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# Optimal power mean bounds for the second Yang mean

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**Abstract**

In this paper, we present the best possible parameters $p$ and $q$ such that the double inequality

$$M_p(a,b) < V(a,b) < M_q(a,b)$$

holds for all $a,b>0$ with $a\neq b$, where $M_r(a,b)=[(a^r+b^r)/2]^{1/r}$ $(r\neq 0)$ and $M_0(a,b)=\sqrt{ab}$ is the $r$th power mean and $V(a,b)=(a-b)/[\sqrt{2}\sinh^{-1}((a-b)/\sqrt{2ab})]$ is the second Yang mean.

**MSC:** 26E60

**Keywords:** power mean; second Yang mean; arithmetic mean; quadratic mean; geometric mean; Lehmer mean; first Seiffert mean; logarithmic mean

## 1 Introduction

For $r\in\mathbb{R}$, the $r$th power mean $M_r(a,b)$ of two distinct positive real numbers $a$ and $b$ is defined by

$$M_r(a,b)=\begin{cases}(\frac{a^r+b^r}{2})^{1/r}, & r\neq 0,\\ \sqrt{ab}, & r=0.\end{cases} \tag{1.1}$$

It is well known that $M_r(a,b)$ is continuous and strictly increasing with respect to $r\in\mathbb{R}$ for fixed $a,b>0$ with $a\neq b$. Many classical means are special cases of the power mean, for example, $M_{-1}(a,b)=2ab/(a+b)=H(a,b)$ is the harmonic mean, $M_0(a,b)=\sqrt{ab}=G(a,b)$ is the geometric mean, $M_1(a,b)=(a+b)/2=A(a,b)$ is the arithmetic mean, and $M_2(a,b)=\sqrt{(a^2+b^2)/2}=Q(a,b)$ is the quadratic mean. The main properties for the power mean are given in [1].

Let

$$L(a,b)=\frac{a-b}{\log a-\log b},\qquad I(a,b)=\frac{1}{e}\left(\frac{a^a}{b^b}\right)^{1/(a-b)},\qquad P(a,b)=\frac{a-b}{2\arcsin(\frac{a-b}{a+b})},$$

$$U(a,b)=\frac{a-b}{\sqrt{2}\arctan(\frac{a-b}{\sqrt{2ab}})},\qquad T^*(a,b)=\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\,d\theta,$$

$$NS(a,b) = \frac{a-b}{2\sinh^{-1}(\frac{a-b}{a+b})}, \qquad X(a,b) = A(a,b)e^{G(a,b)/P(a,b)-1},$$
$$T(a,b) = \frac{a-b}{2\arctan(\frac{a-b}{a+b})}, \qquad B(a,b) = Q(a,b)e^{A(a,b)/T(a,b)-1},$$

and

$$V(a,b) = \frac{a-b}{\sqrt{2}\sinh^{-1}(\frac{a-b}{\sqrt{2ab}})} \tag{1.2}$$

be, respectively, the logarithmic mean, identric mean, first Seiffert mean [2], first Yang mean [3], Toader mean [4], Neuman-Sándor mean [5, 6], Sándor mean [7], second Seiffert mean [8], Sándor-Yang mean [3], and second Yang mean [3] of two distinct positive real numbers $a$ and $b$, where $\sinh^{-1}(x) = \log(x + \sqrt{x^2+1})$ is the inverse hyperbolic sine function.

Recently, the bounds for certain bivariate means in terms of the power mean have attracted the attention of many mathematicians. Radó [9] (see also [10–12]) proved that the double inequalities

$$\begin{aligned} &M_p(a,b) < L(a,b) < M_q(a,b), \\ &M_\lambda(a,b) < I(a,b) < M_\mu(a,b) \end{aligned} \tag{1.3}$$

hold for all $a,b>0$ with $a \neq b$ if and only if $p \leq 0$, $q \geq 1/3$, $\lambda \leq 2/3$, and $\mu \geq \log 2$.

In [13–16], the authors proved that the double inequality

$$M_p(a,b) < T^*(a,b) < M_q(a,b)$$

holds for all $a,b>0$ with $a \neq b$ if and only if $p \leq 3/2$ and $q \geq \log 2/(\log\pi - \log 2)$.

