=m=r=1$ in Theorem 3.3. Then we obtain the Berwald inequalities for the fuzzy integral of concave functions [32].

As in the proofs of Theorems 3.1 and 3.3, we can similarly obtain some reverse inequalities for the Sugeno integral based on $(\alpha,m,r)_g$-convex functions.

Remark 3.11 Let $(\alpha,m)\in(0,1]^2$, $r\in\mathbb{R}$, $r\neq0$, g be a continuous and monotonous function, $f:[0,1]\to[0,\infty)$ be an $(\alpha,m,r)_g$-convex function, and μ be the Lebesgue measure on $\mathbb{R}$. Then:

Case (i). If $(g\circ f)^r(1)-m(g\circ f)^r(0)>0$, then

$$\left((\mathrm{S})\int_0^1 f^u(x)\,d\mu\right)^{\frac{1}{u}} \le \beta\vee\left(1-\left(\frac{g^r(\beta)-m(g\circ f)^r(0)}{(g\circ f)^r(1)-m(g\circ f)^r(0)}\right)^{\frac{1}{\alpha r}}\right)^{\frac{1}{u}},$$

where $\beta=\frac{(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}((\mathrm{S})\int_0^1 f^v(x)\,d\mu)^{\frac{1}{v}}$.

Case (ii). If $(g\circ f)^r(1) - m(g\circ f)^r(0) = 0$, then

$$\left((S)\int_0^1 f^u(x)\,d\mu\right)^{\frac{1}{u}} \le \min\{f(1),1\}.$$

Case (iii). If $(g\circ f)^r(1) - m(g\circ f)^r(0) < 0$, then

$$\left((S)\int_0^1 f^u(x)\,d\mu\right)^{\frac{1}{u}} \le \beta \vee \left(\frac{g^r(\beta) - m(g\circ f)^r(0)}{(g\circ f)^r(1) - m(g\circ f)^r(0)}\right)^{\frac{1}{u\alpha r}},$$

where $\beta = \frac{(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}((S)\int_0^1 f^v(x)\,d\mu)^{\frac{1}{v}}$.

Example Consider $X = [0,1]$ with the Lebesgue measure μ on it. Take the function $f(x) = x^2$ and $g(x) = x^3$; then $f(x)$ is a $(\frac{1}{3},\frac{2}{3},3)_3$-convex function. In fact,

$$\begin{aligned} x^2 &= f\left(\left(x^3\cdot 1^3 + \frac{2}{3}(1-x^3)0^3\right)^{\frac{1}{3}}\right) \\ &\le \left(\left(x\cdot 1 + \frac{2}{3}(1-x)\cdot 0\right)^{\frac{1}{3}}\right)^{\frac{1}{3}} = \sqrt[9]{x} \end{aligned}$$

for $x \in [0,1]$.

Let $u = \frac{1}{2}$ and $v = 2$. A straightforward calculus shows that

$$\begin{aligned} &(S)\int_0^1 f^{\frac{1}{2}}(x)\,d\mu = \bigvee_{\beta\in[0,1]} \beta \wedge \mu\bigl([0,1]\cap\{x\ge\beta\}\bigr) = 0.5, \\ &(S)\int_0^1 f^2(x)\,d\mu = \bigvee_{\beta\in[0,1]} \beta \wedge \mu\bigl([0,1]\cap\{x^4\ge\beta\}\bigr) = 0.2755, \\ &\left(\frac{(1+2)^{\frac{1}{2}}}{(1+\frac{1}{2})^2}\right)\left((S)\int_0^1 f^2(x)\,d\mu\right)^{\frac{1}{2}} = 0.4041. \end{aligned}$$

By Remark 3.11 we have

$$\begin{aligned} 0.9994 &= 0.4041 \vee 0.9994 \\ &= \left(\frac{(1+2)^{\frac{1}{2}}}{(1+\frac{1}{2})^2}\right)\left((S)\int_0^1 f^2(x)\,d\mu\right)^{\frac{1}{2}} \\ &\quad \vee \left(1 - \left(\frac{\bigl((\frac{(1+2)^{\frac{1}{2}}}{(1+\frac{1}{2})^2})((S)\int_0^1 f^2(x)\,d\mu)^{\frac{1}{2}}\bigr)^9 - 0}{1-0}\right)\right)^2 \\ &\ge \left((S)\int_0^1 f^{\frac{1}{2}}(x)\,d\mu\right)^2 = 0.25. \end{aligned}$$

