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Existence result and error bounds for a new class of inverse mixed quasi-variational inequalities

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Abstract

In this paper, a new class of inverse mixed quasi-variational inequalities (IMQVI) is introduced and studied in Hilbert spaces. This type of inequalities includes many quasi-variational inequalities and inverse variational inequalities as its special cases. We first prove some properties of generalized f-projection operators in Hilbert spaces. Then we use these properties to obtain the existence and uniqueness result. Moreover, error bounds for IMQVI in terms of the residual function are also established. The results presented in this paper are new and improve some results in the recent literature.

Keywords: mixed variational inequality; inverse mixed quasi-variational inequality; generalized f-projection operator; error bound; residual function

1 Introduction

It the past decades, variational inequalities and their generalizations have been widely used in finance, economics, transportation, optimization, operations research, and the engineering sciences. For instance, Lescarret [1] and Browder [2] introduced mixed variational inequalities in 1960s. Later, Konnov and Volotskaya [3] applied mixed variational inequalities to several classes of general economic equilibrium problems and oligopolistic equilibrium problems. In 2006, He *et al.* [4, 5] studied a class of inverse variational inequalities and also found their applications in practical world, such as normative flow control problems, which require the network equilibrium state to be in a linearly constrained set, and bipartite market equilibrium problems. Some other generalizations such as quasi-variational inequalities also have been studied extensively. For details, we refer to [1, 6–21] and the references therein.

Motivated and inspired by the work mentioned above, in this paper, we introduce a new class of inverse mixed quasi-variational inequalities (IMQVI) in Hilbert spaces: find an $x \in H$, such that $Ax \in K(x)$ and

$$\langle g(x), y - Ax \rangle + \rho f(y) - \rho f(Ax) \geq 0, \quad \forall y \in K(x), \tag{1.1}$$

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where $g, A : H \to H$ are two continuous mappings, $K : H \to 2^H$ is a set-valued mapping such that, for each $x \in H$, $K(x) \subset H$ is a closed convex set and $f : H \to R \cup \{+\infty\}$ is a proper, convex, and lower semicontinuous on $K(x)$ for each $x \in H$.

We first prove some properties of generalized f-projection operators in Hilbert spaces and then explore the existence and uniqueness results of the IMQVI. Furthermore, we study error bounds for the IMQVI in terms of the residual function. Since IMQVI naturally encompasses many types of quasi-variational inequalities and inverse variational inequalities, the results presented in this paper therefore generalize and improve some results in the existing literature.

2 Preliminaries

Let H be a real Hilbert space with scalar product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Now we recall the concept of the generalized f-projector operator introduced by Wu and Huang [22]. More properties and applications on generalized projection operators can be found in [23–26]. Let $G : H \times K \to R \cup \{+\infty\}$ be a functional defined as follows:

$$G(x, \xi) = \|x\|^2 - 2\langle x, \xi \rangle + \|\xi\|^2 + 2\rho f(\xi),$$

where $\xi \in K$, $x \in H$, ρ is a positive number and $f : K \to R \cup \{+\infty\}$ is proper, convex, and lower semicontinuous.

Definition 2.1 [22] Let H be a real Hilbert space, and K be a nonempty closed and convex subset of H. We say that $P_K^{f,\rho} : H \to 2^K$ is a generalized f-projection operator if

$$P_K^{f,\rho} x = \left\{ u \in K : G(x, u) = \inf_{\xi \in K} G(x, \xi) \right\}, \quad \forall x \in H.$$

From the work of Wu and Huang [22] and Fan *et al.* [11], we know that the generalized f-projection operator has the following properties.

Lemma 2.1 [11, 22] *Let H be a real Hilbert space, and K be a nonempty closed and convex subset of H. Then the following statements hold:*
   (i) *$P_K^{f,\rho} x$ is nonempty and $P_K^{f,\rho}$ is a single valued mapping;*
  (ii) *for all $x \in H$, $x^* = P_K^{f,\rho} x$ if and only if*

$$\langle x^* - x, y - x^* \rangle + \rho f(y) - \rho f(x^*) \geq 0, \quad \forall y \in K;$$

 (iii) *$P_K^{f,\rho}$ is continuous.*

Definition 2.2 Let H be a real Hilbert space, and $g, A : H \to H$ be two single-valued mappings.
   (i) A is said to be λ-strongly monotone on H if there exists a constant λ such that

$$\langle Ax - Ay, x - y \rangle \geq \lambda \|x - y\|^2, \quad \forall x, y \in H;$$

  (ii) A is said to be γ-Lipschitz continuous on H if there exists a constant $\gamma > 0$ such that

$$\|Ax - Ay\| \leq \gamma \|x - y\|, \quad \forall x, y \in H;$$

(iii) A is said to be co-coercive on H if there exists a positive constant $\tau > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \tau \|Ax - Ay\|^2, \quad \forall x, y \in H;$$

(iv) (A, g) is said to be a μ-strongly monotone couple on H if there exists a positive constant $\mu > 0$ such that

$$\langle Ax - Ay, g(x) - g(y) \rangle \geq \mu \|x - y\|^2, \quad \forall x, y \in H.$$

