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Duality for nonsmooth mathematical programming problems with equilibrium constraints

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Abstract

In this paper, we consider the mathematical programs with equilibrium constraints (MPECs) in Banach space. The objective function and functions in the constraint part are assumed to be lower semicontinuous. We study the Wolfe-type dual problem for the MPEC under the convexity assumption. A Mond-Weir-type dual problem is also formulated and studied for the MPEC under convexity and generalized convexity assumptions. Conditions for weak duality theorems are given to relate the MPEC and two dual programs in Banach space, respectively. Also conditions for strong duality theorems are established in an Asplund space.

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Keywords: mathematical programming problems with equilibrium constraints; Wolfe-type dual; Mond-Weir dual; convexity; nonsmooth analysis

1 Introduction

Luo *et al.* [1] presented a comprehensive study of MPEC. Flegel and Kanzow [2] obtained short and elementary proof of the optimality conditions for MPEC using the standard Fritz-John conditions. Further, Flegel and Kanzow [3] introduced a new Abadie-type constraint qualification and a new Slater-type constraint qualification for the MPEC and proved that new Slater-type CQ implies new Abadie-type CQ. Ye [4] considered MPEC and introduced various stationary conditions and established that it is sufficient for being globally or locally optimal under some generalized convexity assumption and obtained new constraint qualifications.

Outrata *et al.* [5] derived necessary optimality conditions for those MPECs which can be treated by the implicit programming approach and proposed a solution method based on the bundle technique of nonsmooth optimization. Flegel *et al.* [6] considered optimization problems with a disjunctive structure of the feasible set and obtained optimality conditions for disjunctive programs with application to MPEC using Guignard-type constraint qualifications. Movahedian and Nobakhtian [7] introduced nonsmooth strong stationarity, M-stationarity and generalized the Abadie and Guignard-type constraint qualifications for nonsmooth MPEC. Movahedian and Nobakhtian [8] introduced a nonsmooth type of the M-stationary condition based on the Michel-Penot subdifferential and established the Fritz-John-type, Kuhn-Tucker-type M-stationary necessary conditions for the

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nonsmooth MPEC. Further, Movahedian and Nobakhtian [9] established necessary optimality conditions for Lipschitz MPEC on Asplund space and sufficient optimality conditions for nonsmooth MPEC in Banach space. We refer to the recent results of Ardali *et al.* [10], Chieu and Lee [11], Guo and Lin [12], Guo *et al.* [13, 14] and Ye and Zhang [15], and the references therein for more details related to the MPEC.

Following Luo *et al.* [1] and Movahedian and Nobakhtian [9], we consider the following mathematical programming problem with equilibrium constraints (MPEC):

(MPEC) $\min f(z)$

subject to: $g(z) \leq 0$, $h(z) = 0$,

$$G(z) \geq 0, \qquad H(z) \geq 0, \qquad \langle G(z), H(z) \rangle = 0,$$

where X is a Banach space, $f : X \to \mathbb{R}$ is a lower semi-continuous (lsc) function, $g : X \to \mathbb{R}^k$, $h : X \to \mathbb{R}^p$, $G : X \to \mathbb{R}^l$ and $H : X \to \mathbb{R}^l$ are functions with lsc components.

The use of equilibrium constraints in modeling process engineering problems is a relatively new and exciting field of research; see Raghunathan and Biegler [16]. Hydroeconomic river basin models (HERBM) based on mathematical programming are conventionally formulated as explicit aggregate optimization problems with a single, aggregate objective function. Britz *et al.* [17] proposed a new solution format for hydroeconomic river basin models, based on a multiobjective optimization problem with equilibrium constraints, which allowed, *inter alia*, to express spatial externalities resulting from asymmetric access to water use.

Wolfe [18] formulated a dual program for a nonlinear programming problem. Motivated by a specific problem, namely the mathematical description of the rotating heavy chain, Toland [19, 20] introduced the notion of duality and established the duality theory for nonconvex optimization problems. Rockafellar [21, 22] studied fundamental duality theory for convex programs using a conjugate function and established a generalized version of the Fenchelís duality theorem. In the last four decades there has been an extensive interest in the duality theory of nonlinear programming problems; see Mangasarian [23] and Mishra and Giorgi [24].

To the best of our knowledge, the dual problem to a nonsmooth MPEC has not been given in the literature as yet.

In this paper, we introduce Wolfe-type and Mond-Weir-type dual programs to the nonsmooth MPEC. We have established weak and strong duality theorems relating the nonsmooth MPEC and the two dual programs. The paper is organized as follows: in Section 2, we give some preliminaries, definitions, and results. In Section 3, we derive weak and strong duality theorems relating to the nonsmooth MPEC and the two dual models under convexity and generalized convexity assumptions.

2 Preliminaries

In this section, we give some notations, basic definitions, and preliminary results, which will be used later in the paper.

The Clarke-Rockafellar subdifferential of f is defined by

$$\partial_c f(x) = \left\{ x^* \in X^* : \langle x^*, v \rangle \leq f^\uparrow(x; v), \forall v \in X \right\},$$

where

$$f^\uparrow(x;v) = \sup_{\epsilon>0} \inf_{\substack{\gamma>0 \\ \delta>0 \\ \lambda>0}} \sup_{\substack{y\in B(x;\gamma) \\ f(y)\leq f(x)+\delta \\ t\in(0,\lambda)}} \inf_{w\in B(v;\epsilon)} \frac{f(y+tw)-f(y)}{t}$$

is the Clarke-Rockafellar directional derivative.

Definition 2.1 (Rockafellar [25]) The lsc function $f : X \to \mathbb{R} \cup \{+\infty\}$ is directionally Lipschitzian at $\bar{x}$ if for some $y \in X$,

$$\limsup_{\substack{x'\xrightarrow{f}\bar{x} \\ t\downarrow 0}} \sup_{y'\to y} \frac{f(x'+ty')-f(x')}{t} < \infty.$$

The function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be radially nonconstant (rnc) if $\forall x, y \in X$, $\exists z \in (x, y)$, with $f(z) \neq f(x)$, *i.e.*, one cannot find any line segment on which f is constant.

Definition 2.2 (Avriel *et al.* [26]) The lsc function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be a quasiconvex function, if for any $x, y \in X$, one has

$$f(z) \leq \max\{f(x), f(y)\}, \quad \forall x, y \in X, z \in [x, y],$$

where $[x, y] = \{x + t(y - x) : t \in (0, 1)\}$.

