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Partially-shared pessimistic bilevel multi-follower programming: concept, algorithm, and application

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Abstract

When multiple followers are involved in a bilevel programming problem, the leader's decision will be affected by the reactions of these followers. For actual problems, the leader in general cannot obtain complete information from the followers so that he may be risk-averse. Then he would need a safety margin to bound the damage resulting from the undesirable selections of the followers. This situation is called a pessimistic bilevel multi-follower (PBLMF) programming problem. This research considers a partially-shared linear PBLMF programming in which there is a partially-shared variable among the followers. The concept and solution algorithm of such a problem are developed. As an illustration, the partially-shared linear PBLMF programming model is applied to a company making venture investments.

Keywords: bilevel programming; pessimistic formulation; multiple followers

1 Introduction

Bilevel programming plays an exceedingly important role in different application fields, such as transportation, economics, ecology, engineering and others; see [1] and the references therein. It has been developed and researched by many authors; *e.g.*, see the monographs [2–5].

When the set of solutions of the lower level problem does not reduce to a singleton, the leader can hardly optimize his choice unless he knows the follower's reaction to his choice. In this situation, at least two approaches have been suggested: optimistic (or strong) formulation and pessimistic (or weak) formulation [3, 6, 7]. The pessimistic bilevel programming problem is very difficult [8]. As a result, most research on bilevel programming focuses on the optimistic formulation. Interested readers can refer to [1, 9] and the references therein.

This research focuses on the concept, algorithms and applications of the pessimistic bilevel programming problem. Several relative studies are reviewed for existing results of solutions and approximations results for pessimistic bilevel programming; see [10–15]. For papers discussing optimality conditions, see [16, 17]. Recently, Wiesemann *et al.* [8] analyzed the structural properties and presented a solvable ϵ-approximation algorithm for the independent pessimistic bilevel programming problem. Based on an exact penalty function, Zheng *et al.* [18] proposed an algorithm for the pessimistic linear bilevel program-

ming problem. Červinka, Matonoha and Outrata [19] developed a new numerical method to compute approximate and so-called relaxed pessimistic solutions to mathematical programming with equilibrium constraints which is a generalized bilevel programming problem.

As is well known, most theoretical and algorithmic contributions to bilevel programming are limited to a specific situation with one leader and one follower. For the actual bilevel programming problems, however, the lower level problem often involves multiple decision makers. For example, in a university, the dean of a faculty is the leader, and aims to minimize the faculty annual budget. All the heads of departments in the faculty are the followers whose aims are maximizing their respective annual budget. The leader chooses an optimal strategy knowing how the followers will react. This is a typical bilevel multi-follower (BLMF) programming problem. Note that the research on BLMF has been concentrated in its optimistic formulation. For example, Calvete and Galé [20] discussed the linear BLMF with independent followers and transformed such a problem into a linear bilevel problem with one leader and one follower. Lu *et al.* [21] generalized a framework for a special kind of BLMF, and identified nine main types of relations among followers. Lu *et al.* [22] considered a trilevel multi-follower programming problem, and analyzed various kinds of relations between decision entities. However, some practical problems need to be modeled as a partially-shared pessimistic BLMF programming model. Let us consider a simple example as follows.

Example 1.1 Assume that a company (*i.e.*, leader) undertakes M projects which will be performed by M construction teams (*i.e.*, followers), respectively. In addition to his own resources, each team usually need to use some shared resources, such as piling machines and cranes, in the company. Furthermore, assume that the construction teams are competitive, and the company cannot obtain complete information from these teams. Then the company may be risk-averse, and consequently, he protects himself against the possible worst choice of the teams. That is, his aim is to minimize the worst-case cost. Let the cost function of the corporation be $H(x, y_1, y_2, \ldots, y_M, z)$. The cost function of the ith construction team is $f_i(x, y_i, z)$ and the ith team subjects constraints are $G_i(x, y_i, z) \leq 0$ in which z is a shared resource among teams. Then the model is given as follows:

$$\min_{x} \sup_{y_1, y_2, \ldots, y_M, z} H(x, y_1, y_2, \ldots, y_M, z),$$

where (y_i, z) is a solution of the ith team's problem ($i = 1, 2, \ldots, M$)

$$\min_{y_i, z} f_i(x, y_i, z)$$

$$\text{s.t.} \quad G_i(x, y_i, z) \leq 0.$$

Note that the above problem cannot be modeled from the existing approaches. To model such a problem, the proposed study considers a partially-shared PBLMF programming problem. The main contributions of this study are three-fold: (i) the concept of a solution of the general PBLMF programming problem is presented and the related existence theorem is established; (ii) a simple algorithm based on penalty function is developed for

solving a partially-shared linear PBLMF programming problem; and (iii) we apply the proposed partially-shared PBLMF programming problem to a company making venture investments.