Let $g, A : H \to H$ be two continuous mappings, and $K : H \to 2^H$ be a set-valued mapping such that, for each $x \in H, K(x) \subset H$ is a closed convex set. Let $f : H \to R \cup \{+\infty\}$ be proper, convex, and lower semicontinuous on $K(x)$ for each $x \in H$. Now we consider the inverse mixed quasi-variational inequality (for short, IMQVI) (1.1): find an $x \in H$, such that $Ax \in K(x)$ and

$$\langle g(x), y - Ax \rangle + \rho f(y) - \rho f(Ax) \geq 0, \quad \forall y \in K(x). \tag{2.1}$$

Remark IMQVI encompasses several models of quasi-variational inequalities and inverse variational inequalities. For example:

(1) If g is the identity mapping and $K(x)$ is a constant set $\overline{K}$ for all $x \in H$, then IMQVI reduces immediately to the inverse mixed variational inequality [27], which is defined as follows: find an $x \in H$, such that $Ax \in \overline{K}$ and

$$\langle g(x), y - Ax \rangle + \rho f(y) - \rho f(Ax) \geq 0, \quad \forall y \in \overline{K}.$$

(2) If $H = R^n$, g is the identity mapping, $f(x) = 0$, and $K(x)$ is a constant set $\overline{K}$ for all $x \in R^n$, then IMQVI reduces to the following inverse variational inequality: find an $x \in R^n$, such that $Ax \in \overline{K}$ and

$$\langle x, y - Ax \rangle \geq 0, \quad \forall y \in \overline{K},$$

which was first proposed by He and Liu [4].

(3) If $H = R^n$ and $f(x) = 0$ for all $x \in R^n$, then IMQVI reduces to the inverse quasi-variational inequality: find an $x \in R^n$, such that $Ax \in K(x)$ and

$$\langle g(x), y - Ax \rangle \geq 0, \quad \forall y \in K(x). \tag{2.2}$$

The inverse quasi-variational inequality (2.2) was introduced and studied by Aussel *et al.* [28]. Moreover, if g is the identity mapping, then IMQVI reduces to the following inverse quasi-variational inequality: find an $x \in R^n$, such that $Ax \in K(x)$ and

$$\langle x, y - Ax \rangle \geq 0, \quad \forall y \in K(x). \tag{2.3}$$

(4) If $H = R^n$, A is the identity mapping and $f(x) = 0$ for all $x \in R^n$, then IMQVI becomes the classic quasi-variational inequality.

3 Some properties of generalized *f*-projection operators

Let $K : H \to 2^H$ be a set-valued mapping such that $K(x)$ is a closed convex set in H, for each $x \in H$. The Hausdorff distance between $K(x)$ and $K(y)$ is defined as follows:

$$\mathcal{H}\big[K(x), K(y)\big] = \max\left\{ \sup_{u \in K(x)} \inf_{v \in K(y)} \|u - v\|, \ \sup_{v \in K(y)} \inf_{u \in K(x)} \|u - v\| \right\}.$$

From Definition 2.1 and Lemma 2.1, we see that the generalized *f*-projection of any $z \in H$ on the set $K(x)$ is defined by

$$P_{K(x)}^{f,\rho} z = \arg \inf_{\xi \in K(x)} G(z, \xi).$$

Now we apply the basic inequality in Lemma 2.1 to prove some properties of the operator $P_{K(x)}^{f,\rho}$ in Hilbert spaces.

Theorem 3.1 *Let H be a real Hilbert space and M be a nonempty bounded subset of H. Let $K : H \to 2^H$ be a set-valued mapping such that, for each $x \in H$, $K(x) \subset H$ is a closed convex set and $f : H \to R \cup \{+\infty\}$ be proper, convex, and lower semicontinuous on $K(x)$. Assume that*

  (i) *there exists a constant $\gamma > 0$ such that $\mathcal{H}[K(x), K(y)] \le \gamma \|x - y\|^2$, $\forall x, y \in H$;*

  (ii) *$0 \in \bigcap_{u \in H} K(u)$;*

  (iii) *f is l-Lipschitz continuous on H.*

Then there exists a constant $k > 0$ such that

$$\big\| P_{K(x)}^{f,\rho} z - P_{K(y)}^{f,\rho} z \big\| \le k \|x - y\|, \quad \forall x, y \in H, z \in M.$$

Proof For any $x, y \in H$ and $z \in M$, denote $\bar{x} = P_{K(x)}^{f,\rho} z$ and $\bar{y} = P_{K(y)}^{f,\rho} z$. Since $\bar{x} \in K(x)$, $\bar{y} \in K(y)$ and $\mathcal{H}[K(x), K(y)] \le \gamma \|x - y\|^2$, we know that there exist $\xi \in K(x)$ and $\eta \in K(y)$ such that $\|\bar{x} - \eta\| \le \gamma \|x - y\|^2$ and $\|\bar{y} - \xi\| \le \gamma \|x - y\|^2$. From Lemma 2.1, we have

$$\langle \bar{x} - z, \xi - \bar{x} \rangle + \rho f(\xi) - \rho f(\bar{x}) \ge 0 \tag{3.1}$$

and

$$\langle \bar{y} - z, \eta - \bar{y} \rangle + \rho f(\eta) - \rho f(\bar{y}) \ge 0. \tag{3.2}$$

