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Decay properties of the discrete wavelet transform in n dimensions with independent dilation parameters

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Abstract

The purpose of this paper is to study the decay properties of the discrete wavelet transform with n independent dilation parameters for functions f in $L^2(\mathbb{R}^n)$ and a relationship with its continuity.

The method we used to study the discrete wavelet transform of f with respect to a radially symmetric admissible function was through the fact of considering two parameters in $\mathbb{Z}^n$.

We conclude that the continuity of f at $x = 0$ is determined by the existence of the limit of the discrete wavelet transform when each one of the independent dilation parameters tends to zero.

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Keywords: discrete wavelet transform; tight frames; reconstruction formula

1 Introduction

It is already known that the discrete wavelet transform has been applied to detect local singularities of signals. In [1] the discrete wavelet transform has been used to study the local continuity of functions f in $L^2(\mathbb{R})$. In [2] the Lipschitz exponent, together with the discrete wavelet transform, has been applied in the estimation of function differentiability, and the type of wavelet helps to detect local discontinuities.

In the n-dimensional case, the study of singularities of functions f in $L^2(\mathbb{R}^n)$ has been also studied by means of the discrete wavelet transform [3].

In fact, in [3] the discrete wavelet transform for a given function f in $L^2(\mathbb{R}^n)$ with respect to an *admissible* function h in $L^2(\mathbb{R}^n)$ is defined as:

$$(L_h f)(a^l, a^l bk) = \langle f, T_{a^l bk} J_{a^l} h\rangle = \int_{\mathbb{R}^n} f(x) \frac{1}{\sqrt{a^{nl}}} \overline{h\left(\frac{x - a^l bk}{a^l}\right)}\, dx, \tag{1.1}$$

where $a > 1$ and $b > 0$ are fixed, $l \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, and $T_{a^l bk} h$ and $J_{a^l} h$ are the translation and dilation operators, respectively. Note that in [3] only one parameter $k \in \mathbb{Z}^n$ is used, and the same dilation parameter a^l is used in all the dimensions.

In this paper, we use again the discrete wavelet transform in n dimensions to study the continuity of functions f in $L^2(\mathbb{R}^n)$, but the main difference with the result given in [3] is that now we use two parameters j, k in $\mathbb{Z}^n$ instead of only one parameter in $\mathbb{Z}^n$. We should say that even though the discrete wavelet transform depends of j, k, where $(j,k) \in \mathbb{Z}^n \times \mathbb{Z}^n$, our approach to prove the continuity of $f \in L^2(\mathbb{R}^n)$ is by considering the space $M_n(\mathbb{R}) \times \mathbb{R}^n$. That is,

(1) for $j = (j_1, j_2, \ldots, j_n)$ in $\mathbb{Z}^n$ and for $a > 1$, we define the matrix $M(a^j)$ as the diagonal matrix whose entries are a^{j_i}, $i = 1, 2, \ldots, n$, and
(2) for $b = (b_1, b_2, \ldots, b_n)$ in $\mathbb{R}^n$ with $b_i > 0$, $i = 1, 2, \ldots, n$, we define the matrix $M(b)$ as the diagonal matrix whose entries are b_i, $i = 1, 2, \ldots, n$.

Then in this case, the discrete wavelet transform for $f \in L^2(\mathbb{R}^n)$ with respect to a radially symmetric *admissible* function $h \in C_0^\infty(\mathbb{R}^n)$ is now defined as

$$(L_h f)\big(M^{-1}(a^j), M^{-1}(a^j)M(b)k\big) = \langle f, T_{M^{-1}(a^j)M(b)k} J_{M^{-1}(a^j)} h\rangle, \tag{1.2}$$

where $j, k \in \mathbb{Z}^n$, and $T_{M^{-1}(a^j)M(b)k}$ and $J_{M^{-1}(a^j)}$ are the translation and dilation operators, respectively.

Thus, by using (1.2) we establish a relationship of local continuity of functions $f \in L^2(\mathbb{R}^n)$ with the decay property of their discrete wavelet transform. To study this fact, we use the following reconstruction formula:

$$f = \frac{1}{A} \sum_{j \in \mathbb{Z}^n} \sum_{k \in \mathbb{Z}^n} \langle f, T_{M^{-1}(a^j)M(b)k} J_{M^{-1}(a^j)} h\rangle T_{M^{-1}(a^j)M(b)k} J_{M^{-1}(a^j)} h, \tag{1.3}$$

where A is a positive constant, and the convergence is in the weak sense [4].

Also, matrix operators have been applied to define extensions of wavelet transforms. For instance, in [5] tight frames of curvelets are presented to address the problem of finding optimally sparse representation of objects with discontinuities along C^2 edges. In [6] a parabolic directional dilation matrix in R^2 is used to derive a discrete tight frame by sampling the continuous curvelet transform (CCT) [7] at dyadic intervals in scale $a_j = 2^{-j}$, at equispaced intervals in direction, $\theta_{j,l} = 2\pi 2^{-\frac{j}{2}} l$, and equispaced sampling on a rotated anisotropic grid in space. In [8] the shearlet transform is presented by using an anisotropic dilation matrix and a shear matrix, which allow one to construct Parseval frames whose elements range not only at various scales and locations, but also at various orientations to deal with two-dimensional signals containing discontinuities such as edges.