The paper is organized as follows. In the next section, the concept of a partially-shared PBLMF programming problem is introduced, and an equivalently penalty problem inspired from [10, 23–25] is given. In Section 3, we analyze the relationships between the original problem and its penalty problem, and then present a solution algorithm. To illustrate the feasibility and rationality of the proposed partially-shared linear PBLMF programming model, an example of venture investments is proposed in Section 4. Finally, concluding remarks are provided in Section 5.

2 Concept and penalty function of partially-shared linear PBLMF

Consider the following partially-shared linear PBLMF programming problem in which $M \geq 2$ followers are involved and there is a partially shared decision variable z among followers:

$$\min_{x \in X} \sup_{(y_i, z) \in \Psi_i(x)} \left[c^T x + \sum_{i=1}^{M} d_i^T y_i + s^T z \right], \tag{1}$$

where $\Psi_i(x)$ is the set of solutions of the ith follower's problem

$$\min_{y_i, z} u_i^T y_i + v_i^T z$$

$$\text{s.t.} \quad A_i x + B_i y_i + C_i z \leq b_i,$$

$$y_i, z \geq 0.$$

Here, $x, c \in \mathbb{R}^n$, $y_i, d_i, u_i \in \mathbb{R}^{m_i}$, $s, z, v_i \in \mathbb{R}^l$, $A_i \in \mathbb{R}^{q_i \times n}$, $B_i \in \mathbb{R}^{q_i \times m_i}$, $C_i \in \mathbb{R}^{q_i \times l}$, $b_i \in \mathbb{R}^{q_i}$, $i = 1, 2, \ldots, M$, X is a closed subset of $\mathbb{R}^n$, and T stands for transpose.

Definition 1

(a) Constraint region of problem (1):

$$S = \left\{ (x, y_1, y_2, \ldots, y_M, z) : x \in X, A_i x + B_i y_i + C_i z \leq b_i, y_i, z \geq 0, i = 1, 2, \ldots, M \right\}.$$

(b) Projection of S onto the leader's decision space:

$$S(X) = \left\{ x \in X : \exists (y_1, y_2, \ldots, y_M, z), \text{such that } (x, y_1, y_2, \ldots, y_M, z) \in S \right\}.$$

(c) Feasible set for the ith follower $\forall x \in S(X)$:

$$S_i(x) = \left\{ (y_i, z) : B_i y_i + C_i z \leq b_i - A_i x, y_i, z \geq 0 \right\}.$$

(d) The ith follower's rational reaction set for $x \in S(X)$:

$$\Psi_i(x) = \left\{ (y_i, z) : (y_i, z) \in \text{Arg} \min \left[u_i^T y_i + v_i^T z : (y_i, z) \in S_i(x) \right] \right\}.$$

(e) Inducible region or feasible region of the leader:

$$IR = \{(x, y_1, y_2, \ldots, y_M, z) : (x, y_1, y_2, \ldots, y_M, z) \in S, (y_i, z) \in \Psi_i(x), i = 1, 2, \ldots, M\}.$$

To introduce the concept of a solution of problem (1) (also called pessimistic solution), one usually employs the following value function $\varphi(x)$:

$$\varphi(x) := c^T x + \sup_{(y_i, z) \in \Psi_i(x)} \left[\sum_{i=1}^{M} d_i^T y_i + s^T z \right].$$

Definition 2 A point $(x^*, y_1^*, y_2^*, \ldots, y_M^*, z^*) \in IR$ is called a pessimistic solution to problem (1), if

$$\varphi(x^*) = c^T x^* + \sum_{i=1}^{M} d_i^T y_i^* + s^T z^*,$$

$$\varphi(x^*) \leq \varphi(x), \quad \forall (x, y_1, y_2, \ldots, y_M, z) \in IR.$$