By the Lipschitz continuity of f, it follows from (3.1) and (3.2) that

$$\begin{aligned}
\|\bar{x} - \bar{y}\|^2 &= \langle \bar{x} - \bar{y}, \bar{x} - \bar{y} \rangle \\
&= \langle \bar{x} - z, \bar{x} - \bar{y} \rangle + \langle \bar{y} - z, \bar{y} - \bar{x} \rangle \\
&= \langle \bar{x} - z, \bar{x} - \xi \rangle + \langle \bar{x} - z, \xi - \bar{y} \rangle + \langle \bar{y} - z, \bar{y} - \eta \rangle + \langle \bar{y} - z, \eta - \bar{x} \rangle \\
&\le \rho f(\xi) - \rho f(\bar{y}) + \|\bar{x} - z\|\|\xi - \bar{y}\| + \rho f(\eta) - \rho f(\bar{x}) + \|\bar{y} - z\|\|\eta - \bar{x}\| \\
&\le \rho l \|\xi - \bar{y}\| + \|\bar{x} - z\|\|\xi - \bar{y}\| + \rho l \|\eta - \bar{x}\| + \|\bar{y} - z\|\|\eta - \bar{x}\| \\
&\le \gamma \big(2\rho l + \|\bar{x} - z\| + \|\bar{y} - z\|\big)\|x - y\|^2,
\end{aligned} \tag{3.3}$$

and so

$$\|\bar{x} - \bar{y}\| \le \sqrt{\gamma\left(2\rho l + \|\bar{x} - z\| + \|\bar{y} - z\|\right)} \|x - y\|. \tag{3.4}$$

On the other hand, since M is bounded, we know that there exists a positive number m such that $\|v\| \le m$ for all $v \in M$. Noticing that $0 \in K(u)$ for all $u \in H$, we have

$$\begin{aligned}
G\left(z, P_{K(u)}^{f,\rho}z\right) &= \|z\|^2 - 2\langle z, P_{K(u)}^{f,\rho}z\rangle + \left\|P_{K(u)}^{f,\rho}z\right\|^2 + 2\rho f\left(P_{K(u)}^{f,\rho}z\right) \\
&\le G(z, 0) \\
&= \|z\|^2 + 2\rho f(0).
\end{aligned} \tag{3.5}$$

From the Lipschitz continuity of f, it follows from (3.5) that

$$\begin{aligned}
\left\|P_{K(u)}^{f,\rho}z\right\|^2 &\le 2\langle z, P_{K(u)}^{f,\rho}z\rangle + 2\rho f(0) - 2\rho f\left(P_{K(u)}^{f,\rho}z\right) \\
&\le 2\|z\|\left\|P_{K(u)}^{f,\rho}z\right\| + 2\rho l\left\|P_{K(u)}^{f,\rho}z\right\| \\
&\le 2(m + \rho l)\left\|P_{K(u)}^{f,\rho}z\right\|,
\end{aligned}$$

and so $\|P_{K(u)}^{f,\rho}z\| \le 2(m + \rho l)$ for all $u \in H$, $z \in M$. Since $\|P_{K(x)}^{f,\rho}z\| \le 2(m + \rho l)$, $\|P_{K(y)}^{f,\rho}z\| \le 2(m + \rho l)$ and $\|z\| \le m$, we have

$$\left\|P_{K(x)}^{f,\rho}z - z\right\| + \left\|P_{K(y)}^{f,\rho}z - z\right\| \le 6m + 4\rho l.$$

Now (3.4) implies that

$$\left\|P_{K(x)}^{f,\rho}z - P_{K(y)}^{f,\rho}z\right\| \le k\|x - y\|, \quad \forall x, y \in H, z \in M,$$

where $k = \sqrt{6\gamma(m + \rho l)}$. This completes the proof. □

Remark 3.1 Theorem 3.1 shows that the generalized f-projection operator $P_{K(x)}^{f,\rho}$ is k-Lipschitz continuous with respect to x on each bounded set of the Hilbert space H under some suitable conditions.

If $H = R^n$ and $f(x) = 0$ for all $x \in R^n$, then the generalized f-projection operator $P_{K(x)}^{f,\rho}$ reduces to the classic metric projection operator $P_{K(x)}$. By Theorem 3.1, we can obtain the following theorem.