Jagers [17], Hästö [18, 19], Yang [20], and Costin and Toader [21] proved that $p_1 = \log 2/\log\pi$, $q_1 = 2/3$, $p_2 = \log 2/(\log\pi - \log 2)$, and $q_2 = 5/3$ are the best possible parameters such that the double inequalities

$$\begin{aligned} &M_{p_1}(a,b) < P(a,b) < M_{q_1}(a,b), \\ &M_{p_2}(a,b) < T(a,b) < M_{q_2}(a,b) \end{aligned} \tag{1.4}$$

hold for all $a,b>0$ with $a \neq b$.

In [21–26], the authors proved that the double inequalities

$$\begin{aligned} &M_{\lambda_1}(a,b) < NS(a,b) < M_{\mu_1}(a,b), \\ &M_{\lambda_2}(a,b) < U(a,b) < M_{\mu_2}(a,b), \\ &M_{\lambda_3}(a,b) < X(a,b) < M_{\mu_3}(a,b), \end{aligned}$$

hold for all $a,b>0$ with $a \neq b$ if and only if $\lambda_1 \leq \log 2/\log[2\log(1+\sqrt{2})]$, $\mu_1 \geq 4/3$, $\lambda_2 \leq 2\log 2/(2\log\pi - \log 2)$, $\mu_2 \geq 4/3$, $\lambda_3 \leq 1/3$, and $\mu_3 \geq \log 2/(1+\log 2)$.

Very recently, Yang and Chu [27] showed that $p = 4\log 2/(4 + 2\log 2 - \pi)$ and $q = 4/3$ are the best possible parameters such that the double inequality

$$M_p(a,b) < B(a,b) < M_q(a,b)$$

holds for all $a, b > 0$ with $a \neq b$.

The main purpose of this paper is to present the best possible parameters $p$ and $q$ such that the double inequality

$$M_p(a,b) < V(a,b) < M_q(a,b)$$

holds for all $a, b > 0$ with $a \neq b$.

## 2 Lemmas

In order to prove our main results we need three lemmas, which we present in this section.

**Lemma 2.1** *Let $t > 0$, $p \in \mathbb{R}$, and*

$$\begin{aligned} f(t,p) = {} & 2\sinh\bigl[2(p-1)t\bigr] + \sinh\bigl[2(p+1)t\bigr] + \sinh\bigl[2(p-2)t\bigr] \\ & + p\sinh(4t) - \sinh(2t). \end{aligned} \tag{2.1}$$

*Then the following statements are true*:

(i) $f(t,p) > 0$ *for all* $t > 0$ *if and only if* $p \geq 2/3$;

(ii) $f(t,p) < 0$ *for all* $t > 0$ *if and only if* $p \leq 0$.

*Proof* It follows from (2.1) that

$$\begin{aligned} \frac{\partial f(t,p)}{\partial t} = {} & \sinh(4t) + 4t\cosh\bigl[2(p-1)t\bigr] + 2t\cosh\bigl[2(p+1)t\bigr] + 2t\cosh\bigl[2(p-2)t\bigr] \\ & > 0 \end{aligned} \tag{2.2}$$

for all $t > 0$ and $p \in \mathbb{R}$.

(i) If $f(t,p) > 0$ for all $t > 0$, then (2.1) leads to

$$\lim_{t\to 0^+} \frac{f(t,p)}{t} = 12\left(p - \frac{2}{3}\right) \geq 0,$$

which gives $p \geq 2/3$.

If $p \geq 2/3$, then (2.1) and (2.2) lead to the conclusion that

$$\begin{aligned} f(t,p) \geq f\left(t, \frac{2}{3}\right) &= \frac{2}{3}\sinh(4t) - \sinh(2t) - 2\sinh\left(\frac{2}{3}t\right) - \sinh\left(\frac{8}{3}t\right) + \sinh\left(\frac{10}{3}t\right) \\ &= \frac{8}{3}\sinh^3\left(\frac{2}{3}t\right)\cosh\left(\frac{2}{3}t\right)\left[8\cosh^2\left(\frac{2}{3}t\right) + 6\cosh\left(\frac{2}{3}t\right) - 3\right] > 0 \end{aligned}$$

for all $t > 0$.