Integral transforms have several applications once they can be defined in the space of continuous functions, and several types of convergence can be studied. For instance, in [9] the fractional Sumudu transform of arbitrary order is investigated on some space of integrable Boehmians. The fractional Sumudu transform of an integrable Boehmian is well defined, linear, and sequentially complete in the space of continuous functions, and its convergence is studied. In [10] some generalization of the S-transform is presented in a space of rapidly decreasing Boehmians, and its continuity with respect to the δ and Δ-convergence is discussed. In [11] some generalization of a class of Hankel-Clifford transformations having Fox H-function as part of its kernel on a class of Boehmians is investigated, as well as its continuity with respect to the δ and Δ-convergence.

In fact, our main result is the following:

Theorem 1 *Let f be in $L^2(\mathbb{R}^n)$ and consider $h \in C_0^\infty(\mathbb{R}^n)$ with h admissible and radially symmetric so that $T_{M^{-1}(a^j)M(b)k}J_{M^{-1}(a^j)}h$ is a tight frame that satisfies* (1.3). *If*

$$\lim_{(M^{-1}(a^j),x)\to(0,x_1)} \sqrt{\det M(a^j)}(L_h f)(M^{-1}(a^j),x)$$

exists for any x_1 in an open neighborhood of 0, *then f is continuous at $x = 0$.*

2 Notation and definitions

In this section we define two operators in n dimensions, namely, the dilation operator and the translation operator. In this case, both operators are defined as diagonal matrices with n independent entries.

Definition 1 For $a > 1$ and $j = (j_1, j_2, \dots, j_n)$ in $\mathbb{Z}^n$, define the $n \times n$ matrix $M(a^j)$ as

$$M(a^j) = \operatorname{diag}(a^{j_1}, a^{j_2}, \dots, a^{j_n}). \tag{2.1}$$

Note that for $a > 1$ and j in $\mathbb{Z}^n$,

$$M^{-1}(a^j) = \operatorname{diag}\left(\frac{1}{a^{j_1}}, \frac{1}{a^{j_2}}, \dots, \frac{1}{a^{j_n}}\right). \tag{2.2}$$

Definition 2 For $b = (b_1, b_2, \dots, b_n)$ in $\mathbb{R}_+^n$, define the $n \times n$ matrix $M(b)$ as:

$$M(b) = \operatorname{diag}(b_1, b_2, \dots, b_n). \tag{2.3}$$

Under this approach, an independent dilation parameter a^{j_i} is used in each dimension as a generalization of the case where a single dilation parameter is used in all the dimensions [3].

Now, for fixed $a > 1$ and $b = (b_1, b_2, \dots, b_n)$ in $\mathbb{R}_+^n$, define for $j = (j_1, j_2, \dots, j_n)$ in $\mathbb{Z}^n$ and $k = (k_1, k_2, \dots, k_n)$ also in $\mathbb{Z}^n$, the dilation and translation operators.

Definition 3 Given h in $L^2(\mathbb{R}^n)$, define

(1) the dilation operator $J_{M(a^j)} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ as

$$(J_{M(a^j)}h)(x) = \frac{1}{\sqrt{\det M(a^j)}} h(M^{-1}(a^j)x), \tag{2.4}$$

where $x \in \mathbb{R}^n$, $a > 1$, and $j \in \mathbb{Z}^n$;

(2) the translation operator $T_{M^{-1}(a^j)M(b)k} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ as

$$(T_{M^{-1}(a^j)M(b)k}h)(x) = h(x - M^{-1}(a^j)M(b)k), \tag{2.5}$$

where $x \in \mathbb{R}^n$, $b \in \mathbb{R}_+^n$, and $k \in \mathbb{Z}^n$.

Note that the operators $J_{M(a^j)}$, and $T_{M^{-1}(a^j)M(b)k}$ satisfy, for h in $L^2(\mathbb{R}^n)$,

$$\begin{aligned} &(1)\quad \|T_{M^{-1}(a^j)M(b)k}h\|_2 = \|h\|_2, \quad \text{and} \quad \|J_{M(a^j)}h\|_2 = \|h\|_2; \\ &(2)\quad T^*_{M^{-1}(a^j)M(b)k} = T^{-1}_{M^{-1}(a^j)M(b)k} = T_{-M^{-1}(a^j)M(b)k}; \quad \text{and} \\ &(3)\quad J^*_{M(a^j)} = J^{-1}_{M(a^j)} = J_{M^{-1}(a^j)}. \end{aligned} \tag{2.6}$$

Hence, for f and g in $L^2(\mathbb{R}^n)$,

$$\langle f, T_{M^{-1}(a^j)M(b)k}J_{M^{-1}(a^j)}g\rangle = \langle J_{M(a^j)}T_{-M^{-1}(a^j)M(b)k}f, g\rangle. \tag{2.7}$$

3 The discrete wavelet transform

In this section we define the discrete wavelet transform for functions $f \in L^2(\mathbb{R}^n)$ with respect to a radially *admissible* function in n dimensions using an independent dilation parameter for each dimension. Moreover, we will prove that the continuity of f at $x = 0$ will determine the decay of its discrete wavelet transform.