Theorem 3.2 *Let M be a nonempty bounded subset of R^n, and $K : R^n \to 2^{R^n}$ be a set-valued mapping such that $K(x)$ is a closed convex set in R^n, for each $x \in R^n$. Assume that*

(i) *there exists $\gamma > 0$ such that $\mathcal{H}[K(x), K(y)] \le \gamma \|x - y\|^2, \forall x, y \in R^n$;*

(ii) *$0 \in \bigcap_{u \in R^n} K(u)$.*

Then there exists a constant $k > 0$ such that

$$\|P_{K(x)}z - P_{K(y)}z\| \le k\|x - y\|, \quad \forall x, y \in R^n, z \in M.$$

Theorem 3.3 *Let H be a real Hilbert space, and $K : H \to 2^H$ be a set-valued mapping such that, for each $x \in H$, $K(x) \subset H$ is a closed convex set. Let $f : H \to R \cup \{+\infty\}$ be proper, convex, and lower semicontinuous on $K(x)$ for each $x \in H$. Then, for any $x, y, u, v \in H$,*

$$\langle P_{K(x)}^{f,\rho} u - P_{K(x)}^{f,\rho} v, u - v \rangle \geq \left\| P_{K(x)}^{f,\rho} u - P_{K(x)}^{f,\rho} v \right\|^2$$

and

$$\left\| u - P_{K(x)}^{f,\rho} u - \left(v - P_{K(y)}^{f,\rho} v \right) \right\| \leq \| u - v \| + \left\| P_{K(x)}^{f,\rho} v - P_{K(y)}^{f,\rho} v \right\|.$$

Proof Based on Lemma 2.1, we have

$$\langle P_{K(x)}^{f,\rho} v - v, P_{K(x)}^{f,\rho} u - P_{K(x)}^{f,\rho} v \rangle + \rho f \left(P_{K(x)}^{f,\rho} u \right) - \rho f \left(P_{K(x)}^{f,\rho} v \right) \geq 0 \tag{3.6}$$

and

$$\langle P_{K(x)}^{f,\rho} u - u, P_{K(x)}^{f,\rho} v - P_{K(x)}^{f,\rho} u \rangle + \rho f \left(P_{K(x)}^{f,\rho} v \right) - \rho f \left(P_{K(x)}^{f,\rho} u \right) \geq 0. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\left\| P_{K(x)}^{f,\rho} u - P_{K(x)}^{f,\rho} v \right\|^2 \leq \langle P_{K(x)}^{f,\rho} u - P_{K(x)}^{f,\rho} v, u - v \rangle. \tag{3.8}$$

By (3.8), we obtain

$$\begin{aligned}
\left\| u - P_{K(x)}^{f,\rho} u - \left(v - P_{K(x)}^{f,\rho} v \right) \right\|^2 &= \| u - v \|^2 - 2\langle P_{K(x)}^{f,\rho} u - P_{K(x)}^{f,\rho} v, u - v \rangle + \left\| P_{K(x)}^{f,\rho} u - P_{K(x)}^{f,\rho} v \right\|^2 \\
&\leq \| u - v \|^2 - \left\| P_{K(x)}^{f,\rho} u - P_{K(x)}^{f,\rho} v \right\|^2 \\
&\leq \| u - v \|^2
\end{aligned}$$

and so

$$\begin{aligned}
\left\| u - P_{K(x)}^{f,\rho} u - \left(v - P_{K(y)}^{f,\rho} v \right) \right\| &\leq \left\| u - P_{K(x)}^{f,\rho} u - \left(v - P_{K(x)}^{f,\rho} v \right) \right\| + \left\| P_{K(x)}^{f,\rho} v - P_{K(y)}^{f,\rho} v \right\| \\
&\leq \| u - v \| + \left\| P_{K(x)}^{f,\rho} v - P_{K(y)}^{f,\rho} v \right\|.
\end{aligned}$$

This completes the proof. □

4 The existence and uniqueness result of IMQVI

From the properties of generalized f-projection operators, it is easy to see that $x \in H$ is a solution of IMQVI (2.1) if and only if x satisfies

$$Ax = P_{K(x)}^{f,\rho} \left[Ax - \rho g(x) \right], \quad \forall \rho > 0. \tag{4.1}$$

In this section, we consider the existence and uniqueness result of IMQVI (2.1) in Hilbert spaces.

Theorem 4.1 *Let H be a real Hilbert space, and $g, A : H \to H$ be Lipschitz continuous on H (with constants α and β, respectively). Let $K : H \to 2^H$ be a set-valued mapping such that, for each $x \in H$, $K(x) \subset H$ is a closed convex set and $f : H \to R \cup \{+\infty\}$ be proper, convex, and lower semicontinuous on $K(x)$. Assume that*

  (i) *g is λ-strongly monotone and (A, g) is a μ-strongly monotone couple on H;*

  (ii) *there exists $k > 0$ such that*

$$\left\| P^{f,\rho}_{K(x)} z - P^{f,\rho}_{K(y)} z \right\| \le k \|x - y\|, \quad \forall x, y \in H, z \in \{ v \mid v = Ax - \rho g(x), x \in H \};$$

  (iii) *$\sqrt{\beta^2 - 2\rho\mu + \rho^2\alpha^2} + \rho\sqrt{1 - 2\lambda + \alpha^2} < \rho - k$.*

Then IMQVI (2.1) has a unique solution in H.