Definition 2.3 (Clarke [27]) The lsc function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be a convex function at $\bar{x} \in X$, if, for all $x \in X$,

$$f(x) \geq f(\bar{x}) + \langle \xi, x - \bar{x} \rangle, \quad \forall \xi \in \partial_c f(\bar{x}).$$

Definition 2.4 (Aussel [28]) The lsc function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be pseudoconvex function at $\bar{x} \in X$, if, for all $x \in X$,

$$\langle \xi, x - \bar{x} \rangle \geq 0, \quad \text{for some } \xi \in \partial_c f(\bar{x}) \quad \Rightarrow \quad f(x) \geq f(\bar{x}),$$

$$f(x) < f(\bar{x}) \quad \Rightarrow \quad \langle \xi, x - \bar{x} \rangle < 0, \quad \forall \xi \in \partial_c f(\bar{x}).$$

Theorem 2.5 (Aussel [28]) *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be lsc, quasiconvex and rnc on a convex open set $U \subset X$. Moreover, assume that f is finite at $\bar{x} \in U$ and $f^\uparrow(\bar{x};0) > -\infty$. Then, for each $x \in U$,*

$$f(x) \leq f(\bar{x}) \quad \Rightarrow \quad \forall \xi \in \partial_c f(\bar{x}) : \langle \xi, x - \bar{x} \rangle \leq 0.$$

Given a feasible vector $\bar{z}$ for the MPEC, we define the following index sets:

$$I_g := I_g(\bar{z}) := \{i = 1, 2, \ldots, k : g_i(\bar{z}) = 0\},$$

$$\alpha := \alpha(\bar{z}) = \{i = 1, 2, \ldots, l : G_i(\bar{z}) = 0, H_i(\bar{z}) > 0\},$$

$$\beta := \beta(\bar{z}) = \{i = 1, 2, \ldots, l : G_i(\bar{z}) = 0, H_i(\bar{z}) = 0\},$$

$$\gamma := \gamma(\bar{z}) = \{i = 1, 2, \ldots, l : G_i(\bar{z}) > 0, H_i(\bar{z}) = 0\}.$$

The set β is known as a degenerate set. If β is empty, the vector $\bar{z}$ is said to satisfy the strict complementarity condition. Movahedian and Nobakhtian [8] introduced a nonsmooth type of M-stationary via the Michel-Penot subdifferential for finite-dimensional spaces. Further, Movahedian and Nobakhtian [9] extend the M-stationary notion to nonsmooth MPEC in terms of the Clarke-Rockafellar subdifferential in Banach spaces. The following definition of the M-stationary point for the nonsmooth MPEC is taken from Definition 3.1 in Movahedian and Nobakhtian [9].

Definition 2.6 A feasible point $\bar{z}$ of MPEC is called the Mordukhovich stationary point if there exists $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{k+p+2l}$, such that the following conditions hold:

$$0 \in \partial_c f(\bar{z}) + \sum_{i \in I_g} \lambda_i^g \partial_c g_i(\bar{z}) + \sum_{i=1}^p \lambda_i^h \partial_c h_i(\bar{z}) - \sum_{i=1}^l [\lambda_i^G \partial_c G_i(\bar{z}) + \lambda_i^H \partial_c H_i(\bar{z})],$$

$$\lambda_{I_g}^g \geq 0, \qquad \lambda_\gamma^G = 0, \qquad \lambda_\alpha^H = 0, \quad \text{either } \lambda_i^G > 0, \lambda_i^H > 0 \text{ or } \lambda_i^G \lambda_i^H = 0, \forall i \in \beta.$$

The following definition of the no nonzero abnormal multiplier constraint qualification for MPEC is taken from Definition 3.3 in Movahedian and Nobakhtian [9].

Definition 2.7 Let $\bar{z}$ be a feasible point of MPEC. We say that the No Nonzero Abnormal Multiplier Constraint Qualification (NNAMCQ) is satisfied at $\bar{z}$ if there is no nonzero vector $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{k+p+2l}$, such that

$$0 \in \sum_{i \in I_g} \lambda_i^g \partial_c g_i(\bar{z}) + \sum_{i=1}^p \lambda_i^h \partial_c h_i(\bar{z}) - \sum_{i=1}^l [\lambda_i^G \partial_c G_i(\bar{z}) + \lambda_i^H \partial_c H_i(\bar{z})],$$

$$\lambda_{I_g}^g \geq 0, \qquad \lambda_\gamma^G = 0, \qquad \lambda_\alpha^H = 0, \quad \text{either } \lambda_i^G > 0, \lambda_i^H > 0 \text{ or } \lambda_i^G \lambda_i^H = 0, \forall i \in \beta.$$

Definition 2.8 (Mordukhovich [29]) A Banach space X is Asplund, or it has the Asplund property, if every convex continuous function $\phi : U \to \mathbb{R}$ defined on an open convex subset U of X is a Frechet differential on a dense subset of U.

In the following theorem, Movahedian and Nobakhtian [9] proved a necessary optimality condition for Lipschitz MPEC on Asplund spaces.

Theorem 2.9 *Let $\bar{z}$ be a local optimal point for the MPEC where X is an Asplund space and all of the functions are locally Lipschitz around $\bar{z}$. Then $\bar{z}$ is an M-stationary point provided that the* NNAMCQ *holds at $\bar{z}$.*

Now, divide the index sets as follows. Let

$$J^+ := \{i : \lambda_i^h > 0\}, \qquad J^- := \{i : \lambda_i^h < 0\},$$

$$\beta^+ := \{i \in \beta : \lambda_i^G > 0, \lambda_i^H > 0\},$$

$$\beta_G^+ := \left\{ i \in \beta : \lambda_i^G = 0, \lambda_i^H > 0 \right\}, \qquad \beta_G^- := \left\{ i \in \beta : \lambda_i^G = 0, \lambda_i^H < 0 \right\},$$

$$\beta_H^+ := \left\{ i \in \beta : \lambda_i^H = 0, \lambda_i^G > 0 \right\}, \qquad \beta_H^- := \left\{ i \in \beta : \lambda_i^H = 0, \lambda_i^G < 0 \right\},$$

$$\alpha^+ := \left\{ i \in \alpha : \lambda_i^G > 0 \right\}, \qquad \alpha^- := \left\{ i \in \alpha : \lambda_i^G < 0 \right\},$$

$$\gamma^+ := \left\{ i \in \gamma : \lambda_i^H > 0 \right\}, \qquad \gamma^- := \left\{ i \in \gamma : \lambda_i^H < 0 \right\}.$$