(ii) If $f(t,p) < 0$ for all $t > 0$, then from part (i) we know that $p < 2/3$. We assert that $p \leq 0$, otherwise $0 < p < 2/3$ and (2.1) leads to

$$\begin{aligned}
&\lim_{t\to+\infty} \frac{f(t,p)}{e^{4t}} \\
&= \lim_{t\to+\infty} \frac{-2\sinh[2(1-p)t] + \sinh[2(1+p)t] - \sinh[2(2-p)t] + p\sinh(4t) - \sinh(2t)}{e^{4t}} \\
&= \frac{p}{2} > 0,
\end{aligned}$$

which contradicts with $f(t,p) < 0$ for all $t > 0$.

If $p \leq 0$, then from (2.1) and (2.2) we have

$$f(t,p) \leq f(t,0) = -2\sinh(2t) - \sinh(4t) < 0$$

for all $t > 0$. □

**Lemma 2.2** *The double inequality*

$$\bigl[\cosh(pt)\bigr]^{1/p} < \frac{\sqrt{2}\sinh(t)}{\sinh^{-1}[\sqrt{2}\sinh(t)]} < \bigl[\cosh(qt)\bigr]^{1/q} \tag{2.3}$$

*holds for all $t > 0$ if and only if $p \leq 0$ and $q \geq 2/3$. Here*

$$\bigl[\cosh(pt)\bigr]^{1/p}\big|_{p=0} := \lim_{p\to 0}\bigl[\cosh(pt)\bigr]^{1/p}.$$

*Proof* Let $t > 0$, $p \in \mathbb{R}$ and $F(t,p)$ be defined by

$$F(t,p) = \log\left[\frac{\sqrt{2}\sinh(t)}{\sinh^{-1}(\sqrt{2}\sinh(t))}\right] - \frac{1}{p}\log\bigl[\cosh(pt)\bigr]. \tag{2.4}$$

Then making use of the power series formulas

$$\begin{aligned}
\sinh(t) &= t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}, \\
\cosh(t) &= 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}, \\
\sinh^{-1}(t) &= t - \frac{1}{2} \times \frac{t^3}{3} + \frac{1\times 3}{2\times 4} \times \frac{t^5}{5} - \frac{1\times 3\times 5}{2\times 4\times 6} \times \frac{t^7}{7} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!t^{2n+1}}{2^{2n}(n!)^2(2n+1)}
\end{aligned}$$

we get

$$\log\left[\frac{\sqrt{2}\sinh(t)}{\sinh^{-1}(\sqrt{2}\sinh(t))}\right] = \frac{t^2}{3} + o\bigl(t^2\bigr), \qquad \frac{1}{p}\log\bigl[\cosh(pt)\bigr] = -\frac{1}{2}pt^2 + o\bigl(t^2\bigr) \tag{2.5}$$

for $t \to 0^+$.

It follows from (2.4) and (2.5) that

$$F(0^+,p)=0, \tag{2.6}$$

$$\frac{\partial F(t,p)}{\partial t}=\frac{\cosh[(p-1)t]}{\sinh(t)\cosh(pt)\sinh^{-1}[\sqrt{2}\sinh(t)]}f_1(t,p), \tag{2.7}$$

where

$$f_1(t,p)=\sinh^{-1}\left[\sqrt{2}\sinh(t)\right]-\frac{\sqrt{2}\sinh(t)\cosh(pt)\cosh(t)}{\sqrt{\cosh(2t)}\cosh[(p-1)t]}, \tag{2.8}$$

$$f_1(0,p)=0, \tag{2.9}$$

$$\frac{\partial f_1(t,p)}{\partial t}=-\frac{\sqrt{2}\sinh(t)}{4[\cosh(2t)]^{3/2}\cosh^2[(p-1)t]}f(t,p), \tag{2.10}$$

where $f(t,p)$ is defined by Lemma 2.1.

$$\lim_{t\to 0}\frac{F(t,p)}{t^2}=-\frac{1}{2}\left(p-\frac{2}{3}\right) \tag{2.11}$$

and

$$\lim_{t\to+\infty}F(t,p)=-\infty \tag{2.12}$$

if $p>0$.