Proof Let $h : H \to H$ be defined as follows:

$$h(u) = u - \frac{1}{\rho} Au + \frac{1}{\rho} P^{f,\rho}_{K(u)} [Au - \rho g(u)], \quad \forall u \in H.$$

For any $x, y \in H$, denote $\bar{x} = Ax - \rho g(x)$ and $\bar{y} = Ay - \rho g(y)$, we have

$$
\begin{aligned}
\left\| h(x) - h(y) \right\| &= \left\| x - y - \frac{1}{\rho} Ax + \frac{1}{\rho} Ay + \frac{1}{\rho} P^{f,\rho}_{K(x)} \bar{x} - \frac{1}{\rho} P^{f,\rho}_{K(y)} \bar{y} \right\| \\
&= \left\| x - y - g(x) + g(y) - \frac{1}{\rho} \left[\bar{x} - P^{f,\rho}_{K(x)} \bar{x} - \left(\bar{y} - P^{f,\rho}_{K(y)} \bar{y} \right) \right] \right\| \\
&\le \left\| x - y - g(x) + g(y) \right\| + \frac{1}{\rho} \left\| \bar{x} - P^{f,\rho}_{K(x)} \bar{x} - \left(\bar{y} - P^{f,\rho}_{K(y)} \bar{y} \right) \right\|.
\end{aligned}
\tag{4.2}
$$

Since $g : H \to H$ is λ-strongly monotone and α-Lipschitz continuous on H, we obtain

$$
\begin{aligned}
\left\| x - y - g(x) + g(y) \right\|^2 &= \|x - y\|^2 - 2\langle g(x) - g(y), x - y \rangle + \left\| g(x) - g(y) \right\|^2 \\
&\le \left(1 - 2\lambda + \alpha^2 \right) \|x - y\|^2.
\end{aligned}
\tag{4.3}
$$

On the other hand, it follows from (ii) and Theorem 3.3 that

$$
\begin{aligned}
\left\| \bar{x} - P^{f,\rho}_{K(x)} \bar{x} - \left(\bar{y} - P^{f,\rho}_{K(y)} \bar{y} \right) \right\| &\le \|\bar{x} - \bar{y}\| + \left\| P^{f,\rho}_{K(x)} \bar{y} - P^{f,\rho}_{K(y)} \bar{y} \right\| \\
&\le \left\| Ax - Ay - \rho \left[g(x) - g(y) \right] \right\| + k \|x - y\|.
\end{aligned}
\tag{4.4}
$$

Since A is β-Lipschitz continuous and (A, g) is a μ-strongly monotone couple on H, we have

$$
\begin{aligned}
& \left\| Ax - Ay - \rho \left[g(x) - g(y) \right] \right\|^2 \\
&= \|Ax - Ay\|^2 - 2\rho \langle Ax - Ay, g(x) - g(y) \rangle + \rho^2 \left\| g(x) - g(y) \right\|^2 \\
&\le \left(\beta^2 - 2\rho\mu + \rho^2\alpha^2 \right) \|x - y\|^2.
\end{aligned}
\tag{4.5}
$$

Now (4.2)-(4.5) imply that

$$\left\| h(x) - h(y) \right\| \le \theta \|x - y\|, \tag{4.6}$$

where $\theta = \sqrt{1 - 2\lambda + \alpha^2} + \frac{1}{\rho}(\sqrt{\beta^2 - 2\rho\mu + \rho^2\alpha^2} + k)$. By the assumption, we know that $0 < \theta < 1$ and so (4.6) implies that $h(u) = u - \frac{1}{\rho}Au + \frac{1}{\rho}P^{f,\rho}_{K(u)}[Au - \rho g(u)]$ is a contracting mapping in Hilbert space H. Thus, h has a unique fixed point x^* in H and so x^* is a unique solution of IMQVI (2.1). This completes the proof. □

Remark 4.1 By Theorem 3.1, we know that if $\{v | v = Ax - \rho g(x), x \in H\}$ is bounded and the conditions (i)-(iii) in Theorem 3.1 are satisfied, then there exists $k > 0$ such that

$$\left\| P^{f,\rho}_{K(x)}z - P^{f,\rho}_{K(y)}z \right\| \le k\|x - y\|, \quad \forall x, y \in H, z \in \{v | v = Ax - \rho g(x), x \in H\}.$$

Therefore, the condition (ii) in Theorem 4.1 is suitable.

5 Error bounds for IMQVI

It is well known that error bounds play important roles in the study of variational inequality problems. They allow one to estimate how far a feasible element is from the solution set without even having computed a single solution of the associated variational inequality. In [28], Aussel *et al.* provided the following two error bounds.

Theorem DA1 (Theorem 1 of [28]) *Let* $g, A : R^n \to R^n$ *be Lipschitz continuous on* R^n *(with constants l and L, respectively), and let* $K : R^n \to 2^{R^n}$ *be a set-valued map such that $K(x)$ is a closed convex set in* R^n, *for each* $x \in R^n$. *Suppose the following hold*:

(a) (A, g) *is a strongly monotone couple on* R^n *with constant* μ,

(b) *there exists* $0 < k < \frac{\mu}{l}$ *such that, for any* $\theta > \frac{Lk}{\mu - lk}$,

$$\left\| P^{\theta}_{K(x)}z - P^{\theta}_{K(y)}z \right\| \le k\|x - y\|, \quad \forall x, y, z \in R^n.$$

If x^* *is the solution of* (2.2), *then, for any* $x \in R^n$ *and any* $\theta > \frac{Lk}{\mu - lk}$, *we have*

$$\|x - x^*\| \le \frac{(\theta l + L)}{\theta\mu - (\theta l + L)k}\|R^{\theta}(x)\|,$$

where $R^{\theta}(x) = Ax - P_{K(x)}[Ax - \theta g(x)]$.