3 Duality

In this section, we formulate and study a Wolfe-type dual problem for the MPEC under the convexity assumption. A Mond-Weir-type dual problem is also formulated and studied for the MPEC under convexity and generalized convexity assumptions. The Wolfe-type dual problem is formulated as follows:

$$\text{WDMPEC}(\bar{z}) \max_{u,\lambda} f(u) + \sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{p} \lambda_i^h h_i(u) - \sum_{i=1}^{l} \left[\lambda_i^G G_i(u) + \lambda_i^H H_i(u) \right]$$

subject to

$$0 \in \partial_c f(u) + \sum_{i \in I_g} \lambda_i^g \partial_c g_i(u) + \sum_{i=1}^{p} \lambda_i^h \partial_c h_i(u) - \sum_{i=1}^{l} \left[\lambda_i^G \partial_c G_i(u) + \lambda_i^H \partial_c H_i(u) \right], \tag{3.1}$$

$$\lambda_{I_g}^g \geq 0, \qquad \lambda_\gamma^G = 0, \qquad \lambda_\alpha^H = 0, \quad \text{either } \lambda_i^G > 0, \lambda_i^H > 0 \text{ or } \lambda_i^G \lambda_i^H = 0, \forall i \in \beta,$$

where $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{k+p+2l}$.

Theorem 3.1 (Weak duality) *Let $\bar{z}$ be feasible for* MPEC *where X is a Banach space, (u, λ) feasible for* WDMPEC($\bar{z}$), *and index sets I_g, α, β, γ defined accordingly. Suppose that f, g_i ($i \in I_g$), h_i ($i \in J^+$), G_i ($i \in \alpha^- \cup \beta_H^-$), H_i ($i \in \gamma^- \cup \beta_G^-$) are convex at u and radially nonconstant. Also, assume that $-h_i$ ($i \in J^-$), $-G_i$ ($i \in \alpha^+ \cup \beta_H^+ \cup \beta^+$), $-H_i$ ($i \in \gamma^+ \cup \beta_G^+ \cup \beta^+$) are directionally Lipschitzian, convex at u, and radially nonconstant. If $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$, then, for any z feasible for the* MPEC, *we have*

$$f(z) \geq f(u) + \sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{p} \lambda_i^h h_i(u) - \sum_{i=1}^{l} \left[\lambda_i^G G_i(u) + \lambda_i^H H_i(u) \right].$$

Proof Let z be any feasible point for MPEC. Then we have

$$g_i(z) \leq 0, \quad \forall i \in I_g \text{ and } h_i(z) = 0, i = 1, 2, \dots, p.$$

Since f is convex at u,

$$f(z) - f(u) \geq \langle \xi, z - u \rangle, \quad \forall \xi \in \partial_c f(u). \tag{3.2}$$

Similarly, we have

$$g_i(z) - g_i(u) \geq \langle \xi_i^g, z - u \rangle, \quad \forall \xi_i^g \in \partial_c g_i(u), \forall i \in I_g, \tag{3.3}$$

$$h_i(z) - h_i(u) \geq \langle \xi_i^h, z - u \rangle, \quad \forall \xi_i^h \in \partial_c h_i(u), \forall i \in J^+, \tag{3.4}$$

$$-h_i(z) + h_i(u) \geq -\langle \xi_i^h, z - u \rangle, \quad \forall \xi_i^h \in \partial_c h_i(u), \forall i \in J^-, \tag{3.5}$$

$$-G_i(z) + G_i(u) \geq -\langle \xi_i^G, z - u \rangle, \quad \forall \xi_i^G \in \partial_c G_i(u), \forall i \in \alpha^+ \cup \beta_H^+ \cup \beta^+, \tag{3.6}$$

$$-H_i(z) + H_i(u) \geq -\langle \xi_i^H, z - u \rangle, \quad \forall \xi_i^H \in \partial_c H_i(u), \forall i \in \gamma^+ \cup \beta_G^+ \cup \beta^+. \tag{3.7}$$

If $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$, multiplying (3.3)-(3.7) by $\lambda_i^g \geq 0$ ($i \in I_g$), $\lambda_i^h > 0$ ($i \in J^+$), $-\lambda_i^h > 0$ ($i \in J^-$), $\lambda_i^G > 0$ ($i \in \alpha^+ \cup \beta_H^+ \cup \beta^+$), $\lambda_i^H > 0$ ($i \in \gamma^+ \cup \beta_G^+ \cup \beta^+$), respectively, and adding (3.2)-(3.7), we get

$$f(z) - f(u) + \sum_{i \in I_g} \lambda_i^g g_i(z) - \sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^p \lambda_i^h h_i(z) - \sum_{i=1}^p \lambda_i^h h_i(u) - \sum_{i=1}^l \lambda_i^G G_i(z)$$

$$+ \sum_{i=1}^l \lambda_i^G G_i(u) - \sum_{i=1}^l \lambda_i^H H_i(z) + \sum_{i=1}^l \lambda_i^H H_i(u)$$

$$\geq \left\langle \xi + \sum_{i \in I_g} \lambda_i^g \xi_i^g + \sum_{i=1}^p \lambda_i^h \xi_i^h - \sum_{i=1}^l [\lambda_i^G \xi_i^G + \lambda_i^H \xi_i^H], z - u \right\rangle.$$

From (3.1), there exist $\bar{\xi} \in \partial_c f(u)$, $\bar{\xi}_i^g \in \partial_c g_i(u)$, $\bar{\xi}_i^h \in \partial_c h_i(u)$, $\bar{\xi}_i^G \in \partial_c G_i(u)$, and $\bar{\xi}_i^H \in \partial_c H_i(u)$, such that

$$\bar{\xi} + \sum_{i \in I_g} \lambda_i^g \bar{\xi}_i^g + \sum_{i=1}^p \lambda_i^h \bar{\xi}_i^h - \sum_{i=1}^l [\lambda_i^G \bar{\xi}_i^G + \lambda_i^H \bar{\xi}_i^H] = 0.$$