Definition 4 Suppose that h in $L^2(\mathbb{R}^n)$ is radially symmetric and satisfies the admissibility condition $\int_0^\infty |\eta(k)|^2 \frac{1}{|k|}\, dk < \infty$, where $\widehat{h}(y) = \eta(|y|)$. Then the discrete wavelet transform of f in $L^2(\mathbb{R}^n)$ with respect to h is defined as

$$(L_hf)\big(M^{-1}(a^j), M^{-1}(a^j)M(b)k\big) = \langle f, T_{M^{-1}(a^j)M(b)k}J_{M^{-1}(a^j)}h\rangle, \tag{3.1}$$

where $a > 1$ and $b \in \mathbb{R}^n_+$ are fixed, and j and k are in $\mathbb{Z}^n$.

Note that in this case, $M^{-1}(a^j) \in M_n(\mathbb{R})$, where $M_n(\mathbb{R})$ denotes the space of all $n \times n$ matrices with entries in $\mathbb{R}$, and $M^{-1}(a^j)M(b)k \in \mathbb{R}^n$. Furthermore, from (2.5) and (2.4) we have

$$\begin{aligned} (L_hf)\big(M^{-1}(a^j), M^{-1}(a^j)M(b)k\big) &= \langle f, T_{M^{-1}(a^j)M(b)k}J_{M^{-1}(a^j)}h\rangle \\ &= \int_{\mathbb{R}^n} f(x)\overline{(T_{M^{-1}(a^j)M(b)k}J_{M^{-1}(a^j)}h)(x)}\, dx \\ &= \int_{\mathbb{R}^n} f(x)\overline{(J_{M^{-1}(a^j)}h)\big(x - M^{-1}(a^j)M(b)k\big)}\, dx \\ &= \int_{\mathbb{R}^n} f(x)\sqrt{\det M(a^j)}\,\overline{h\big(M(a^j)x - M(b)k\big)}\, dx. \end{aligned} \tag{3.2}$$

Also, note that by (3.2) the discrete wavelet transform can be written as a convolution. That is, we have the following result.

Lemma 1 *If h in $L^2(\mathbb{R}^n)$ is admissible and the support of h is compact, then for f in $L^2(\mathbb{R}^n)$,*

$$(L_hf)\big(M^{-1}(a^j), M^{-1}(a^j)M(b)k\big) = \big((J_{M^{-1}(a^j)}\overline{h})^{\sim} * f\big)\big(M^{-1}(a^j)M(b)k\big), \tag{3.3}$$

where $\sim$ means $h^{\sim}(x) = h(-x)$.

Proof According to (3.2),

$$\begin{aligned}
&\big((J_{M^{-1}(a^j)}\overline{h})^{\sim} * f\big)\big(M^{-1}(a^j)M(b)k\big)\\
&\quad= \int_{\mathbb{R}^n} (J_{M^{-1}(a^j)}\overline{h})^{\sim}\big(M^{-1}(a^j)M(b)k - x\big)f(x)\,dx\\
&\quad= \int_{\mathbb{R}^n} (J_{M^{-1}(a^j)}\overline{h})\big(x - M^{-1}(a^j)M(b)k\big)f(x)\,dx\\
&\quad= \int_{\mathbb{R}^n} \sqrt{\det M(a^j)}\,\overline{h\big(M(a^j)\big(x - M^{-1}(a^j)M(b)k\big)\big)}f(x)\,dx\\
&\quad= (L_h f)\big(M^{-1}(a^j), M^{-1}(a^j)M(b)k\big). \qquad \square
\end{aligned}$$

Then we have the following result.

Lemma 2 *Suppose that $h \in C_0^\infty(\mathbb{R}^n)$ is admissible, radially symmetric, $h \not\equiv 0$, and $\widehat{h}(0) = 0$. Let*

$$\begin{aligned}
&(\mathcal{L}_h f)\big(M^{-1}(a^j), M^{-1}(a^j)M(b)k\big)\\
&\quad\equiv \sqrt{\det M(a^j)}(L_h f)\big(M^{-1}(a^j), M^{-1}(a^j)M(b)k\big).
\end{aligned} \tag{3.4}$$

Then $\mathcal{L}_h f$ is continuous at any point $(M^{-1}(a^j), M^{-1}(a^j)M(b)k)$ in $M_n(\mathbb{R}) \times \mathbb{R}^n$. Furthermore, if $f \in L^2(\mathbb{R}^n)$ is continuous in a neighborhood of $x = 0 \in \mathbb{R}^n$, then for any k in $\mathbb{Z}^n$,

$$(\mathcal{L}_h f)\big(M^{-1}(a^j), M^{-1}(a^j)M(b)k\big) \to 0 \quad \textit{as } \big(M^{-1}(a^j), M^{-1}(a^j)M(b)k\big) \to (0,0).$$