Theorem DA2 (Lemma 1 of [28]) *Let* $A : R^n \to R^n$ *be Lipschitz continuous on* R^n *(with constant L) on* R^n *and let* $K : R^n \to 2^{R^n}$ *be a set-valued map such that $K(x)$ is a closed convex set in* R^n, *for each* $x \in R^n$. *Assume the following hold*:

(a) A *is strongly monotone on* R^n *with constant* μ,

(b) *there exists* $0 < k < \mu$ *such that, for any* $\theta > \frac{L(8k+L)}{4(\mu - k)}$,

$$\left\| P^{\theta}_{K(x)}z - P^{\theta}_{K(y)}z \right\| \le k\|x - y\|, \quad \forall x, y, z \in R^n.$$

If x^* *is the solution of* (2.3), *then, for any* $x \in R^n$ *and any* $\theta > \frac{L(8k+L)}{4(\mu - k)}$, *we have*

$$\|x - x^*\| \le \frac{4\theta}{4\theta(\mu - k) - L(8k + L)}\|R^{\theta}(x)\|,$$

where $R^{\theta}(x) = Ax - P_{K(x)}(Ax - \theta x)$.

In this section, we will develop some error bounds measuring the distance between any point and the exact solution of IMQVI (2.1) in Hilbert spaces. Let

$$e(x, \rho) = Ax - P^{f,\rho}_{K(x)}\big[Ax - \rho g(x)\big]$$

denote the residue of the generalized f-projection equation (4.1). Then solving an IMQVI (2.1) problem is equivalent to finding a zero point of $e(x, \rho)$. For any given $x \in H$, the magnitude of $\|e(x, \rho)\|$ depends on the value of ρ. Now we give the error bounds in terms of the residual function $\|e(x, \rho)\|$.

Theorem 5.1 *Let H be a real Hilbert space, and $g, A : H \to H$ be Lipschitz continuous on H (with constants α and β, respectively). Let $K : H \to 2^H$ be a set-valued mapping such that, for each $x \in H$, $K(x) \subset H$ is a closed convex set and $f : H \to R \cup \{+\infty\}$ be proper, convex, and lower semicontinuous on $K(x)$. Assume that*

(i) *(A, g) is a μ-strongly monotone couple on H;*

(ii) *there exists $0 < k < \frac{\mu}{\alpha}$ such that, for any $\rho > \frac{\beta k}{\mu - \alpha k}$,*

$$\big\| P^{f,\rho}_{K(x)} z - P^{f,\rho}_{K(y)} z \big\| \le k \|x - y\|, \quad \forall x, y \in H, z \in \{ v \mid v = Ax - \rho g(x), x \in H \}.$$

If x^* is the solution of IMQVI (2.1), then, for any $x \in H$ and any $\rho > \frac{\beta k}{\mu - \alpha k}$, we have

$$\big\| x - x^* \big\| \le \frac{(\alpha \rho + \beta)}{\mu \rho - (\alpha \rho + \beta) k} \big\| e(x, \rho) \big\|.$$

Proof Denote $u = P^{f,\rho}_{K(x^*)}[Ax - \rho g(x)]$. Since x^* is the solution of IMQVI (2.1), it follows that

$$\big\langle \rho g(x^*), u - Ax^* \big\rangle + \rho f(u) - \rho f\big(Ax^*\big) \ge 0, \tag{5.1}$$

for all $\rho > 0$. From the definition of u and $Ax^* \in K(x^*)$, we know that

$$\big\langle u - \big[Ax - \rho g(x)\big], Ax^* - u \big\rangle + \rho f\big(Ax^*\big) - \rho f(u) \ge 0. \tag{5.2}$$

By (5.1) and (5.2), we have

$$\begin{aligned}
0 &\le \big\langle \rho\big[g(x^*) - g(x)\big] + Ax - u, u - Ax^* \big\rangle \\
&= \rho \big\langle g(x^*) - g(x), u - Ax \big\rangle + \rho \big\langle g(x^*) - g(x), Ax - Ax^* \big\rangle \\
&\quad + \langle Ax - u, u - Ax \rangle + \big\langle Ax - u, Ax - Ax^* \big\rangle.
\end{aligned} \tag{5.3}$$

Since (A, g) is a μ-strongly monotone couple, it follows from (5.3) that

$$\rho \big\langle g(x^*) - g(x), u - Ax \big\rangle + \big\langle Ax - u, Ax - Ax^* \big\rangle \ge \rho \mu \big\| x^* - x \big\|^2 + \| Ax - u \|^2. \tag{5.4}$$