So,

$$f(z) - f(u) + \sum_{i \in I_g} \lambda_i^g g_i(z) - \sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^p \lambda_i^h h_i(z) - \sum_{i=1}^p \lambda_i^h h_i(u)$$

$$- \sum_{i=1}^l \lambda_i^G G_i(z) + \sum_{i=1}^l \lambda_i^G G_i(u) - \sum_{i=1}^l \lambda_i^H H_i(z) + \sum_{i=1}^l \lambda_i^H H_i(u) \geq 0.$$

Now, using the feasibility of z for MPEC, that is, $g_i(z) \leq 0$, $h_i(z) = 0$, $G_i(z) \geq 0$, $H_i(z) \geq 0$, we get

$$f(z) - f(u) - \sum_{i \in I_g} \lambda_i^g g_i(u) - \sum_{i=1}^p \lambda_i^h h_i(u) + \sum_{i=1}^l \lambda_i^G G_i(u) + \sum_{i=1}^l \lambda_i^H H_i(u) \geq 0.$$

Hence,

$$f(z) \geq f(u) + \sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^p \lambda_i^h h_i(u) - \sum_{i=1}^l \left[\lambda_i^G G_i(u) + \sum_{i=1}^l \lambda_i^H H_i(u) \right].$$

This completes the proof. □

The following corollary is a direct consequence of Theorem 3.1.

Corollary 3.2 *Let $\bar{z}$ be feasible for* MPEC *where all constraint functions g_i, h_i, G_i, H_i are affine and index sets I_g, α, β, γ defined accordingly. Then, for any z feasible for the* MPEC *and (u, λ) feasible for* WDMPEC$(\bar{z})$, *we have*

$$f(z) \geq f(u) + \sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^p \lambda_i^h h_i(u) - \sum_{i=1}^l \left[\lambda_i^G G_i(u) + \lambda_i^H H_i(u) \right].$$

Analogously, we have the following result for Asplund spaces.

Theorem 3.3 (Weak duality) *Let $\bar{z}$ be feasible for* MPEC *where X is an Asplund space, (u, λ) feasible for* WDMPEC$(\bar{z})$ *and index sets I_g, α, β, γ defined accordingly. Suppose that f, g_i $(i \in I_g)$, h_i $(i \in J^+)$, G_i $(i \in \alpha^- \cup \beta_H^-)$, H_i $(i \in \gamma^- \cup \beta_G^-)$ are convex at u and radially nonconstant. Also, assume that $-h_i$ $(i \in J^-)$, $-G_i$ $(i \in \alpha^+ \cup \beta_H^+ \cup \beta^+)$, $-H_i$ $(i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$ are directionally Lipschitzian, convex at u, and radially nonconstant. If $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$, then, for any z feasible for the* MPEC, *we have*

$$f(z) \geq f(u) + \sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^p \lambda_i^h h_i(u) - \sum_{i=1}^l \left[\lambda_i^G G_i(u) + \lambda_i^H H_i(u) \right].$$

Proof The proof follows the lines of the proof of Theorem 3.1.   □

Theorem 3.4 (Strong duality) *Assume $\bar{z}$ is a locally optimal solution of* MPEC *where X is an Asplund space, such that* NNAMCQ *is satisfied at $\bar{z}$ and the index sets I_g, α, β, γ are defined accordingly. Let f, g_i $(i \in I_g)$, h_i $(i \in J^+)$, $-h_i$ $(i \in J^-)$, G_i $(i \in \alpha^- \cup \beta_H^-)$, $-G_i$ $(i \in \alpha^+ \cup \beta_H^+ \cup \beta^+)$, H_i $(i \in \gamma^- \cup \beta_G^-)$, $-H_i$ $(i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$ satisfy the assumption of the Theorem* 3.3. *Then there exists $\bar{\lambda}$, such that $(\bar{z}, \bar{\lambda})$ is an optimal solution of* WDMPEC$(\bar{z})$ *and the respective objective values are equal.*

Proof Since $\bar{z}$ is a locally optimal solution of MPEC and the NNAMCQ is satisfied at $\bar{z}$, hence, by Theorem 2.9, $\exists \bar{\lambda} = (\bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \mathbb{R}^{k+p+2l}$, such that the nonsmooth M-stationarity conditions for MPEC are satisfied, that is, there exist $\bar{\xi} \in \partial_c f(\bar{z})$, $\bar{\xi}_i^g \in \partial_c g_i(\bar{z})$, $\bar{\xi}_i^h \in \partial_c h_i(\bar{z})$, $\bar{\xi}_i^G \in \partial_c G_i(\bar{z})$, and $\bar{\xi}_i^H \in \partial_c H_i(\bar{z})$, such that

$$0 = \bar{\xi} + \sum_{i \in I_g} \bar{\lambda}_i^g \bar{\xi}_i^g + \sum_{i=1}^p \bar{\lambda}_i^h \bar{\xi}_i^h - \sum_{i=1}^l \left[\bar{\lambda}_i^G \bar{\xi}_i^G + \bar{\lambda}_i^H \bar{\xi}_i^H \right],$$

$$\bar{\lambda}_{I_g}^g \geq 0, \qquad \bar{\lambda}_\gamma^G = 0, \bar{\lambda}_\alpha^H = 0, \quad \text{either } \bar{\lambda}_i^G > 0, \bar{\lambda}_i^H > 0 \text{ or } \bar{\lambda}_i^G \bar{\lambda}_i^H = 0, \forall i \in \beta.$$

Therefore, $(\bar{z}, \bar{\lambda})$ is feasible for WDMPEC$(\bar{z})$. By Theorem 3.3, we have

$$f(\bar{z}) \geq f(u) + \sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^p \lambda_i^h h_i(u) - \sum_{i=1}^l \left[\lambda_i^G G_i(u) + \lambda_i^H H_i(u) \right], \tag{3.8}$$

for any feasible solution (u, λ) for WDMPEC$(\bar{z})$. Also, from the feasibility condition of MPEC and WDMPEC$(\bar{z})$, that is, for $i \in I_g(\bar{z})$, $g_i(\bar{z}) = 0$, and $h_i(\bar{z}) = 0$, $G_i(\bar{z}) = 0$, $\forall i \in \alpha \cup \beta$,

$H_i(\bar{z}) = 0$, $\forall i \in \beta \cup \gamma$, we have

$$f(\bar{z}) = f(\bar{z}) + \sum_{i \in I_g} \bar{\lambda}_i^g g_i(\bar{z}) + \sum_{i=1}^{p} \bar{\lambda}_i^h h_i(\bar{z}) - \sum_{i=1}^{l} \left[\bar{\lambda}_i^G G_i(\bar{z}) + \bar{\lambda}_i^H H_i(\bar{z}) \right]. \tag{3.9}$$