Proof First, we show that $\mathcal{L}_h f$ is continuous at any point $(M^{-1}(a^j), M^{-1}(a^j)M(b)k)$ in $M_n(\mathbb{R}) \times \mathbb{R}^n$. Since $f \in L^2(\mathbb{R}^n)$ and $h \in C_0^\infty(\mathbb{R}^n)$, it follows that $(J_{M^{-1}(a^j)}\overline{h})^{\sim} * f \in C^\infty(\mathbb{R}^n)$, [12]. Thus, from (3.3) we have that

$$\begin{aligned}
&(\mathcal{L}_h f)\big(M^{-1}(a^j), M^{-1}(a^j)M(b)k\big)\\
&\quad= \sqrt{\det M(a^j)}\big((J_{M^{-1}(a^j)}\overline{h})^{\sim} * f\big)\big(M^{-1}(a^j)M(b)k\big)
\end{aligned}$$

is continuous at any point $(M^{-1}(a^j), M^{-1}(a^j)M(b)k)$ in $M_n(\mathbb{R}) \times \mathbb{R}^n$.

Next, we will show that the continuity of $f \in L^2(\mathbb{R}^n)$ implies the existence of the limit of $\mathcal{L}_h f$ as $(M^{-1}(a^j), M^{-1}(a^j)M(b)k) \to (0,0)$.

Note that by considering the Frobenius norm in $M_n(\mathbb{R})$ given by $\|Q\| = \langle Q, Q\rangle^{\frac{1}{2}}$, from (2.2) we have that $M^{-1}(a^j) \to 0$ is equivalent to $j \to +\infty$.

Then, since $\operatorname{supp} h$ is compact, there is $L > 0$ such that $\operatorname{supp} h \subset B_L(0)$. Thus, by (2.7),

$$\begin{aligned}
&(L_h f)\big(M^{-1}(a^j), M^{-1}(a^j)M(b)k\big)\\
&\quad= \int_{B_L(0)} \frac{1}{\sqrt{\det M(a^j)}} f\big(M^{-1}(a^j)x + M^{-1}(a^j)M(b)k\big)\overline{h(x)}\,dx.
\end{aligned} \tag{3.5}$$

Hence, by (3.4),

$$(\mathcal{L}_h f)\big(M^{-1}(a^j), M^{-1}(a^j)M(b)k\big) = \int_{B_L(0)} f\big(M^{-1}(a^j)x + M^{-1}(a^j)M(b)k\big)\overline{h(x)}\,dx. \tag{3.6}$$

Thus, since f is continuous near $0 \in \mathbb{R}^n$, it follows that for any $k \in \mathbb{Z}^n$,

$$(\mathcal{L}_h f)\big(M^{-1}(a^j), M^{-1}(a^j)M(b)k\big) \to f(0)\bigg[\int_{B_L(0)} \overline{h(x)}\,dx\bigg] \to f(0)\cdot 0 = 0.$$

This completes the proof of Lemma 2. □

4 Proof of main theorem

In this section we will prove the converse of Lemma 2, which is our main result given in Theorem 1. That is, the convergence of the discrete wavelet transform determines the continuity of f at zero. So, in order to analyze this result, we first give the definition of a tight frame [13].

Definition 5 A family of functions $(h_\alpha)_{\alpha\in\Omega}$ in a Hilbert space H is called a frame if there are constants $A > 0$ and $B < +\infty$ such that for all f in H,

$$A\|f\|^2 \le \sum_{\alpha\in\Omega} \big|\langle f, h_\alpha\rangle\big|^2 \le B\|f\|^2. \tag{4.1}$$

If $A = B$, then the frame $(h_\alpha)_{\alpha\in\Omega}$ is called a tight frame. In this case,

$$A\|f\|^2 = \sum_{\alpha\in\Omega} \big|\langle f, h_\alpha\rangle\big|^2.$$

Hence,

$$f = \frac{1}{A}\sum_{\alpha\in\Omega} \langle f, h_\alpha\rangle h_\alpha, \tag{4.2}$$

where the convergence is in the weak sense.

On the other hand, if there is $b_0 > 0$ such that for any $b = (b_1, b_2, \dots, b_n)$ in $\mathbb{R}^n$ with $b_i \in (0, b_0)$, the family $T_{M^{-1}(a^j)M(b)k} J_{M^{-1}(a^j)} h$ is a frame in $L^2(\mathbb{R}^n)$, and if conditions (2), (3), and (4) given in [4] are satisfied, then by Theorem 2 in [4] there exist $A > 0$ and $B < +\infty$ such that

$$A\|f\|^2 \le \sum_{j,k} \big|\langle T_{M^{-1}(a^j)M(b)k} J_{M^{-1}(a^j)} h, f\rangle\big| \le B\|f\|^2.$$

Hence, if $A = B$, then by (4.2), if $h \in L^2(\mathbb{R}^n)$ is admissible, then we get the inversion formula given in (1.3).

Proof of Theorem 1 Suppose that

$$\lim_{(M^{-1}(a^j), z)\to(0, z_1)} (\mathcal{L}_h f)\big(M^{-1}(a^j), z\big) \equiv F(z_1) \tag{4.3}$$

exists for any z_1 in an open neighborhood containing the closed ball $\overline{B_R(0)}$, with $R > 0$.