In light of the facts that g is α-Lipschitz continuous and A is β-Lipschitz continuous, (5.4) implies that

$$
\begin{aligned}
\mu\rho\left\|x^* - x\right\|^2 &\leq \rho\left\|g\left(x^*\right) - g(x)\right\|\|u - Ax\| + \|Ax - u\|\left\|Ax - Ax^*\right\| \\
&\leq (\rho\alpha + \beta)\left\|x^* - x\right\|\left\|P_{K(x^*)}^{f,\rho}\left[Ax - \rho g(x)\right] - Ax\right\| \\
&\leq (\rho\alpha + \beta)\left\|x^* - x\right\|\left(\left\|P_{K(x^*)}^{f,\rho}\left[Ax - \rho g(x)\right] - P_{K(x)}^{f,\rho}\left[Ax - \rho g(x)\right]\right\|\right. \\
&\quad + \left.\left\|P_{K(x)}^{f,\rho}\left[Ax - \rho g(x)\right] - Ax\right\|\right) \\
&\leq (\rho\alpha + \beta)\left\|x^* - x\right\|\left(k\left\|x^* - x\right\| + \left\|e(x,\rho)\right\|\right),
\end{aligned}
\tag{5.5}
$$

for any $\rho > \frac{\beta k}{\mu - \alpha k}$. Since $\alpha k < \mu$ and $\rho > \frac{\beta k}{\mu - \alpha k}$, it follows from (5.5) that

$$
\left\|x^* - x\right\| \leq \frac{(\alpha\rho + \beta)}{\mu\rho - (\alpha\rho + \beta)k}\left\|e(x,\rho)\right\|.
$$

This completes the proof. □

If $H = R^n$ and $f(x) = 0$ for all $x \in R^n$, from Theorem 5.1, we obtain the following theorem.

Theorem 5.2 *Let $g, A : R^n \to R^n$ be Lipschitz continuous on R^n (with constants l and L, respectively). Let $K : R^n \to 2^{R^n}$ be a set-valued mapping such that, for each $x \in R^n$, $K(x) \subset R^n$ is a closed convex set. Assume that*

  (i) *(A, g) is a μ-strongly monotone couple on R^n;*
  (ii) *there exists $0 < k < \frac{\mu}{l}$ such that, for any $\theta > \frac{Lk}{\mu - lk}$,*

$$
\|P_{K(x)}z - P_{K(y)}z\| \leq k\|x - y\|, \quad \forall x, y \in R^n, z \in \left\{v \mid v = Ax - \theta g(x), x \in R^n\right\}.
$$

If x^* is the solution of (2.2), then, for any $x \in R^n$ and any $\theta > \frac{Lk}{\mu - lk}$, we have

$$
\left\|x - x^*\right\| \leq \frac{(\theta l + L)}{\theta\mu - (\theta l + L)k}\left\|R^\theta(x)\right\|,
$$

where $R^\theta(x) = Ax - P_{K(x)}[Ax - \theta g(x)]$.

For any $x \in H$, based on Theorem 3.3, we know that $P_{K(x)}^{f,\rho}$ is co-coercive mapping with modulus 1 on H. Applying the co-coercivity of $P_{K(x)}^{f,\rho}$, we prove another error bound for IMQVI (2.1).

Theorem 5.3 *Let H be a real Hilbert space, and $g, A : H \to H$ be Lipschitz continuous on H (with constants α and β, respectively). Let $K : H \to 2^H$ be a set-valued mapping such that, for each $x \in H$, $K(x) \subset H$ is a closed convex set and $f : H \to R \cup \{+\infty\}$ be proper, convex, and lower semicontinuous on $K(x)$. Assume that*

  (i) *(A, g) is a μ-strongly monotone couple on H;*
  (ii) *there exists $0 < k < \frac{\mu}{\alpha}$ such that, for any $\rho > \frac{\beta(\beta + 8k)}{4(\mu - \alpha k)}$,*

$$
\left\|P_{K(x)}^{f,\rho}z - P_{K(y)}^{f,\rho}z\right\| \leq k\|x - y\|, \quad \forall x, y \in H, z \in \left\{v \mid v = Ax - \rho g(x), x \in H\right\}.
$$

If x^* is the solution of IMQVI (2.1), then, for any $x \in H$ and any $\rho > \frac{\beta(\beta+8k)}{4(\mu-\alpha k)}$, we have

$$\|x - x^*\| \leq \frac{4\alpha\rho}{4\rho(\mu - \alpha k) - \beta(\beta + 8k)}\|e(x, \rho)\|.$$

Proof Denote $v = Ax - \rho g(x)$ and $v^* = Ax^* - \rho g(x^*)$. From the definition of $e(x, \rho)$, we know that