For the sake of simplicity, this study only considers a special case of $M = 2$ in problem (1), *i.e.*,

$$\min_{x \in X} \sup_{(y_i, z) \in \Psi_i(x)} \left[c^T x + d_1^T y_1 + d_2^T y_2 + s^T z \right], \tag{2}$$

where $\Psi_i(x)$ is the set of solutions of the ith follower's problem

$$\min_{y_i, z} u_i^T y_i + v_i^T z$$
$$\text{s.t.} \quad A_i x + B_i y_i + C_i z \leq b_i, \tag{3}$$
$$y_i, z \geq 0.$$

For each $x \in S(X)$, denote by $f(x)$ the optimal value of the following problem $P(x)$:

$$\sup_{(y_i, z) \in \Psi_i(x)} \left[d_1^T y_1 + d_2^T y_2 + s^T z \right].$$

Then problem (2) is equivalently transformed into the following problem P:

$$\min_{x \in S(X)} \left[c^T x + f(x) \right].$$

The dual problem of (3) is written as

$$\max_{w_i \geq 0} -(b_i - A_i x)^T w_i$$
$$\text{s.t.} \quad -B_i^T w_i \leq u_i,$$
$$-C_i^T w_i \leq v_i.$$

Denote by $\pi_i(x, y_i, w_i, z) = u_i^T y_i + v_i^T z + (b_i - A_i x)^T w_i$ the ith ($i = 1, 2$) follower's duality gap.

For $\rho > 0$, we now consider the following penalized problem $P_\rho(x)$:

$$\max_{y_1, y_2, w_1, w_2, z} \left\{ d_1^T y_1 + d_2^T y_2 + s^T z - \rho \sum_{i=1}^{2} \pi_i(x, y_i, w_i, z) \right\}$$

$$\text{s.t.} \quad -B_i^T w_i \leq u_i,$$

$$-C_i^T w_i \leq v_i, \tag{4}$$

$$A_i x + B_i y_i + C_i z \leq b_i,$$

$$y_i, w_i, z \geq 0, \quad i = 1, 2,$$

and denote the optimal value function by $f_\rho(x)$.

The dual problem of (4) is

$$\min_{t_1, t_2, \ldots, t_5} \left[u_1^T t_1 + u_2^T t_2 + v_1^T t_3 + v_2^T t_3 + (b_1 - A_1 x)^T t_4 + (b_2 - A_2 x)^T t_5 \right]$$

$$\text{s.t.} \quad -B_1^T t_4 \leq \rho u_1 - d_1,$$

$$-B_2^T t_5 \leq \rho u_2 - d_2,$$

$$-C_1^T t_4 - C_2^T t_5 \leq \rho v_1 + \rho v_2 - s, \tag{5}$$

$$B_1 t_1 + C_1 t_3 \leq \rho(b_1 - A_1 x),$$

$$B_2 t_2 + C_2 t_3 \leq \rho(b_2 - A_2 x),$$

$$t_i \geq 0, \quad i = 1, 2, \ldots, 5.$$

Furthermore, for each $x \in S(X)$, if $f_\rho(x)$ exists, then it is also the optimal value function of problem (5).

Finally, we find two penalized problems of problem P as follows.

Problem P_ρ:

$$\min_{x, t_1, t_2, \ldots, t_5} \left[c^T x + u_1^T t_1 + u_2^T t_2 + v_1^T t_3 + v_2^T t_3 + (b_1 - A_1 x)^T t_4 + (b_2 - A_2 x)^T t_5 \right]$$

$$\text{s.t.} \quad -B_1^T t_4 \leq \rho u_1 - d_1,$$

$$-B_2^T t_5 \leq \rho u_2 - d_2,$$

$$-C_1^T t_4 - C_2^T t_5 \leq \rho v_1 + \rho v_2 - s,$$

$$B_1 t_1 + C_1 t_3 \leq \rho(b_1 - A_1 x),$$

$$B_2 t_2 + C_2 t_3 \leq \rho(b_2 - A_2 x),$$

$$x \in X, \quad t_i \geq 0, \quad i = 1, 2, \ldots, 5.$$

Problem $\tilde{P}_\rho$:

$$\min_{x \in S(X)} \left[c^T x + f_\rho(x) \right].$$

The following section outlines the existence of solutions to problems $P_\rho(x), P_\rho, \tilde{P}_\rho$, and P, gives the relationships among them and presents a solution algorithm.