On the other hand, for fixed x in $\overline{B_R(0)}$ and y in $\mathbb{R}^n$, let

$$\mathcal{I}\bigl(M^{-1}(a^j),x,y-M(b)k\bigr) = \begin{cases} h(-y+M(b)k)(\mathcal{L}_h f)(M^{-1}(a^j),x+M^{-1}(a^j)(y-M(b)k)) & \text{if } j\nrightarrow\infty, \\ h(-y+M(b)k)F(x) & \text{if } j\to\infty. \end{cases} \tag{4.4}$$

Note that for such x in $\overline{B_R(0)}$, the function $\mathcal{I}$ is well defined for all $M(a^j)$ in $M_n(\mathbb{R})$ and $y-M(b)k$ in $\mathbb{R}^n$. Then we have the following three claims (see Appendix for the proofs).

Claim 4.1 *The function $\mathcal{I}$ is continuous on $M_n(\mathbb{R})\times\overline{B_R(0)}\times\mathbb{R}^n$.*

Claim 4.2 *For fixed x in $\overline{B_R(0)}$, the double series*

$$\sum_{j\in\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n}\mathcal{I}\bigl(M^{-1}(a^j),x,y-M(b)k\bigr)$$

converges uniformly on $\overline{B_R(0)}$.

Claim 4.3 *For fixed x in $\overline{B_R(0)}$ and $(M^{-1}(a^j),y-M(b)k)$ in $M_n(\mathbb{R})\times\mathbb{R}^n$, let*

$$w(x)=\sum_{j\in\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n}\mathcal{I}\bigl(M^{-1}(a^j),x,y-M(b)k\bigr).$$

Then the function w is continuous at $x=0$.

Coming back to the proof of Theorem 1, for any $p=(p_1,p_2,\dots,p_n)\in Z^n$ with $p_i\geq 0$ and any x in $\mathbb{R}^n$, define

$$U_p(x)=\sum_{j_1=-p_1}^{p_1}\cdots\sum_{j_n=-p_n}^{p_n}\sum_{k\in\mathbb{Z}^n}(L_h f)\bigl(M^{-1}(a^j),M^{-1}(a^j)M(b)k\bigr) \cdot\sqrt{\det M(a^j)}\,h\bigl(M(a^j)x-M(b)k\bigr).$$

Then by Claim 4.3, for any $x\in\mathbb{R}^n$,

$$\lim_{p\to\infty}U_p(x)=w(x).$$

That is, $U_p\to w$ pointwise as $p\to\infty$.

On the other hand, by (1.3) we have $U_p\to Af$ weakly on $L^2(M_n(\mathbb{R})\times\mathbb{R}^n)$. Hence, $f=A^{-1}w$ almost everywhere. Thus, by Claim 4.3 the function f is continuous at $x=0$.

This completes the proof of Theorem 1. □

5 Examples

Example of Lemma 2. Consider $h\in C_0^\infty(\mathbb{R}^n)$ a radially symmetric admissible function such that $h\neq 0$ and $\widehat{h}(0)=0$. Since $h\in C_0^\infty(\mathbb{R}^n)$, there is $L>0$ such that $\operatorname{supp}h\subset B_L(0)$.

Now, take $a=2$, $M(b)=I\in M_n(\mathbb{R})$, $\mathbf{X}=(x_1,\dots,x_n)\in\mathbb{R}^n$, $j=(j_1,\dots,j_n)\in\mathbb{Z}^n$, and $k=(k_1,\dots,k_n)\in\mathbb{Z}^n$.

Let us consider $f(x_1,\dots,x_n) = \sum_{i=1}^n x_i^2 + \sum_{i=1}^n x_i + c$ with $c \neq 0$. Note that f is continuous at $(0,\dots,0)$.

Now, by (3.6),

$$(\mathcal{L}_h f)\big(M^{-1}(2^j), M^{-1}(2^j)Ik\big) = \int_{B_L(0)} f\big(M^{-1}(2^j)x + M^{-1}(2^j)Ik\big)\overline{h(x)}\,dx = \int_{B_L(0)} f\big(M^{-1}(2^j)(x+k)\big)\overline{h(x)}\,dx.$$

Since

$$f\big(M^{-1}(2^j)(x+k)\big) = f\left(\frac{x_1+k_1}{2^{j_1}},\dots,\frac{x_n+k_n}{2^{j_n}}\right) = \sum_{i=1}^n \left(\frac{x_i+k_i}{2^{j_i}}\right)^2 + \sum_{i=1}^n \frac{x_i+k_i}{2^{j_i}} + c,$$

we have

$$\begin{aligned}&(\mathcal{L}_h f)\big(M^{-1}(2^j), M^{-1}(2^j)Ik\big)\\ &= \sum_{i=1}^n \frac{1}{(2^{j_i})^2}\int_{B_L(0)} x_i^2\overline{h(\mathbf{X})}\,d\mathbf{X} + \sum_{i=1}^n \frac{2k_i}{(2^{j_i})^2}\int_{B_L(0)} x_i\overline{h(\mathbf{X})}\,d\mathbf{X} + \sum_{i=1}^n \frac{k_i^2}{(2^{j_i})^2}\int_{B_L(0)} \overline{h(\mathbf{X})}\,d\mathbf{X}\\ &\quad + \sum_{i=1}^n \frac{1}{2^{j_i}}\int_{B_L(0)} x_i\overline{h(\mathbf{X})}\,d\mathbf{X} + \sum_{i=1}^n \frac{k_i}{2^{j_i}}\int_{B_L(0)} \overline{h(\mathbf{X})}\,d\mathbf{X} + c\int_{B_L(0)} \overline{h(\mathbf{X})}\,d\mathbf{X}.\end{aligned}$$

Because of $\widehat{h}(0) = 0$ and $h \in C_0^\infty(\mathbb{R}^n)$,

$$\lim_{(j\to\infty)} (\mathcal{L}_h f)\big(M^{-1}(2^j), M^{-1}(2^j)Ik\big) = 0,$$

which is consistent with the result given in Lemma 2.