We first prove that the inequality

$$\frac{\sqrt{2}\sinh(t)}{\sinh^{-1}[\sqrt{2}\sinh(t)]}<\left[\cosh(pt)\right]^{1/p} \tag{2.13}$$

holds for all $t>0$ if and only if $p\geq 2/3$.

If $p\geq 2/3$, then inequality (2.13) holds for all $t>0$ follows easily from Lemma 2.1(i), (2.4), (2.6), (2.7), (2.9), and (2.10).

If inequality (2.13) holds for all $t>0$, then (2.4) and (2.11) lead to $p\geq 2/3$.

Next, we prove that the inequality

$$\frac{\sqrt{2}\sinh(t)}{\sinh^{-1}[\sqrt{2}\sinh(t)]}>\left[\cosh(pt)\right]^{1/p} \tag{2.14}$$

holds for all $t>0$ if and only if $p\leq 0$.

If $p\leq 0$, then that inequality (2.14) holds for all $t>0$ follows easily from Lemma 2.1(ii), (2.4), (2.6), (2.7), (2.9), and (2.10).

If inequality (2.14) holds for all $t>0$, then (2.4) leads to $F(t,p)>0$. We assert that $p\leq 0$, otherwise $p>0$ and (2.12) implies that there exists large enough $T_0>0$ such that $F(t,p)<0$ for $t\in(T_0,\infty)$. □

**Lemma 2.3** *Let $t>0$, $p\in\mathbb{R}$, and $f_1(t,p)$ be defined by* (2.8). *Then the following statements are true*:

(i) $f_1(t,p) < 0$ *for all* $t > 0$ *if and only if* $p \geq 2/3$;
(ii) $f(t,p) > 0$ *for all* $t > 0$ *if and only if* $p \leq 0$.

*Proof* (i) If $p \geq 2/3$, then $f_1(t,p) < 0$ for all $t > 0$ follows easily from (2.9) and (2.10) together with Lemma 2.1(i).

If $f_1(t,p) < 0$ for all $t > 0$, then (2.8) leads to

$$\lim_{t\to 0}\frac{f_1(t,p)}{t^3} = \frac{-\sqrt{2}(p-\frac{2}{3})t^3 + o(t^3)}{t^3} = -\sqrt{2}\left(p-\frac{2}{3}\right) \leq 0,$$

which gives $p \geq 2/3$.

(ii) If $p \leq 0$, then $f_1(t,p) > 0$ for all $t > 0$ follows easily from (2.9) and (2.10) together with Lemma 2.1(ii).

Note that

$$\begin{aligned}
&\frac{f_1(t,p)}{e^{(|p|-|p-1|)t}\sinh(t)} \\
&\quad= \frac{\sinh^{-1}[\sqrt{2}\sinh(t)]}{e^{(|p|-|p-1|)t}\sinh(t)} - \frac{\sqrt{2}\cosh(t)\cosh(pt)}{e^{(|p|-|p-1|)t}\cosh[(p-1)t]\sqrt{\cosh(2t)}} \\
&\quad= \frac{\log[\sqrt{2}\sinh(t)+\sqrt{\cosh(2t)}]}{e^{(|p|-|p-1|)t}\sinh(t)} - \frac{\sqrt{2}(1+e^{-2|p|t})\cosh(t)}{(1+e^{-2|p-1|t})\sqrt{\cosh(2t)}}. \qquad (2.15)
\end{aligned}$$