3 Algorithm of partially-shared linear PBLMF

In order to establish theoretical results, we state the main assumption throughout the paper.

Assumption (A) S is a non-empty compact polyhedron.

For convenience, we denote

$$
\begin{aligned}
Z_1(\rho) := \big\{(t_4, t_5) : &-B_1^T t_4 \le \rho u_1 - d_1, -B_2^T t_5 \le \rho u_2 - d_2, \\
&-C_1^T t_4 - C_2^T t_5 \le \rho v_1 + \rho v_2 - s, t_4, t_5 \ge 0 \big\},
\end{aligned}
$$

$$
\begin{aligned}
Z_2(\rho) := \big\{(x, t_1, t_2, t_3) : &B_1 t_1 + C_1 t_3 \le \rho(b_1 - A_1 x), \\
&B_2 t_2 + C_2 t_3 \le \rho(b_2 - A_2 x), x \in X, t_1, t_2, t_3 \ge 0 \big\},
\end{aligned}
$$

$$
Z_3(\rho, x) := \big\{(t_1, t_2, t_3) : \exists x \in X, \text{such that } (x, t_1, t_2, t_3) \in Z_2(\rho)\big\},
$$

$$
Z_4^i := \big\{ w_i : -B_i^T w_i \le u_i, -C_i^T w_i \le v_i, w_i \ge 0 \big\},
$$

$$
Z_5(x) := \big\{(y_1, y_2, z) : \exists x \in X, \text{such that } A_i x + B_i y_i + C_i z \le b_i, y_i, z \ge 0, i = 1, 2\big\}.
$$

In the sequel, denote by $V(\mathcal{A})$ the set of vertices of $\mathcal{A}$ for a set $\mathcal{A}$.

The following three lemmas provide the existence of solutions to problems $P_\rho(x)$, P_ρ, and $\tilde{P}_\rho$, respectively.

Lemma 3.1 *Under Assumption* (A), *for each $x \in S(X)$ and a fixed value of $\rho > 0$, problem $P_\rho(x)$ has at least one solution in $V(Z_5(x)) \times V(Z_4^1) \times V(Z_4^2)$.*

Proof For each $x \in S(X)$ and fixed $\rho > 0$, we have

$$
\sup_{(y_i, z) \in S_i(x), w_i \in Z_4^i} \left\{ d_1^T y_1 + d_2^T y_2 + s^T z - \rho \sum_{i=1}^{2} \pi_i(x, y_i, w_i, z) \right\}
$$

$$
\le \max_{(y_i, z) \in S_i(x)} \big[d_1^T y_1 + d_2^T y_2 + s^T z \big].
$$

It follows from Assumption (A) that the objective function of the linear programming problem $P_\rho(x)$ is bounded from above, and hence it has at least one solution in $V(Z_5(x)) \times V(Z_4^1) \times V(Z_4^2)$. This completes the proof. □

Lemma 3.2 *Under Assumption* (A), *for a fixed value of $\rho > 0$, problem P_ρ has at least one solution in $V(Z_2(\rho)) \times V(Z_1(\rho))$.*

Proof Clearly, problem P_ρ is a disjoint bilinear programming problem whose solution occurs at a vertex of its constraint region [26]. This completes the proof. □

Lemma 3.3 *Under Assumption* (A), *for a fixed value of $\rho > 0$, problem $\tilde{P}_\rho$ has at least one solution.*

Proof Under Assumption (A), it follows from Theorem 4.3 in [3] that $f_\rho(x)$ is continuous. Hence, the result follows immediately from the Weierstrass theorem. □

For any $\eta > 0$, let $Z_6 := \{(x, t) : Bt \le \eta(b - Ax)\}$ and $Z_7 := \{(x, y) : By \le b - Ax\}$. To prove Theorem 3.1, we first provide the following lemma.