Using (3.8) and (3.9), we have

$$f(\bar{z}) + \sum_{i \in I_g} \bar{\lambda}_i^g g_i(\bar{z}) + \sum_{i=1}^{p} \bar{\lambda}_i^h h_i(\bar{z}) - \sum_{i=1}^{l} \left[\bar{\lambda}_i^G G_i(\bar{z}) + \bar{\lambda}_i^H H_i(\bar{z}) \right]$$

$$\geq f(u) + \sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{p} \lambda_i^h h_i(u) - \sum_{i=1}^{l} \left[\lambda_i^G G_i(u) + \lambda_i^H H_i(u) \right].$$

Hence, $(\bar{z}, \bar{\lambda})$ is an optimal solution for WDMPEC($\bar{z}$) and the respective objective values are equal. □

Example 3.1 Consider the following MPEC in $\mathbb{R}^2$:

MPEC(1) $\min |z_1| + z_2^2$

subject to : $|z_1| + z_2 \geq 0$,

$$- z_2 \geq 0,$$

$$z_2 \left(|z_1| + z_2 \right) = 0.$$

Now, we formulate Wolfe-type dual problem WDMPEC($\bar{z}$) for MPEC(1):

$$\max_{u, \lambda} |u_1| + u_2^2 - \left[\lambda^G \left(|u_1| + u_2 \right) + \lambda^H (-u_2) \right]$$

subject to

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \xi \\ 2u_2 \end{pmatrix} - \lambda^G \begin{pmatrix} \eta \\ 1 \end{pmatrix} - \lambda^H \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

where $\xi, \eta \in [-1, 1]$.

If β is non-empty, then either

$$\lambda^G > 0, \qquad \lambda^H > 0, \quad \text{or} \quad \lambda^G \lambda^H = 0.$$

If we take the point $\bar{z} = (0, 0)$ from the feasible region, then the index sets $\alpha(0, 0)$ and $\gamma(0, 0)$ are empty sets, but $\beta := \beta(0, 0)$ is non-empty. Also, from solving a constraint equation in the feasible region of WDMPEC(0, 0), we get $\lambda^G = \frac{\xi}{\eta}$ and $\lambda^H = \frac{\xi}{\eta} - 2u_2$, where $\eta \neq 0$. Since β is non-empty, we consider a β^+, β_G^+, β_H^+ to decide the feasible region of WDMPEC(0, 0). It is clear that the assumptions of Theorem 3.1 are satisfied, so Theorem 3.1 holds between MPEC(1) and WDMPEC(0, 0).

It is clear that $\bar{z} = (0, 0)$ is the optimal solution of MPEC(1) and NNAMCQ is satisfied at $\bar{z}$. Hence, the assumptions of the Theorem 3.4 are satisfied. Then, by Theorem 3.4, there

exists $\bar{\lambda}$ such that $(\bar{z}, \bar{\lambda})$ is an optimal solution of WDMPEC$(0, 0)$ and the respective values are equal.

We now prove the duality relation between the mathematical programming problem with equilibrium constraints (MPEC) and the following Mond-Weir-type dual problem

$$\text{MWDMPEC}(\bar{z}) \max_{u, \lambda} f(u)$$

subject to

$$0 \in \partial_c f(u) + \sum_{i \in I_g} \lambda_i^g \partial_c g_i(u) + \sum_{i=1}^{p} \lambda_i^h \partial_c h_i(u) - \sum_{i=1}^{l} \left[\lambda_i^G \partial_c G_i(u) + \lambda_i^H \partial_c H_i(u) \right], \tag{3.10}$$

$$g_i(u) \geq 0 \quad (i \in I_g), \qquad h_i(u) = 0 \quad (i = 1, \ldots, p),$$

$$G_i(u) \leq 0 \quad (i \in \alpha \cup \beta), \qquad H_i(u) \leq 0 \quad (i \in \beta \cup \gamma),$$

$$\lambda_{I_g}^g \geq 0, \qquad \lambda_\gamma^G = 0, \qquad \lambda_\alpha^H = 0, \quad \text{either } \lambda_i^G > 0, \lambda_i^H > 0 \text{ or } \lambda_i^G \lambda_i^H = 0, \forall i \in \beta,$$

where $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{k+p+2l}$.

Theorem 3.5 (Weak duality) *Let $\bar{z}$ be feasible for* MPEC *where X is a Banach space, (u, λ) be feasible for* MWDMPEC$(\bar{z})$, *and the index sets $I_g, \alpha, \beta, \gamma$ are defined accordingly. Suppose that f, g_i $(i \in I_g), h_i$ $(i \in J^+), G_i$ $(i \in \alpha^- \cup \beta_H^-), H_i$ $(i \in \gamma^- \cup \beta_G^-)$ are convex at u and radially nonconstant. Also, assume that $-h_i$ $(i \in J^-), -G_i$ $(i \in \alpha^+ \cup \beta_H^+ \cup \beta^+), -H_i$ $(i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$ are directionally Lipschitzian, convex at u, and radially nonconstant. If $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$, then, for any z feasible for the* MPEC, *we have*

$$f(z) \geq f(u).$$

Proof Since f is convex at u,

$$f(z) - f(u) \geq \langle \xi, z - u \rangle, \quad \forall \xi \in \partial_c f(u). \tag{3.11}$$

Similarly, we have

$$g_i(z) - g_i(u) \geq \langle \xi_i^g, z - u \rangle, \quad \forall \xi_i^g \in \partial_c g_i(u), \forall i \in I_g, \tag{3.12}$$

$$h_i(z) - h_i(u) \geq \langle \xi_i^h, z - u \rangle, \quad \forall \xi_i^h \in \partial_c h_i(u), \forall i \in J^+, \tag{3.13}$$

$$-h_i(z) + h_i(u) \geq -\langle \xi_i^h, z - u \rangle, \quad \forall \xi_i^h \in \partial_c h_i(u), \forall i \in J^-, \tag{3.14}$$

$$-G_i(z) + G_i(u) \geq -\langle \xi_i^G, z - u \rangle, \quad \forall \xi_i^G \in \partial_c G_i(u), \forall i \in \alpha^+ \cup \beta_H^+ \cup \beta^+, \tag{3.15}$$

$$-H_i(z) + H_i(u) \geq -\langle \xi_i^H, z - u \rangle, \quad \forall \xi_i^H \in \partial_c H_i(u), \forall i \in \gamma^+ \cup \beta_G^+ \cup \beta^+. \tag{3.16}$$