Example of main theorem. Suppose that $h \in C_0^\infty(\mathbb{R}^2)$ is a radially symmetric admissible function with $\widehat{h}(0) = 0$ and $h \neq 0$.

Take $a = 2$ and $M(b) = I \in M_n(\mathbb{R})$ and consider $f(x_1,x_2) = \frac{1}{|x_1|}$, which is not continuous at $(0,0)$.

Since $h \in C_0^\infty(\mathbb{R}^2)$, there is $L > 0$ such that $\operatorname{supp} h \subset B_L(0)$.

Now, by (3.6),

$$(\mathcal{L}_h f)\big(M^{-1}(2^j), M^{-1}(2^j)Ik\big) = \int_{B_L(0)} f\big(M^{-1}(2^j)x + M^{-1}(2^j)Ik\big)\overline{h(x)}\,dx = \int_{B_L(0)} f\big(M^{-1}(2^j)(x+k)\big)\overline{h(x)}\,dx.$$

Note that $f(M^{-1}(2^j)(x+k)) = f(\frac{x_1+k_1}{2^{j_1}}, \frac{x_2+k_2}{2^{j_2}}) = \frac{2^{j_1}}{|x_1+k_1|}$.

Then

$$(\mathcal{L}_h f)\big(M^{-1}(2^j), M^{-1}(2^j)Ik\big) = 2^{j_1}\int\int_{B_L(0)} \frac{1}{|x_1+k_1|}\overline{h(x_1,x_2)}\,dx_1\,dx_2.$$

Choose $k = (k_1, k_2) \in \mathbb{Z}^2$ with $k_1 > L$ such that

$$0 < \int\int_{B_L(0)} \frac{1}{|x_1 + k_1|} \overline{h(x_1, x_2)}\, dx_1\, dx_2 < \infty.$$

Then

$$\lim_{(j\to\infty)} (\mathcal{L}_h f)\bigl(M^{-1}(2^j), M^{-1}(2^j)Ik\bigr)$$

does not exist for $j = (j_1, j_2)$.

This is consistent with the result given in Theorem 1.

Appendix

Proof of Claim 4.1 Case (1) If $M^{-1}(a^j) \nrightarrow 0$, then

$$\begin{aligned}&\mathcal{I}\bigl(M^{-1}(a^j), x, y - M(b)k\bigr)\\ &\quad = h\bigl(-y + M(b)k\bigr)(\mathcal{L}_h f)\bigl(M^{-1}(a^j), x + M^{-1}(a^j)\bigl(y - M(b)k\bigr)\bigr).\end{aligned}$$

Now, since $f \in L^2(\mathbb{R}^n)$ and $h \in C_0^\infty(\mathbb{R}^n)$, it follows that $(J_{M^{-1}(a^j)}\overline{h})^\sim * f$ is of class C^∞ [12]. Then, by (3.3), the function $\mathcal{I}$ is continuous on $M_n(\mathbb{R}) \times \overline{B_R(0)} \times \mathbb{R}^n$.

Case (2) If $M^{-1}(a^j) \to 0$, then by (4.3), for any x_1 in $\overline{B_R(0)}$,

$$\begin{aligned}&\lim_{(M^{-1}(a^j),x,y)\to(0,x_1,y_1)} \mathcal{I}\bigl(M^{-1}(a^j), x, y - M(b)k\bigr)\\ &\quad = \lim_{(M^{-1}(a^j),x,y)\to(0,x_1,y_1)} h\bigl(-y + M(b)k\bigr)(\mathcal{L}_h f)\bigl(M^{-1}(a^j), x + M^{-1}(a^j)\bigl(y - M(b)k\bigr)\bigr)\\ &\quad = h\bigl(-y_1 + M(b)k\bigr) \lim_{(M^{-1}(a^j),z)\to(0,x_1)} (\mathcal{L}_h f)\bigl(M^{-1}(a^j), z\bigr)\\ &\quad = h\bigl(-y_1 + M(b)k\bigr)F(x_1) = \mathcal{I}\bigl(0, x_1, y_1 - M(b)k\bigr).\end{aligned}$$

This completes the proof of Claim 4.1. □

Proof of Claim 4.2 In the case that $j \nrightarrow \infty$, from (4.4) and (3.4) we have

$$\begin{aligned}&\mathcal{I}\bigl(M^{-1}(a^j), x, y - M(b)k\bigr)\\ &\quad = h\bigl(-y + M(b)k\bigr)\sqrt{\det M\bigl(a^j\bigr)}(L_h f)\bigl(M^{-1}(a^j), x + M^{-1}(a^j)\bigl(y - M(b)k\bigr)\bigr).\end{aligned}$$