Lemma 3.4 *For any $\eta > 0$, if $(x_\eta^*, t_\eta^*) \in V(Z_6)$, then there exists $(x^*, y^*) \in V(Z_7)$, such that $x^* = x_\eta^*$ and $t_\eta^* = \eta y^*$.*

Proof It is easy to verify that $(x_\eta^*, \frac{t_\eta^*}{\eta}) \in Z_7$. Let $(x^1, y^1), \ldots, (x^r, y^r)$ be the distinct vertices of Z_7. Since any point in Z_7 can be written as a convex combination of these vertices, let $(x_\eta^*, \frac{t_\eta^*}{\eta}) = \sum_{i=1}^{\hat{r}} \alpha^i(x^i, y^i)$, where $\sum_{i=1}^{\hat{r}} \alpha^i = 1$, $\alpha^i > 0$, $i = 1, \ldots, \hat{r}$, and $\hat{r} \le r$. Then we have

$$\left(x_\eta^*, t_\eta^*\right) = \sum_{i=1}^{\hat{r}} \alpha^i\left(x^i, \eta y^i\right). \tag{6}$$

Note that $(x^i, \eta y^i) \in Z_6$. Hence, (6) implies that $\hat{r} = 1$. Because (x_η^*, t_η^*) is a vertex of Z_6, a contradiction results unless $\hat{r} = 1$.

Therefore, there exists a point $(x^*, y^*) \in V(Z_7)$, such that $x^* = x_\eta^*$ and $t_\eta^* = \eta y^*$. This completes the proof. □

Next, the following result relates the solution between problems P_ρ and $\tilde{P}_\rho$.

Theorem 3.1 *Under Assumption* (A), *for a fixed value of $\rho > 0$, if $(x_\rho, t_1^\rho, t_2^\rho, t_3^\rho, t_4^\rho, t_5^\rho)$ is a solution of problem P_ρ, x_ρ solves problem $\tilde{P}_\rho$. Furthermore, $x_\rho \in Q := \{x : (x, y_1, y_2, z) \in V(Q^\dagger)\}$ where*

$$Q^\dagger := \left\{(x, y_1, y_2, z) : x \in X, A_i x + B_i y_i + C_i z \le b_i, y_i, z \ge 0, i = 1, 2\right\}.$$

Proof Denote the objective function of problem P_ρ by $F(x, t_1, t_2, t_3, t_4, t_5)$. Suppose that x_ρ^* solves problem $\tilde{P}_\rho$. Then there exist $(t_1^*, t_2^*, t_3^*) \in Z_3(\rho, x_\rho^*)$ and $(t_4^*, t_5^*) \in Z_1(\rho)$, such that

$$f_\rho\left(x_\rho^*\right) = u_1^T t_1^* + u_2^T t_2^* + v_1^T t_3^* + v_2^T t_3^* + \left(b_1 - A_1 x_\rho^*\right)^T t_4^* + \left(b_2 - A_2 x_\rho^*\right)^T t_5^*. \tag{7}$$

Moreover, $(x_\rho^*, t_1^*, t_2^*, t_3^*, t_4^*, t_5^*)$ is a feasible point of problem P_ρ.

Then we have

$$c^T x_\rho + f_\rho(x_\rho) \le F\left(x_\rho, t_1^\rho, t_2^\rho, t_3^\rho, t_4^\rho, t_5^\rho\right) \tag{8}$$

$$\le F\left(x_\rho^*, t_1^*, t_2^*, t_3^*, t_4^*, t_5^*\right) \tag{9}$$

$$= c^T x_\rho^* + f_\rho\left(x_\rho^*\right), \tag{10}$$

where (8) holds due to the definition of $f_\rho(x_\rho)$, (9) holds because of the optimality of $(x_\rho, t_1^\rho, t_2^\rho, t_3^\rho, t_4^\rho, t_5^\rho)$ and (10) follows from (7).

Thus, (8)-(10) implies that x_ρ is a solution of problem $\tilde{P}_\rho$.

Using the result of Lemma 3.2, we can obtain

$$\left(x_\rho, t_1^\rho, t_2^\rho, t_3^\rho\right) \in V\left(Z_2(\rho)\right).$$

Furthermore, by the result of Lemma 3.4, the definitions of $Z_2(\rho)$ and $Q^\dagger$, we find that $x_\rho \in Q$. This completes the proof. □

Note that $Q^\dagger$ can be referred to as the constraint region of problem (2) based on Definition 1(a).

Finally, we provide the following result which demonstrates that our penalty method is exact, and also presents the relationships between problems $\tilde{P}_\rho$ and P.