Remark 3.12 Let $(\alpha,m)\in(0,1]^2$, $r\in\mathbb{R}$, $r\ne 0$, g be a continuous and monotonous function, $f:[a,b]\to[0,\infty)$ be an $(\alpha,m,r)_g$-convex function, and μ be the Lebesgue measure on $\mathbb{R}$. Then:

Case (i). If $(g\circ f)^r(b) - m(g\circ f)^r(a) > 0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \leq \beta \vee \left(b - \left(\left(\frac{g^r(\beta) - m(g\circ f)^r(a)}{(g\circ f)^r(b) - m(g\circ f)^r(a)}\right)^{1/\alpha}\left(b^r - ma^r\right) + ma^r\right)^{1/r}\right)^{\frac{1}{u}},$$

where $\beta = \frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}\left(\frac{(\mathrm{S})\int_a^b f^v(x)\,d\mu}{b-a}\right)^{\frac{1}{v}}$.

Case (ii). If $(g\circ f)^r(b) - m(g\circ f)^r(a) = 0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \leq \min\left\{f(a)\sqrt[r]{m}, (b-a)^{\frac{1}{u}}\right\}.$$

Case (iii). If $(g\circ f)^r(b) - m(g\circ f)^r(a) < 0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \leq \beta \vee \left(\left(\left(\frac{g^r(\beta) - m(g\circ f)^r(a)}{(g\circ f)^r(b) - m(g\circ f)^r(a)}\right)^{1/\alpha}\left(b^r - ma^r\right) + ma^r\right)^{1/r} - a\right)^{\frac{1}{u}},$$

where $\beta = \frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}\left(\frac{(\mathrm{S})\int_a^b f^v(x)\,d\mu}{b-a}\right)^{\frac{1}{v}}$.

Remark 3.13 If $\alpha = 0$ in Remark 3.12, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \leq \min\left\{f(b), (b-a)^{\frac{1}{u}}\right\}.$$

Example Consider $X = [1,4]$ with the Lebesgue measure μ on it. Take the function $f(x) = \sqrt{x^3}$ and $g(x) = \sqrt[3]{x}$; then $f(x)$ is a $(\frac{1}{3}, 0, 3)_{\frac{1}{3}}$-convex function. In fact,

$$\begin{aligned}\sqrt{x^3} &= f\left(\left[\left(\frac{x^3 - 0\cdot 1^3}{4^3 - 0\cdot 1^3}\right)\cdot 4^3 + 0\left(1 - \left(\frac{x^3 - 0\cdot 1^3}{4^3 - 0\cdot 1^3}\right)\right)\cdot 1^3\right]^{1/3}\right) \\ &\leq \left(\left[\left(\frac{x^3 - 0\cdot 1^3}{4^3 - 0\cdot 1^3}\right)^{1/3}\cdot 8 + 0\left(1 - \left(\frac{x^3 - 0\cdot 1^3}{4^3 - 0\cdot 1^3}\right)^{1/3}\right)\cdot 1\right]^{1/3}\right)^3 = 2x\end{aligned}$$

for $x \in [1,4]$.

Let $u = \frac{1}{2}$ and $v = 2$. A straightforward calculus shows that

$$(\mathrm{S})\int_1^4 f^2(x)\,d\mu = \bigvee_{\beta\in[1,4]} \beta \wedge \mu\left([0,1]\cap\left\{x^3 \geq \beta\right\}\right) = 2.6212,$$

$$(\mathrm{S})\int_1^4 f^{\frac{1}{2}}(x)\,d\mu = \bigvee_{\beta\in[1,4]} \beta \wedge \mu\left([0,1]\cap\left\{x^{\frac{3}{4}} \geq \beta\right\}\right) = 1.8040,$$

$$\left(\frac{3^2\cdot(1+2)^{\frac{1}{2}}}{(1+\frac{1}{2})^2}\right)\left(\frac{(\mathrm{S})\int_1^4 f^2(x)\,d\mu}{4-1}\right)^{\frac{1}{2}} = 6.4760.$$

By Remark 3.12 we have

$$\begin{aligned}
6.4760 &= 6.4760 \vee 0.0740 \\
&= \left(\frac{3^2 \cdot (1+2)^{\frac{1}{2}}}{(1+\frac{1}{2})^2}\right)\left(\frac{(S)\int_1^4 f^2(x)\,d\mu}{4-1}\right)^{\frac{1}{2}} \\
&\quad \vee \left(4 - \left(\left(\frac{(\frac{3^2\cdot(1+2)^{\frac{1}{2}}}{(1+\frac{1}{2})^2})(\frac{(S)\int_1^4 f^2(x)\,d\mu}{4-1})^{\frac{1}{2}} - 0}{8-0}\right)\cdot 4^3\right)^{\frac{1}{3}}\right)^2 \\
&\geq \left((S)\int_1^4 f^{\frac{1}{2}}(x)\,d\mu\right)^2 = 3.2544.
\end{aligned}$$

Now we consider some special cases of $(\alpha, m, r)_g$-convex functions in Remark 3.12.