Then following (1) of equation (2.6), we have

$$\begin{aligned}&\bigl|\mathcal{I}\bigl(M^{-1}(a^j), x, y - M(b)k\bigr)\bigr|\\ &\quad = \bigl|h\bigl(-y + M(b)k\bigr)\bigr|\sqrt{\det M\bigl(a^j\bigr)}\bigl|\langle f, T_{x+M^{-1}(a^j)(y-M(b)k)} J_{M^{-1}(a^j)} h\rangle\bigr|\\ &\quad \le \bigl|h\bigl(-y + M(b)k\bigr)\bigr|\sqrt{\det M\bigl(a^j\bigr)}\|f\|_2\|h\|_2.\end{aligned}$$

Now, for $p = (p_1, p_2, \ldots, p_n) \in \mathbb{Z}_+^n$ and $j = (j_1, j_2, \ldots, j_n) \in Z^n$, define, for $i = 1, 2, \ldots, n$,

$$G\bigl(M^{-1}\bigl(a^j\bigr), y - M(b)k\bigr) = \begin{cases} |\mathcal{I}(M^{-1}(a^j), x, y - M(b)k)| & \text{if } j_i \in [-p_i, p_i], \\ |h(-y + M(b)k)|\sqrt{\det M(a^j)}\|f\|_2\|h\|_2 & \text{if } j_i \notin [-p_i, p_i]. \end{cases}$$

Then, for all $(M^{-1}(a^j), y - M(b)k) \in M_n(\mathbb{R}) \times \mathbb{R}^n$,

$$\bigl|\mathcal{I}\bigl(M^{-1}\bigl(a^j\bigr), x, y - M(b)k\bigr)\bigr| \le G\bigl(M^{-1}\bigl(a^j\bigr), y - M(b)k\bigr). \tag{A.1}$$

Then

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n}\bigl|G\bigl(M^{-1}\bigl(a^j\bigr), y - M(b)k\bigr)\bigr| \\
&\quad = \left(\sum_{j_1=-\infty}^{-p_1-1} + \sum_{j_1=-p_1}^{p_1} + \sum_{j_1=p_1+1}^{\infty}\right)\left(\sum_{j_2=-\infty}^{-p_2-1} + \sum_{j_2=-p_2}^{p_2} + \sum_{j_2=p_2+1}^{\infty}\right)\cdots\left(\sum_{j_n=-\infty}^{-p_n-1} + \sum_{j_n=-p_n}^{p_n} + \sum_{j_n=p_n+1}^{\infty}\right) \\
&\qquad \sum_{k\in\mathbb{Z}^n}\bigl|G\bigl(M^{-1}\bigl(a^j\bigr), y - M(b)k\bigr)\bigr|.
\end{aligned} \tag{A.2}$$

Note that for the case that $j \nrightarrow \infty$, it follows that $j_i \nrightarrow +\infty$ for $i = 1, 2, \ldots, n$. Thus, we will consider only summations over j_i from $-\infty$ to p_i.

Now, with respect to the sum over $k \in \mathbb{Z}^n$, since $\operatorname{supp} h$ is compact, there is $L > 0$ such that $\operatorname{supp} h \subset B_L(0)$. So, there is a positive integer $N > L$ such that $h(-y + M(b)k) = 0$ for $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$ with $k_i > N$ for $i = 1, 2, \ldots, n$. Hence, we can consider the series over $k_i \in \mathbb{Z}$ only from $-N$ to N.

Thus, if we set

$$H := \left(\sum_{k_1=-N}^{N}\cdots\sum_{k_n=-N}^{N}\right)\bigl|h\bigl(-y + M(b)k\bigr)\bigr|\sqrt{\det M\bigl(a^j\bigr)}\|f\|_2\|h\|_2$$

and

$$I := \left(\sum_{k_1=-N}^{N}\cdots\sum_{k_n=-N}^{N}\right)\bigl|\mathcal{I}\bigl(M^{-1}\bigl(a^j\bigr), x, y - M(b)k\bigr)\bigr|,$$

then equation (A.2) can be written as

$$\begin{aligned}
&\sum_{j\in Z^n}\sum_{k\in Z^n}\bigl|G\bigl(M^{-1}\bigl(a^j\bigr), y - M(b)k\bigr)\bigr| \\
&\quad = \left(\sum_{j_1=-\infty}^{-p_1-1}\sum_{j_2=-\infty}^{-p_2-1}\cdots\sum_{j_n=-\infty}^{-p_n-1}\right)H \\
&\qquad + \left(\sum_{j_1=-\infty}^{-p_1-1}\sum_{j_2=-\infty}^{-p_2-1}\cdots\sum_{j_n=-p_n}^{p_n}\right)H + \cdots + \left(\sum_{j_1=-p_1}^{p_1}\sum_{j_2=-\infty}^{-p_2-1}\cdots\sum_{j_n=-\infty}^{-p_n-1}\right)H \\
&\qquad + (\text{sequence of series that contain two series from } j_i = -p_i \text{ to } p_i)H
\end{aligned}$$