Theorem 3.2 *Let Assumption* (A) *hold, and* $\{x_\rho\}$ *be a sequence of solutions of problem* $\tilde{P}_\rho$. *Then there exists* $\rho^* > 0$, *such that for all* $\rho > \rho^*$, x_ρ *is a solution of problem P.*

Proof Let $(y_1(x), y_2(x), z(x), w_1(x), w_2(x)) \in V(Z_5(x)) \times V(Z_4^1) \times V(Z_4^2)$ be a solution of problem $P_\rho(x)$. For any $(\hat{y}_1, \hat{y}_2, \hat{z}, \hat{w}_1, \hat{w}_2) \in V(Z_5(x)) \times V(Z_4^1) \times V(Z_4^2)$, we have

$$f_\rho(x) = d_1^T y_1(x) + d_2^T y_2(x) + s^T z(x) - \rho \sum_{i=1}^{2} \pi_i\big(x, y_i(x), w_i(x), z(x)\big)$$

$$\geq d_1^T \hat{y}_1 + d_2^T \hat{y}_2 + s^T \hat{z} - \rho \sum_{i=1}^{2} \pi_i(x, \hat{y}_i, \hat{w}_i, \hat{z}). \tag{11}$$

In particular, choose $(\hat{y}_i, \hat{z})$ and $\hat{w}_i$ $(i = 1, 2)$, such that they are solutions of problem (3) and its dual problem respectively. Then we obtain

$$\sum_{i=1}^{2} \pi_i(x, \hat{y}_i, \hat{w}_i, \hat{z}) = 0.$$

Hence, we have

$$0 \leq \sum_{i=1}^{2} \pi_i\big(x, y_i(x), w_i(x), z(x)\big)$$

$$\leq \frac{d_1^T y_1(x) + d_2^T y_2(x) + s^T z(x) - d_1^T \hat{y}_1 - d_2^T \hat{y}_2 - s^T \hat{z}}{\rho}$$

$$\leq \frac{\|d_1^T y_1(x) + d_2^T y_2(x) + s^T z(x) - d_1^T \hat{y}_1 - d_2^T \hat{y}_2 - s^T \hat{z}\|_2}{\rho},$$

where $\|\cdot\|_2$ denotes the Euclidean norm. Moreover, it follows from Assumption (A) that there exists a constant $\delta > 0$, such that

$$\left\| d_1^T y_1(x) + d_2^T y_2(x) + s^T z(x) - d_1^T \hat{y}_1 - d_2^T \hat{y}_2 - s^T \hat{z} \right\|_2 \leq \delta.$$

We then find that

$$0 \leq \sum_{i=1}^{2} \pi_i\big(x, y_i(x), w_i(x), z(x)\big) \leq \frac{\delta}{\rho}.$$

For each $x \in S(X)$, the number of elements of the set $V(Z_5(x)) \times V(Z_4^1) \times V(Z_4^2)$ is finite, and then there exists $0 < \rho^*(x) < +\infty$, such that

$$\sum_{i=1}^{2} \pi_i \big(x, y_i(x), w_i(x), z(x) \big) = 0, \quad \forall \rho \geq \rho^*(x). \tag{12}$$

Since $S(X)$ is a bounded non-empty polyhedron, there exists a constant $\rho^* > 0$, such that

$$\sum_{i=1}^{2} \pi_i \big(x, y_i(x), w_i(x), z(x) \big) = 0, \quad \forall \rho \geq \rho^*, x \in S(X),$$

and then $(y_1(x), y_2(x), z(x))$ is a feasible point of problem $P(x)$. Hence, for any $x \in S(X)$, we have

$$d_1^T y_1(x) + d_2^T y_2(x) + s^T z(x) \leq f(x). \tag{13}$$

Moreover, for any $x \in S(X)$ and $\rho > 0$, it follows from the definitions of $f_\rho(x)$ and $f(x)$ that

$$f(x) \leq f_\rho(x). \tag{14}$$

Combining (11)-(14), we have

$$f(x) = f_\rho(x), \quad \forall \rho \geq \rho^*, x \in S(X). \tag{15}$$

Therefore, for all $\rho > \rho^*$, we find

$$c^T x_\rho + f(x_\rho) \leq c^T x_\rho + f_\rho(x_\rho) \tag{16}$$

$$\leq c^T x + f_\rho(x) \tag{17}$$

$$= c^T x + f(x), \tag{18}$$

where (16) and (18) follow from (15), and (17) holds because of the optimality of x_ρ. Equations (16)-(18) imply that x_ρ is a solution for problem P for all $\rho > \rho^*$. This concludes the proof. □

Combining the results of Theorems 3.1 and 3.2, we can characterize problem (2) as a particular kind of nonlinear programming problem whose solution is related to a vertex of the constraint region.