Remark 3.14 Let $(\alpha, m) \in [0,1]^2$, $r \in \mathbb{R}$, $r \neq 0$, $g = \mathrm{id}$, $f : [a,b] \to [0,\infty)$ be an (α, m, r)-convex function, and μ be the Lebesgue measure on $\mathbb{R}$. Then:

Case (i). If $f^r(b) - mf^r(a) > 0$, then

$$\left((S)\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \leq \beta \wedge \left(b - \left(\left(\frac{\beta^r - mf^r(a)}{f^r(b) - mf^r(a)}\right)^{1/\alpha}\left(b^r - ma^r\right) + ma^r\right)^{1/r}\right)^{\frac{1}{u}},$$

where $\beta = \frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}(\frac{(S)\int_a^b f^v(x)\,d\mu}{b-a})^{\frac{1}{v}}$.

Case (ii). If $f^r(b) - mf^r(a) = 0$, then

$$\left((S)\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \leq \min\left\{f(a)\sqrt[r]{m}, (b-a)^{\frac{1}{u}}\right\}.$$

Case (iii). If $f^r(b) - mf^r(a) < 0$, then

$$\left((S)\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \leq \beta \wedge \left(\left(\left(\frac{\beta^r - mf^r(a)}{f^r(b) - mf^r(a)}\right)^{1/\alpha}\left(b^r - ma^r\right) + ma^r\right)^{1/r} - a\right)^{\frac{1}{u}},$$

where $\beta = \frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}(\frac{(S)\int_a^b f^v(x)\,d\mu}{b-a})^{\frac{1}{v}}$.

Remark 3.15 Let $\alpha = m = 1$, $r \in \mathbb{R}$, $r \neq 0$, g be a continuous and monotonous function, $f : [a,b] \to [0,\infty)$ be an r_g-mean convex function, and μ be the Lebesgue measure on $\mathbb{R}$. Then:

Case (i). If $(g \circ f)^r(b) - (g \circ f)^r(a) > 0$, then

$$\left((S)\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \leq \beta \wedge \left(b - \left(\left(\frac{g^r(\beta) - (g \circ f)^r(a)}{(g \circ f)^r(b) - (g \circ f)^r(a)}\right)\left(b^r - a^r\right) + a^r\right)^{1/r}\right)^{\frac{1}{u}},$$

where $\beta = \frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}(\frac{(S)\int_a^b f^v(x)\,d\mu}{b-a})^{\frac{1}{v}}$.

Case (ii). If $(g \circ f)^r(b) - (g \circ f)^r(a) = 0$, then

$$\left((S)\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \leq \min\left\{f(a)\sqrt[r]{m}, (b-a)^{\frac{1}{u}}\right\}.$$

Case (iii). If $(g\circ f)^r(b) - (g\circ f)^r(a) < 0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \le \beta \wedge \left(\left(\left(\left(\frac{g^r(\beta) - (g\circ f)^r(a)}{(g\circ f)^r(b) - (g\circ f)^r(a)}\right)\left(b^r - a^r\right) + a^r\right)^{1/r} - a\right)^{\frac{1}{u}},$$

where $\beta = \frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}\left(\frac{(\mathrm{S})\int_a^b f^v(x)\,d\mu}{b-a}\right)^{\frac{1}{v}}$.

Remark 3.16 Let $\alpha = m = 1$, $r \in \mathbb{R}$, $r \neq 0$, $g = \mathrm{id}$, $f : [a,b] \to [0,\infty)$ be an r-mean convex function, and μ be the Lebesgue measure on $\mathbb{R}$. Then:

Case (i). If $f^r(b) - f^r(a) > 0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \le \beta \wedge \left(b - \left(\left(\frac{\beta^r - f^r(a)}{f^r(b) - f^r(a)}\right)\left(b^r - a^r\right) + a^r\right)^{1/r}\right)^{\frac{1}{u}},$$

where $\beta = \frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}\left(\frac{(\mathrm{S})\int_a^b f^v(x)\,d\mu}{b-a}\right)^{\frac{1}{v}}$.