If $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$, multiplying (3.12)-(3.16) by $\lambda_i^g \geq 0$ $(i \in I_g), \lambda_i^h > 0$ $(i \in J^+), -\lambda_i^h > 0$ $(i \in J^-), \lambda_i^G > 0$ $(i \in \alpha^+ \cup \beta_H^+ \cup \beta^+), \lambda_i^H > 0$ $(i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$, respectively, and adding

(3.11)-(3.16), we get

$$
f(z) - f(u) + \sum_{i \in I_g} \lambda_i^g g_i(z) - \sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^p \lambda_i^h h_i(z) - \sum_{i=1}^p \lambda_i^h h_i(u) - \sum_{i=1}^l \lambda_i^G G_i(z)
$$

$$
+ \sum_{i=1}^l \lambda_i^G G_i(u) - \sum_{i=1}^l \lambda_i^H H_i(z) + \sum_{i=1}^l \lambda_i^H H_i(u)
$$

$$
\geq \left\langle \xi + \sum_{i \in I_g} \lambda_i^g \xi_i^g + \sum_{i=1}^p \lambda_i^h \xi_i^h - \sum_{i=1}^l [\lambda_i^G \xi_i^G + \lambda_i^H \xi_i^H], z - u \right\rangle.
$$

From (3.10), there exist $\bar{\xi} \in \partial_c f(u)$, $\bar{\xi}_i^g \in \partial_c g_i(u)$, $\bar{\xi}_i^h \in \partial_c h_i(u)$, $\bar{\xi}_i^G \in \partial_c G_i(u)$, and $\bar{\xi}_i^H \in \partial_c H_i(u)$, such that

$$
\bar{\xi} + \sum_{i \in I_g} \lambda_i^g \bar{\xi}_i^g + \sum_{i=1}^p \lambda_i^h \bar{\xi}_i^h - \sum_{i=1}^l [\lambda_i^G \bar{\xi}_i^G + \lambda_i^H \bar{\xi}_i^H] = 0.
$$

So,

$$
f(z) - f(u) + \sum_{i \in I_g} \lambda_i^g g_i(z) - \sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^p \lambda_i^h h_i(z) - \sum_{i=1}^p \lambda_i^h h_i(u)
$$

$$
- \sum_{i=1}^l \lambda_i^G G_i(z) + \sum_{i=1}^l \lambda_i^G G_i(u) - \sum_{i=1}^l \lambda_i^H H_i(z) + \sum_{i=1}^l \lambda_i^H H_i(u) \geq 0.
$$

Now, using the feasibility of z and u for MPEC and MWDMPEC($\bar{z}$), respectively, we get

$$
f(z) \geq f(u).
$$

This completes the proof. □

The following corollary is a direct consequence of Theorem 3.5.

Corollary 3.6 *Let $\bar{z}$ be feasible for* MPEC *where all constraint functions g_i, h_i, G_i, H_i are affine and the index sets I_g, α, β, γ defined accordingly. Then, for any z feasible for the* MPEC *and (u, λ) feasible for* MWDMPEC($\bar{z}$), *we have*

$$
f(z) \geq f(u).
$$

Analogously, we have the following result for Asplund spaces.

Theorem 3.7 (Weak duality) *Let $\bar{z}$ be feasible for* MPEC *where X is an Asplund space, (u, λ) be feasible for* MWDMPEC($\bar{z}$) *and the index sets I_g, α, β, γ are defined accordingly. Suppose that f, g_i ($i \in I_g$), h_i ($i \in J^+$), G_i ($i \in \alpha^- \cup \beta_H^-$), H_i ($i \in \gamma^- \cup \beta_G^-$) are convex at u and radially nonconstant. Also, assume that $-h_i$ ($i \in J^-$), $-G_i$ ($i \in \alpha^+ \cup \beta_H^+ \cup \beta^+$), $-H_i$ ($i \in \gamma^+ \cup \beta_G^+ \cup \beta^+$) are directionally Lipschitzian, convex at u, and radially nonconstant. If $\alpha^- \cup$*

$\gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$, *then, for any z feasible for the* MPEC, *we have*

$$f(z) \geq f(u).$$

Proof The proof follows the lines of the proof of Theorem 3.5. □

Theorem 3.8 (Strong duality) *Assume $\bar{z}$ is a locally optimal solution of* MPEC *where X is an Asplund space, such that* NNAMCQ *is satisfied at $\bar{z}$ and the index sets $I_g, \alpha, \beta, \gamma$ defined accordingly. Let f, g_i $(i \in I_g), h_i$ $(i \in J^+), -h_i$ $(i \in J^-), G_i$ $(i \in \alpha^- \cup \beta_H^-), -G_i$ $(i \in \alpha^+ \cup \beta_H^+ \cup \beta^+)$, H_i $(i \in \gamma^- \cup \beta_G^-), -H_i$ $(i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$ satisfy the assumption of the Theorem 3.7. Then there exists $\bar{\lambda}$, such that $(\bar{z}, \bar{\lambda})$ is an optimal solution of* MWDMPEC($\bar{z}$), *and the respective objective values are equal.*

Proof $\bar{z}$ is a locally optimal solution of MPEC and the NNAMCQ is satisfied at $\bar{z}$, by Theorem 2.9, $\exists \bar{\lambda} = (\bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \mathbb{R}^{k+p+2l}$, such that the nonsmooth M-stationarity conditions for MPEC are satisfied, that is, there exist $\bar{\xi} \in \partial_c f(\bar{z}), \bar{\xi}_i^g \in \partial_c g_i(\bar{z}), \bar{\xi}_i^h \in \partial_c h_i(\bar{z}), \bar{\xi}_i^G \in \partial_c G_i(\bar{z})$ and $\bar{\xi}_i^H \in \partial_c H_i(\bar{z})$, such that

$$0 = \bar{\xi} + \sum_{i \in I_g} \bar{\lambda}_i^g \bar{\xi}_i^g + \sum_{i=1}^{p} \bar{\lambda}_i^h \bar{\xi}_i^h - \sum_{i=1}^{l} \left[\bar{\lambda}_i^G \bar{\xi}_i^G + \bar{\lambda}_i^H \bar{\xi}_i^H \right],$$

$$\bar{\lambda}_{I_g}^g \geq 0, \qquad \bar{\lambda}_\gamma^G = 0, \qquad \bar{\lambda}_\alpha^H = 0, \quad \text{either } \bar{\lambda}_i^G > 0, \bar{\lambda}_i^H > 0 \text{ or } \bar{\lambda}_i^G \bar{\lambda}_i^H = 0, \forall i \in \beta.$$