Theorem 3.3 *Under Assumption* (A), *there exists a solution* (x^*, y_1^*, y_2^*, z^*) *of problem* (2) *such that* $x^* \in Q$.

From the result of Theorem 3.3, we know that a solution of problem (2) may be related to a vertex of Q^+. Hence one possible way to find the solution would be to generate all vertices of Q^+ and test each one as a possible solution by maximizing $P(x)$ in (y_1, y_2, z) for

fixed x. That is, solving problem (2) is equivalent to finding x^* with $(y_1^*, z^*) \in \Psi_1(x^*)$ and $(y_2^*, z^*) \in \Psi_2(x^*)$ such that

$$c^T x^* + d_1^T y_1^* + d_2^T y_2^* + s^T z^* = \min_{1 \le i \le N} \left\{ c^T x_{[j]} + \max_{(y_i, z) \in \Psi_i(x_{[j]})} \left[d_1^T y_1 + d_2^T y_2 + s^T z \right] \right\},$$

where for $x_{[j]}$ $(j = 1, 2, \ldots, N)$ there exists a point $(x_{[j]}, y_1^{[j]}, y_2^{[j]}, z^{[j]})$, such that $(x_{[j]}, y_1^{[j]}, y_2^{[j]}, z^{[j]})$ are the N ordered basic feasible points for the following linear programming problem:

$$\min_{x, y_1, y_2, z} c^T x + d_1^T y_1 + d_2^T y_2 + s^T z$$

$$\text{s.t.} \quad A_1 x + B_1 y_1 + C_1 z \le b_1,$$

$$A_2 x + B_2 y_2 + C_2 z \le b_2,$$

$$x \in X, \qquad y_1, y_2, z \ge 0.$$

Rather than enumerate all vertices of the set Q^+ explicitly, we present the following algorithm which finds a solution to problem (2).

Algorithm

Step 0. Choose $\rho > 0$ and $\gamma > 1$.

Step 1. Solve problem P_ρ, and denote the solution by $(x^\rho, t_1^\rho, t_2^\rho, t_3^\rho, t_4^\rho, t_5^\rho)$.

Step 2. Solve problem $P_\rho(x^\rho)$, and denote the solution by $(y_1^\rho, y_2^\rho, w_1^\rho, w_2^\rho, z^\rho)$.

Step 3. If $\sum_{i=1}^2 \pi_i(x^\rho, y_i^\rho, w_i^\rho, z^\rho) = 0$, then stop, $(x^\rho, y_1^\rho, y_2^\rho, z^\rho)$ is a solution of problem (2). Otherwise, set $\rho = \gamma \rho$ and proceed to Step 1.

4 An application

In this section, we apply the proposed partially-shared PBLMF decision making model to a company making venture investments. Consider the investments with a CEO and the selected two departments of the company. All decision entities have individual objectives, constraints, and variables and do not cooperate with one another. The departments within the company are also in an uncooperative situation, but they need to use the same warehouse in this company. The CEO's decision takes the responses of the selected departments into consideration and aims to maximize the company's profit. At the same time, the departments fully consider the CEO's decision, and make a rational response to maximize their own profit. Under incomplete and asymmetric information, the CEO cannot directly observe both departments' effort and the inventory expense. Then it is difficult for the CEO to design an investment planning model that aims to maximize the company's profit (or minimize the company's cost). In this case, the CEO may want to create a safety margin to bound the damage resulting from undesirable selections of the two departments.

To model the above venture investment problem, the notations are introduced as follows:

(1) *The CEO (Leader)*

- Objective G_1:

    The aim is to maximize the company's profit.

- Variables (x_1, x_2):

x_1: How much is used for the CEO's investment in product 1.

x_2: How much is used for the CEO's investment in product 2.

- Constraint:

  $H_1 \leq 0$: The total investment cost for products 1 and 2.

(2) *Two selected departments (Followers)* Suppose that both department 1 (for product 1) and department 2 (for product 2) share the same warehouse.

  ◇ Department 1:

  - Objective G_{21}:

    The aim is to maximize his total profit which includes his own profit in product 1 and a fraction of the company's revenue.