If $f_1(t,p) > 0$ for all $t > 0$, then

$$\lim_{t\to+\infty}\frac{f_1(t,p)}{e^{(|p|-|p-1|)t}\sinh(t)} \geq 0$$

and we assert that $p \leq 0$. Otherwise, equation (2.15) leads to

$$\lim_{t\to+\infty}\frac{f_1(t,p)}{e^{(|p|-|p-1|)t}\sinh(t)} = -\frac{\sqrt{2}}{2} < 0$$

if $p = 1$ and

$$\lim_{t\to+\infty}\frac{f_1(t,p)}{e^{(|p|-|p-1|)t}\sinh(t)} = -\sqrt{2} < 0$$

if $p \in (0,1) \cup (1,\infty)$. □

## 3 Main results

**Theorem 3.1** *The double inequality*

$$M_p(a,b) < V(a,b) < M_q(a,b)$$

*holds for all* $a, b > 0$ *with* $a \neq b$ *if and only if* $p \leq 0$ *and* $q \geq 2/3$.

*Proof* Since both $M_r(a,b)$ and $V(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b > 0$. Let $t = \frac{1}{2}\log(a/b) > 0$ and $r \in \mathbb{R}$, then (1.1) and (1.2) lead to

$$V(a,b)=\sqrt{ab}V\left(\sqrt{\frac{a}{b}},\sqrt{\frac{b}{a}}\right)=\sqrt{ab}V\left(e^{t},e^{-t}\right)=\frac{\sqrt{2ab}\sinh(t)}{\sinh^{-1}[\sqrt{2}\sinh(t)]} \tag{3.1}$$

and

$$M_r(a,b)=\sqrt{ab}M_r\left(\sqrt{\frac{a}{b}},\sqrt{\frac{b}{a}}\right)=\sqrt{ab}M_r\left(e^{t},e^{-t}\right)=\sqrt{ab}\big[\cosh(rt)\big]^{1/r}. \tag{3.2}$$

Therefore, Theorem 3.1 follows easily from (3.1) and (3.2) together with Lemma 2.2. □

**Theorem 3.2** *The double inequality*

$$\frac{a^{p-1}+b^{p-1}}{a^p+b^p}\frac{ab\sqrt{2(a^2+b^2)}}{a+b}<V(a,b)<\frac{a^{q-1}+b^{q-1}}{a^q+b^q}\frac{ab\sqrt{2(a^2+b^2)}}{a+b}$$

*holds for all $a,b>0$ with $a\neq b$ if and only if $p\geq 2/3$ and $q\leq 0$.*

*Proof* Without loss of generality, we assume that $a>b>0$. Let $t=\frac{1}{2}\log(a/b)>0$ and $r\in\mathbb{R}$, then

$$\frac{a^{r-1}+b^{r-1}}{a^r+b^r}\frac{ab\sqrt{2(a^2+b^2)}}{a+b}=\frac{\sqrt{ab}\cosh[(r-1)t]\sqrt{\cosh(2t)}}{\cosh(t)\cosh(rt)}. \tag{3.3}$$

Therefore, Theorem 3.2 follows easily from (3.1) and (3.3) together with Lemma 2.3. □

Let $p\in\mathbb{R}$ and $a,b>0$. Then the $p$th Lehmer mean [28] $L_p(a,b)=\frac{a^{p+1}+b^{p+1}}{a^p+b^p}$ is strictly increasing with respect to $p\in\mathbb{R}$ for fixed $a,b>0$ with $a\neq b$. From Theorem 3.2 we get Corollary 3.3 immediately.

**Corollary 3.3** *The double inequality*

$$\frac{Q(a,b)G^2(a,b)}{A(a,b)L_{p-1}(a,b)}<V(a,b)<\frac{Q(a,b)G^2(a,b)}{A(a,b)L_{q-1}(a,b)}$$

*holds for all $a,b>0$ with $a\neq b$ if and only if $p\geq 2/3$ and $q\leq 0$.*

Let $p=2/3,1,2,+\infty$ and $q=0,-1/2,-1,-2,-\infty$. Then Corollary 3.3 leads to