$$
\begin{aligned}
&\langle e(x,\rho), g(x) - g(x^*)\rangle \\
&= \langle e(x,\rho) - e(x^*,\rho), g(x) - g(x^*)\rangle \\
&= \langle P^{f,\rho}_{K(x^*)}v^* - P^{f,\rho}_{K(x)}v, g(x) - g(x^*)\rangle + \langle Ax - Ax^*, g(x) - g(x^*)\rangle \\
&= \frac{1}{\rho}\langle P^{f,\rho}_{K(x^*)}v^* - P^{f,\rho}_{K(x)}v, Ax - Ax^*\rangle + \frac{1}{\rho}\langle P^{f,\rho}_{K(x^*)}v^* - P^{f,\rho}_{K(x)}v, v^* - v\rangle \\
&\quad + \langle Ax - Ax^*, g(x) - g(x^*)\rangle \\
&= \frac{1}{\rho}\langle P^{f,\rho}_{K(x)}v^* - P^{f,\rho}_{K(x)}v, Ax - Ax^*\rangle - \frac{1}{\rho}\langle P^{f,\rho}_{K(x)}v^* - P^{f,\rho}_{K(x^*)}v^*, Ax - Ax^*\rangle \\
&\quad + \frac{1}{\rho}\langle P^{f,\rho}_{K(x)}v^* - P^{f,\rho}_{K(x)}v, v^* - v\rangle - \frac{1}{\rho}\langle P^{f,\rho}_{K(x)}v^* - P^{f,\rho}_{K(x^*)}v^*, v^* - v\rangle \\
&\quad + \langle Ax - Ax^*, g(x) - g(x^*)\rangle.
\end{aligned} \tag{5.6}
$$

Since (A, g) is a μ-strongly monotone couple, it follows from (5.6) and Theorem 3.3 that

$$
\begin{aligned}
&\langle e(x,\rho), g(x) - g(x^*)\rangle \\
&\geq \frac{1}{\rho}\langle P^{f,\rho}_{K(x)}v^* - P^{f,\rho}_{K(x)}v, Ax - Ax^*\rangle + \frac{1}{\rho}\|P^{f,\rho}_{K(x)}v^* - P^{f,\rho}_{K(x)}v\|^2 + \mu\|x - x^*\|^2 \\
&\quad - \frac{1}{\rho}\|P^{f,\rho}_{K(x)}v^* - P^{f,\rho}_{K(x^*)}v^*\|\|Ax - Ax^*\| - \frac{1}{\rho}\|P^{f,\rho}_{K(x)}v^* - P^{f,\rho}_{K(x^*)}v^*\|\|v^* - v\| \\
&= \frac{1}{\rho}\left\|P^{f,\rho}_{K(x)}v^* - P^{f,\rho}_{K(x)}v + \frac{1}{2}(Ax - Ax^*)\right\|^2 - \frac{1}{4\rho}\|Ax - Ax^*\|^2 + \mu\|x - x^*\|^2 \\
&\quad - \frac{1}{\rho}\|P^{f,\rho}_{K(x)}v^* - P^{f,\rho}_{K(x^*)}v^*\|(\|Ax - Ax^*\| + \|v^* - v\|) \\
&\geq \mu\|x - x^*\|^2 - \frac{1}{4\rho}\beta^2\|x - x^*\|^2 - \frac{1}{\rho}k(2\beta + \alpha\rho)\|x - x^*\|^2,
\end{aligned} \tag{5.7}
$$

for any $\rho > \frac{\beta(\beta+8k)}{4(\mu-\alpha k)}$. On the other hand, we have

$$
\begin{aligned}
\langle e(x,\rho), g(x) - g(x^*)\rangle &\leq \|e(x,\rho)\|\|g(x) - g(x^*)\| \\
&\leq \alpha\|e(x,\rho)\|\|x - x^*\|.
\end{aligned} \tag{5.8}
$$

Since $\alpha k < \mu$ and $\rho > \frac{\beta(\beta+8k)}{4(\mu-\alpha k)}$, it follows from (5.7) and (5.8) that

$$\|x - x^*\| \leq \frac{4\alpha\rho}{4\rho(\mu - \alpha k) - \beta(\beta + 8k)}\|e(x, \rho)\|.$$

This completes the proof. $\square$

If $H = R^n$, g is identity mapping in R^n, and $f(x) = 0$ for all $x \in R^n$, by using Theorem 5.3, we have the following theorem.

Theorem 5.4 *Let $A : R^n \to R^n$ be Lipschitz continuous on R^n (with constant L) on R^n and let $K : R^n \to 2^{R^n}$ be a set-valued map such that $K(x)$ is a closed convex set in R^n, for each $x \in R^n$. Assume the following hold*:

  (i) *A is strongly monotone on R^n with constant μ,*
  (ii) *there exists $0 < k < \mu$ such that, for any $\theta > \frac{L(8k+L)}{4(\mu-k)}$,*

$$\|P_{K(x)}z - P_{K(y)}z\| \leq k\|x - y\|, \quad \forall x, y \in H, z \in \left\{v | v = Ax - \theta x, x \in R^n\right\}.$$

If x^* is the solution of (2.3), then, for any $x \in R^n$ and any $\theta > \frac{L(8k+L)}{4(\mu-k)}$, we have

$$\|x - x^*\| \leq \frac{4\theta}{4\theta(\mu - k) - L(8k + L)}\|R^\theta(x)\|,$$

where $R^\theta(x) = Ax - P_{K(x)}(Ax - \theta x)$.

Remark 5.1 It is easy to see that the condition (ii) in Theorem 5.4 is weaker than the condition (b) in Theorem DA2 and the condition (ii) in Theorem 5.2 is also weaker than the condition (b) in Theorem DA1.

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