$$
\begin{aligned}
&+ (\text{sequence of series that contain three series from } j_i = -p_i \text{ to } p_i)H \\
&\vdots \\
&+ (\text{sequence of series that contain } n-2 \text{ series from } j_i = -p_i \text{ to } p_i)H \\
&+ \left(\sum_{j_1=-p_1}^{p_1} \sum_{j_2=-p_2}^{p_2} \cdots \sum_{j_n=-\infty}^{-p_n-1}\right)H + \cdots + \left(\sum_{j_1=-\infty}^{-p_1-1} \sum_{j_2=-p_2}^{p_2} \cdots \sum_{j_n=-p_n}^{p_n}\right)H \\
&+ \left(\sum_{j_1=-p_1}^{p_1} \sum_{j_2=-p_2}^{p_2} \cdots \sum_{j_n=-p_n}^{p_n}\right)I.
\end{aligned}
\tag{A.3}
$$

Now, let $S = \operatorname{Sup} |h(-y + M(b)k)|$ where $-L \le -y + M(b)k \le L$, and since $a > 1$, it follows that the double series given in the first term in (A.3) given by

$$
\begin{aligned}
&\left(\sum_{j_1=-\infty}^{-p_1-1} \sum_{j_2=-\infty}^{-p_2-1} \cdots \sum_{j_n=-\infty}^{-p_n-1}\right)H \\
&\quad = \left(\sum_{j_1=-\infty}^{-p_1-1} \cdots \sum_{j_n=-\infty}^{-p_n-1}\right)\left(\sum_{k_1=-N}^{N} \cdots \sum_{k_n=-N}^{N}\right)\sqrt{\det M(a^j)}\,\|f\|_2\|h\|_2\bigl|h(-y + M(b)k)\bigr|
\end{aligned}
$$

converges to

$$
\|f\|_2\|h\|_2 S(2N+1)^n \left(\sum_{j_1=-\infty}^{-p_1-1} \cdots \sum_{j_n=-\infty}^{-p_n-1}\right)\sqrt{a^{j_1}a^{j_2}\cdots a^{j_n}}.
$$

Using the same argument, we can prove that the remaining sequences of series that contain at least one index j_i from $j_i = -\infty$ to $-p_i - 1$ also converge.

On the other hand, note that, by Claim 4.1, the function $\mathcal{I}$ is continuous. Thus, the double series in the last term in (A.3) given by

$$
\begin{aligned}
&\left(\sum_{j_1=-p_1}^{p_1} \sum_{j_2=-p_2}^{p_2} \cdots \sum_{j_n=-p_n}^{p_n}\right)I \\
&\quad = \left(\sum_{j_1=-p_1}^{p_1} \sum_{j_2=-p_2}^{p_2} \cdots \sum_{j_n=-p_n}^{p_n}\right)\left(\sum_{k_1=-N}^{N} \cdots \sum_{k_n=-N}^{N}\right)\bigl|\mathcal{I}(M^{-1}(a^j), x, y - M(b)k)\bigr|
\end{aligned}
$$

also converges.

Thus,

$$
\sum_{j\in\mathbb{Z}^n} \sum_{k\in Z^n} G(M^{-1}(a^j), y - M(b)k) \quad \text{converges.}
$$

Hence, for fixed $x \in \overline{B_R(0)}$, we get from (A.1) that

$$
\sum_{j\in\mathbb{Z}^n} \sum_{k\in Z^n} \mathcal{I}(M^{-1}(a^j), x, y - M(b)k) \quad \text{converges absolutely and uniformly on } \overline{B_R(0)}.
$$

This proves Claim 4.2. □

Proof of Claim 4.3 By Claim 4.1 the function $\mathcal{I}$ is continuous on $M_n(\mathbb{R}) \times \overline{B_L(0)} \times \mathbb{R}^n$, and by Claim 4.2 the double series

$$\sum_{j \in \mathbb{Z}^n} \sum_{k \in \mathbb{Z}^n} \mathcal{I}\bigl(M^{-1}\bigl(a^j\bigr), x, y - M(b)k\bigr)$$

converges uniformly on $\overline{B_L(0)}$. In particular, for $x = 0$, the double series

$$\sum_{j \in \mathbb{Z}^n} \sum_{k \in \mathbb{Z}^n} \mathcal{I}\bigl(M\bigl(a^j\bigr), 0, y - M(b)k\bigr)$$

converges absolutely and uniformly on $\overline{B_L(0)}$.

Hence,

$$\begin{aligned} \lim_{x \to 0} w(x) &= \sum_{j \in \mathbb{Z}^n} \sum_{k \in \mathbb{Z}^n} \lim_{x \to 0} \mathcal{I}\bigl(M^{-1}\bigl(a^j\bigr), x, y - M(b)k\bigr) \\ &= \sum_{j \in \mathbb{Z}^n} \sum_{k \in \mathbb{Z}^n} \mathcal{I}\bigl(M^{-1}\bigl(a^j\bigr), 0, y - M(b)k\bigr) = w(0). \end{aligned}$$

This proves Claim 4.3. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The two authors jointly worked on deriving the results and approved the final manuscript.

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