  - Variables (y_1, z):

    y_1: The effort for completing product 1.

    z: The inventory expense for product 1.

  - Constraints:

    $H_{21} \leq 0$: The effort and cost of department 1 that links with the CEO's investments.

    $H_{22} \leq 0$: The maximum effort of department 1.

    $H_{23} \leq 0$: The maximum inventory expense in product 1.

  ◇ Department 2:

  - Objective G_{31}:

    The aim is to maximize his total profit which includes his own profit in product 2 and a fraction of the company's revenue.

  - Variables (y_2, z):

    y_2: The effort for completing product 2.

    z: The inventory expense for product 2.

  - Constraints:

    $H_{31} \leq 0$: The effort and cost of department 2 that links with the CEO's investments.

    $H_{32} \leq 0$: The maximum effort of department 2.

    $H_{33} \leq 0$: The maximum inventory expense in product 2.

Then a partially-shared linear PBLMF model of the company making venture investments is given as follows:

$$\max_{x_1, x_2} \min_{y_1, y_2, z} [3x_1 + 2.5x_2 - y_1 - 2y_2 + 2z + 5]$$

s.t. $x_1 + x_2 \leq 1$,

$x_1, x_2 \geq 0$,

$$\max_{y_1, z} \left[0.1(3x_1 + 2.5x_2 - y_1 - 2y_2 + 2z + 5) + 2.1y_1 - 1.2z \right]$$

$$\text{s.t.} \quad x_1 + x_2 - 2y_1 + z \geq 1,$$

$$0 \leq y_1 \leq 2, \tag{19}$$

$$0 \leq z \leq 1,$$

$$\max_{y_2,z}\left[0.2(3x_1 + 2.5x_2 - y_1 - 2y_2 + 2z + 5) + 2.4y_2 - 1.4z\right]$$

$$\text{s.t.} \quad x_1 - 2y_2 + z \geq 1,$$

$$0 \leq y_2 \leq 2,$$

$$0 \leq z \leq 1.$$

To use the proposed algorithm, we can equivalently transform problem (19) into the following problem:

$$\min_{x_1,x_2} \max_{y_1,y_2,z} \left[-3x_1 - 2.5x_2 + y_1 + 2y_2 - 2z - 5\right]$$

$$\text{s.t.} \quad x_1 + x_2 \leq 1,$$

$$x_1, x_2 \geq 0,$$

$$\min_{y_1,z}\left[0.1(-3x_1 - 2.5x_2 + y_1 + 2y_2 - 2z - 5) - 2.1y_1 + 1.2z\right]$$

$$\text{s.t.} \quad x_1 + x_2 - 2y_1 + z \geq 1,$$

$$0 \leq y_1 \leq 2, \tag{20}$$

$$0 \leq z \leq 1,$$

$$\min_{y_2,z}\left[0.2(-3x_1 - 2.5x_2 + y_1 + 2y_2 - 2z - 5) - 2.4y_2 + 1.4z\right]$$

$$\text{s.t.} \quad x_1 - 2y_2 + z \geq 1,$$

$$0 \leq y_2 \leq 2,$$

$$0 \leq z \leq 1.$$

Note that both (19) and (20) have the same solutions, and their optimal values are negatives of each other.

Choose $\rho = 1$ and $\gamma = 10$. The disjoint bilinear programming problems at Step 1 are solved by a commercial optimization software package BARON [27, 28]. By using the proposed algorithm, it is easy to see that $(x_1^*, x_2^*, y_1^*, y_2^*, z^*)^T = (0, 1, 0.5, 0, 1)^T$ is a solution of problem (19), and the optimal value is 9.

5 Conclusions

This study addresses a partially-shared linear PBLMF programming problem in which there is a partially-shared variable among followers. Furthermore, the study presents the concept and a solution algorithm for such a problem. Finally, the partially-shared PBLMF model is applied to a company making venture investments. For future research, it will be interesting to propose the modified intelligent algorithms for the partially-shared PBLMF model, to apply the model in various areas, and to explore the concept, algorithms, and applications of a PBLMF model in the referential-uncooperative situation.

Authors' contributions
YZ and ZZ conceived and designed the study. YZ wrote and edited the manuscript. ZZ and LY examined all the steps of the proofs in this research and gave some advice. All authors read and approved the final manuscript.

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