**Corollary 3.4** *The inequalities*

$$\begin{aligned}\min\{a,b\}\frac{Q(a,b)}{A(a,b)}&<\frac{G^2(a,b)}{Q(a,b)}<\frac{G^2(a,b)Q(a,b)}{A^2(a,b)}\\&<\frac{Q(a,b)G^{4/3}(a,b)M_{1/3}^{1/3}(a,b)}{A(a,b)M_{2/3}^{2/3}(a,b)}<V(a,b)<Q(a,b)\\&<\frac{Q(a,b)[2A(a,b)-G(a,b)]}{A(a,b)}\\&<\frac{Q^3(a,b)}{A^2(a,b)}<\frac{2Q^2(a,b)-G^2(a,b)}{Q(a,b)}<\max\{a,b\}\frac{Q(a,b)}{A(a,b)}\end{aligned}$$

*hold for all $a,b>0$ with $a\neq b$.*

From (1.3), (1.4), and Theorem 3.1 we clearly see that $M_{2/3}(a,b)$ is the sharp upper power mean bound for the 2-order generalized logarithmic mean $L^{1/2}(a^2,b^2)$, the first Seiffert mean $P(a,b)$, and the second Yang mean $V(a,b)$. In [29], Theorem 3, Yang and Chu proved that the inequality

$$P(a,b) > L^{1/r}\left(a^r,b^r\right) \tag{3.4}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $r\leq 2$.

As a result of comparing $V(a,b)$ with $L^{1/2}(a^2,b^2)$, we have the following.

**Theorem 3.5** *The inequality*

$$V(a,b) < L^{1/2}\left(a^2,b^2\right)$$

*holds for all $a,b>0$ with $a\neq b$.*

*Proof* We assume that $a>b$. Let $t=\frac{1}{2}\log(a/b)>0$, then

$$L^{1/2}\left(a^2,b^2\right)=\left(\frac{a^2-b^2}{2(\log a-\log b)}\right)^{1/2}=\sqrt{ab}\sqrt{\frac{\sinh(2t)}{2t}}. \tag{3.5}$$

It follows from (3.1) and (3.5) that

$$\begin{aligned} &L^{1/2}\left(a^2,b^2\right)-V(a,b)\\ &\quad=\frac{\sqrt{ab}\sqrt{\sinh(2t)}}{\sqrt{2t}\sinh^{-1}(\sqrt{2}\sinh(t))}\left[\sinh^{-1}\left(\sqrt{2}\sinh(t)\right)-\sqrt{2t}\tanh(t)\right]. \end{aligned} \tag{3.6}$$

Let

$$g(t)=\sinh^{-1}\left(\sqrt{2}\sinh(t)\right)-\sqrt{2t}\tanh(t). \tag{3.7}$$

Then simple computation leads to

$$g(0)=0, \tag{3.8}$$

$$g'(t)=\sqrt{2}\left(\frac{\cosh(t)}{\sqrt{\cosh(2t)}}-\frac{t+\sinh(t)\cosh(t)}{2\cosh^2(t)\sqrt{t\tanh(t)}}\right), \tag{3.9}$$

$$\begin{aligned} &\left(\frac{\cosh(t)}{\sqrt{\cosh(2t)}}\right)^2-\left(\frac{t+\sinh(t)\cosh(t)}{2\cosh^2(t)\sqrt{t\tanh(t)}}\right)^2\\ &\quad=\frac{\cosh^2(t)}{\cosh(2t)}-\frac{(t+\sinh(t)\cosh(t))^2}{4t\sinh(t)\cosh^3(t)}\\ &\quad=\frac{(2t\cosh(2t)-\sinh(2t))(\sinh(2t)\cosh(2t)-2t)}{16t\sinh(t)\cosh(2t)\cosh^3(t)}\\ &\quad=\frac{\sinh(4t)-4t}{16t\sinh(t)\cosh(2t)\cosh^3(t)}\left(\cosh(2t)-\frac{\sinh(2t)}{2t}\right)>0 \end{aligned} \tag{3.10}$$

for $t>0$.

Therefore, Theorem 3.5 follows easily from (3.6)-(3.10). □

**Remark 3.6** From (1.4), (3.4), Theorems 3.1, and 3.5 we get the inequalities

$$M_0(a,b) < V(a,b) < L^{1/2}(a^2,b^2) < P(a,b) < M_{2/3}(a,b)$$

for all $a, b > 0$ with $a \neq b$.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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