Since $\bar{z}$ is an optimal solution for MPEC, we have

$$\sum_{i \in I_g} \bar{\lambda}_i^g g_i(\bar{z}) = 0, \qquad \sum_{i=1}^{p} \bar{\lambda}_i^h h_i(\bar{z}) = 0, \qquad \sum_{i=1}^{l} \bar{\lambda}_i^G G_i(\bar{z}) = 0, \qquad \sum_{i=1}^{l} \bar{\lambda}_i^H H_i(\bar{z}) = 0.$$

Therefore, $(\bar{z}, \bar{\lambda})$ is feasible for MWDMPEC($\bar{z}$). Also, by Theorem 3.7, for any feasible (u, λ), we have

$$f(\bar{z}) \geq f(u).$$

Thus, $(\bar{z}, \bar{\lambda})$ is an optimal solution for MWDMPEC($\bar{z}$) and the respective objective values are equal. This completes the proof. □

Example 3.2 Consider the following MPEC problem in $\mathbb{R}^2$:

MPEC $\min |z_1| + z_2$

   subject to $|z_1| + z_2 \geq 0$,

   $z_2 - |z_1| \geq 0$,

   $(|z_1| + z_2)(z_2 - |z_1|) = 0$.

The Mond-Weir-type dual problem MWDMPEC($\bar{z}$) for the MPEC is

$\max_{u,\lambda} |u_1| + u_2$

subject to

$$
\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ 1 \end{pmatrix} - \lambda^G \begin{pmatrix} \xi_2 \\ 1 \end{pmatrix} - \lambda^H \begin{pmatrix} \eta \\ 1 \end{pmatrix}, \tag{3.17}
$$

where $\xi_1, \xi_2, \eta \in [-1, 1]$,

$$
\lambda^G \big(|u_1| + u_2 \big) \leq 0, \tag{3.18}
$$

$$
\lambda^H \big(u_2 - |u_1| \big) \leq 0, \tag{3.19}
$$

if β is non-empty, then either

$$
\lambda^G > 0, \qquad \lambda^H > 0, \quad \text{or} \quad \lambda^G \lambda^H = 0.
$$

From (3.17), $\lambda^G \xi_2 + \lambda^H \eta = \xi_1$, and $\lambda^G + \lambda^H = 1$, we get $\lambda^H = \frac{\xi_2 - \xi_1}{\xi_2 - \eta}$ and $\lambda^G = \frac{\xi_1 - \eta}{\xi_2 - \eta}$, where $\xi_2 \neq \eta$. If $\bar{z} = (0, 0)$, then the index sets $\alpha(0, 0)$ and $\gamma(0, 0)$ are empty sets, but $\beta(0, 0)$ is non-empty. It is clear that the assumptions of Corollary 3.6 are satisfied. So, Corollary 3.6 holds between MPEC and MWDMPEC(0, 0).

Also, we can see that the NNAMCQ is satisfied at $\bar{z}$. Then by Theorem 3.8 there exists $\bar{\lambda} = (\bar{\lambda}^G, \bar{\lambda}^H)$ such that $(\bar{z}, \bar{\lambda})$ is an optimal solution of MWDMPEC(0, 0) and the optimal values are equal.

Now, we establish weak and strong duality theorems for the MPEC and its Mond-Weir-type dual problem under generalized convexity assumptions.

Theorem 3.9 (Weak duality) *Let $\bar{z}$ be feasible for* MPEC *where X is a Banach space, (u, λ) be feasible for* MWDMPEC($\bar{z}$), *and the index sets $I_g, \alpha, \beta, \gamma$ are defined accordingly. Suppose that f is pseudoconvex at $\bar{z}$, g_i $(i \in I_g)$, h_i $(i \in J^+)$, G_i $(i \in \alpha^- \cup \beta_H^-)$, H_i $(i \in \gamma^- \cup \beta_G^-)$ are quasiconvex at u and radially nonconstant. Also, assume that $-h_i$ $(i \in J^-)$, $-G_i$ $(i \in \alpha^+ \cup \beta_H^+ \cup \beta^+)$, $-H_i$ $(i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$ are directionally Lipschitzian, quasiconvex at u, and radially nonconstant. If $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$, then, for any z feasible for the* MPEC, *we have*

$$
f(z) \geq f(u).
$$

Proof Suppose that, for some feasible point z, such that $f(z) < f(u)$, then, by pseudoconvexity of f at u, we have

$$
\langle \xi, z - u \rangle < 0, \quad \forall \xi \in \partial_c f(u). \tag{3.20}
$$

From (3.10), there exist $\bar{\xi}_i^g \in \partial_c g_i(u)$ $(i \in I_g)$, $\bar{\xi}_i^h \in \partial_c h_i(u)$ $(i = 1, \dots, p)$, $\bar{\xi}_i^G \in \partial_c G_i(u)$ $(i \in \alpha \cup \beta)$ and $\bar{\xi}_i^H \in \partial_c H_i(u)$ $(i \in \beta \cup \gamma)$, such that

$$
-\sum_{i \in I_g} \lambda_i^g \bar{\xi}_i^g - \sum_{i=1}^p \lambda_i^h \bar{\xi}_i^h + \sum_{\alpha \cup \beta} \lambda_i^G \bar{\xi}_i^G + \sum_{\beta \cup \gamma} \lambda_i^H \bar{\xi}_i^H \in \partial_c f(u). \tag{3.21}
$$

By (3.20), we get

$$\left\langle \left(-\sum_{i \in I_g} \lambda_i^g \bar{\xi}_i^g - \sum_{i=1}^{p} \lambda_i^h \bar{\xi}_i^h + \sum_{\alpha \cup \beta} \lambda_i^G \bar{\xi}_i^G + \sum_{\beta \cup \gamma} \lambda_i^H \bar{\xi}_i^H \right), z - u \right\rangle < 0. \tag{3.22}$$