Case (ii). If $f^r(b) - f^r(a) = 0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \le \min\left\{f(a)\sqrt[r]{m}, (b-a)^{\frac{1}{u}}\right\}.$$

Case (iii). If $f^r(b) - f^r(a) < 0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \le \beta \wedge \left(\left(\left(\left(\frac{\beta^r - f^r(a)}{f^r(b) - f^r(a)}\right)\left(b^r - a^r\right) + a^r\right)^{1/r} - a\right)^{\frac{1}{u}},$$

where $\beta = \frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}\left(\frac{(\mathrm{S})\int_a^b f^v(x)\,d\mu}{b-a}\right)^{\frac{1}{v}}$.

Remark 3.17 Let $(\alpha, m) \in [0,1]^2$, $r = 1$, g be a continuous and monotonous function, $f : [a,b] \to [0,\infty)$ be an $(\alpha, m)_g$-convex function, and μ be the Lebesgue measure on $\mathbb{R}$. Then:

Case (i). If $(g\circ f)(b) - m(g\circ f)(a) > 0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \le \beta \wedge \left(\left(1 - \left(\frac{g(\beta) - m(g\circ f)(a)}{(g\circ f)(b) - m(g\circ f)(a)}\right)^{1/\alpha}\right)(b - ma)\right)^{\frac{1}{u}},$$

where $\beta = \frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}\left(\frac{(\mathrm{S})\int_a^b f^v(x)\,d\mu}{b-a}\right)^{\frac{1}{v}}$.

Case (ii). If $(g\circ f)(b) - m(g\circ f)(a) = 0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \le \min\left\{f(a)\sqrt[r]{m}, (b-a)^{\frac{1}{u}}\right\}.$$

Case (iii). If $(g\circ f)(b) - m(g\circ f)(a) < 0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \le \beta \wedge \left(\left(\left(\frac{g(\beta) - m(g\circ f)(a)}{(g\circ f)(b) - m(g\circ f)(a)}\right)^{1/\alpha}(b - ma) + ma\right) - a\right)^{\frac{1}{u}},$$

where $\beta = \frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}\left(\frac{(\mathrm{S})\int_a^b f^v(x)\,d\mu}{b-a}\right)^{\frac{1}{v}}$.

Remark 3.18 Let $(\alpha, m) \in [0,1]^2$, $r = 1$, $g = \mathrm{id}$, $f : [a,b] \to [0,\infty)$ be an (α, m)-convex function, and μ be the Lebesgue measure on $\mathbb{R}$. Then:

Case (i). If $f(b) - mf(a) > 0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \le \beta \wedge \left(\left(1 - \left(\frac{\beta - mf(a)}{f(b) - mf(a)}\right)^{1/\alpha}\right)(b - ma)\right)^{\frac{1}{u}},$$

where $\beta = \frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}(\frac{(\mathrm{S})\int_a^b f^v(x)\,d\mu}{b-a})^{\frac{1}{v}}$.

Case (ii). If $f(b) - mf(a) = 0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \le \min\left\{f(a)\sqrt[r]{m}, (b-a)^{\frac{1}{u}}\right\}.$$

Case (iii). If $f(b) - mf(a) < 0$, then

$$\left((\mathrm{S})\int_a^b f^u(x)\,d\mu\right)^{\frac{1}{u}} \le \beta \wedge \left(\left(\left(\frac{\beta - mf(a)}{f(b) - mf(a)}\right)^{1/\alpha}(b - ma) + ma\right) - a\right)^{\frac{1}{u}},$$

where $\beta = \frac{(b-a)^{\frac{1}{u}}(1+v)^{\frac{1}{v}}}{(1+u)^{\frac{1}{u}}}(\frac{(\mathrm{S})\int_a^b f^v(x)\,d\mu}{b-a})^{\frac{1}{v}}$.

Remark 3.19 Let $g = \mathrm{id}$ and $\alpha = m = r = 1$ in Remark 3.12. Then we obtain the Berwald inequalities for the fuzzy integral of convex functions [32].

4 Conclusion

In this paper, we have discussed the Berwald inequalities for the Sugeno integral based on $(\alpha, m, r)_g$-concave functions. We have provided the reverse inequalities as well. As open problems for future research, it would be interesting to explore Berwald inequalities for other generalizations of the fuzzy integral. We will investigate these problems in the future.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this paper, and they read and approved the final manuscript.

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