For each $i \in I_g$, $g_i(z) \leq 0 \leq g_i(u)$. Hence, by Theorem 4.4 in [9], we have

$$\langle \xi, z - u \rangle \leq 0, \quad \forall \xi \in \partial_c g_i(u), \forall i \in I_g. \tag{3.23}$$

Similarly, we have

$$\langle \xi, z - u \rangle \leq 0, \quad \forall \xi \in \partial_c h_i(u), \forall i \in J^+. \tag{3.24}$$

Now, for any feasible point u of MWDMPEC($\bar{z}$), and for each $i \in J^-$, $0 = -h_i(u) = h_i(z)$. On the other hand, $-G_i(z) \leq -G_i(u)$, $\forall i \in \alpha^+ \cup \beta_H^+$, and $-H_i(z) \leq -H_i(u)$, $\forall i \in \gamma^+ \cup \beta_G^+$. Since all of these functions are directionally Lipschitzian, by Theorem 2.5, we get

$$\langle \xi, z - u \rangle \geq 0, \quad \forall \xi \in \partial_c h_i(u), \forall i \in J^-, \tag{3.25}$$

$$\langle \xi, z - u \rangle \geq 0, \quad \forall \xi \in \partial_c G_i(u), \forall i \in \alpha^+ \cup \beta_H^+, \tag{3.26}$$

$$\langle \xi, z - u \rangle \geq 0, \quad \forall \xi \in \partial_c H_i(u), \forall i \in \gamma^+ \cup \beta_G^+. \tag{3.27}$$

From equation (3.23)-(3.27), it is clear that

$$\langle \bar{\xi}_i^g, z - u \rangle \leq 0 \quad (i \in I_g), \qquad \langle \bar{\xi}_i^h, z - u \rangle \leq 0 \quad (i \in J^+), \qquad \langle \bar{\xi}_i^h, z - u \rangle \geq 0 \quad (i \in J^-),$$

$$\langle \bar{\xi}_i^G, z - u \rangle \geq 0, \quad \forall i \in \alpha^+ \cup \beta_H^+, \qquad \langle \bar{\xi}_i^H, z - u \rangle \geq 0, \quad \forall i \in \gamma^+ \cup \beta_G^+.$$

Since $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$, we have

$$\left\langle \sum_{\alpha \cup \beta} \lambda_i^G \bar{\xi}_i^G, z - u \right\rangle \geq 0, \qquad \left\langle \sum_{\beta \cup \gamma} \lambda_i^H \bar{\xi}_i^H, z - u \right\rangle \geq 0,$$

$$\left\langle \sum_{i \in I_g} \lambda_i^g \bar{\xi}_i^g, z - u \right\rangle \geq 0, \qquad \left\langle \sum_{i=1}^{p} \lambda_i^h \bar{\xi}_i^h, z - u \right\rangle \geq 0.$$

Therefore,

$$\left\langle \left(-\sum_{i \in I_g} \lambda_i^g \bar{\xi}_i^g - \sum_{i=1}^{p} \lambda_i^h \bar{\xi}_i^h + \sum_{\alpha \cup \beta} \lambda_i^G \bar{\xi}_i^G + \sum_{\beta \cup \gamma} \lambda_i^H \bar{\xi}_i^H \right), z - u \right\rangle \geq 0,$$

which contradicts (3.22). Hence, $f(z) \geq f(u)$. This completes the proof. □

Analogously, we have the following result for Asplund spaces.

Theorem 3.10 (Weak duality) *Let $\bar{z}$ be feasible for* MPEC *where X is an Asplund space, (u, λ) be feasible for* MWDMPEC($\bar{z}$), *and the index sets I_g, α, β, γ are defined accordingly.*

Suppose that f is pseudoconvex at $\bar{z}$, g_i ($i \in I_g$), h_i ($i \in J^+$), G_i ($i \in \alpha^- \cup \beta_H^-$), H_i ($i \in \gamma^- \cup \beta_G^-$) are quasiconvex at u and radially nonconstant. Also, assume that $-h_i$ ($i \in J^-$), $-G_i$ ($i \in \alpha^+ \cup \beta_H^+ \cup \beta^+$), $-H_i$ ($i \in \gamma^+ \cup \beta_G^+ \cup \beta^+$) are directionally Lipschitzian, quasiconvex at u, and radially nonconstant. If $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$, then, for any z feasible for the MPEC, we have

$$f(z) \geq f(u).$$

Proof The proof follows the lines of the proof of Theorem 3.9. □

Theorem 3.11 (Strong duality) *Assume $\bar{z}$ is a locally optimal solution of MPEC where X is an Asplund space, such that NNAMCQ is satisfied at $\bar{z}$, and the index sets I_g, α, β, γ are defined accordingly. Let f, g_i ($i \in I_g$), h_i ($i \in J^+$), $-h_i$ ($i \in J^-$), G_i ($i \in \alpha^- \cup \beta_H^-$), $-G_i$ ($i \in \alpha^+ \cup \beta_H^+ \cup \beta^+$), H_i ($i \in \gamma^- \cup \beta_G^-$), $-H_i$ ($i \in \gamma^+ \cup \beta_G^+ \cup \beta^+$) satisfy the assumption of Theorem 3.10. Then there exists $\bar{\lambda}$, such that $(\bar{z}, \bar{\lambda})$ is an optimal solution of MWDMPEC($\bar{z}$), and the respective objective values are equal.*

Proof The proof follows the lines of the proof of Theorem 3.8, invoking Theorem 3.10. □

4 Results and discussion

We have studied mathematical programs with equilibrium constraints (MPECs). The objective function and functions in the constraint part are assumed to be lower semicontinuous. We studied the Wolfe-type dual problem for the MPEC under the convexity assumption. A Mond-Weir-type dual problem was also formulated and studied for the MPEC under convexity and generalized convexity assumptions. Conditions for weak duality theorems were given to relate the MPEC and two dual programs in Banach space, respectively. Also conditions for strong duality theorems were established in an Asplund space. We also discussed the cases when all the constraint functions are affine. Two numerical examples were given to illustrate the Wolfe-type duality and the Mond-Weir-type duality with our MPECs, respectively.

Authors' contributions
YP conceived of the study and drafted the manuscript initially. S-MG participated in its design, coordination and finalized the manuscript. SKM outlined the scope and design of the study. All authors read and approved the final manuscript.

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