Chen et al. *Journal of Inequalities and Applications* (2016) 2016:10
DOI 10.1186/s13660-015-0950-7

Journal of Inequalities and Applications
a SpringerOpen Journal

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Asymptotic expansions of the error for hyper-singular integrals with an interval variable

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Abstract

In this paper, we present high accuracy quadrature formulas for hyper-singular integrals $\int_a^b g(x)q^\alpha(x,t)\,dx$, where $q(x,t) = |x-t|$ (or $x-t$), $t \in (a,b)$, and $\alpha \leq -1$ (or $\alpha < -1$). If $g(x)$ is $2m+1$ times differentiable on $[a,b]$, the asymptotic expansions of the error show that the convergence order is $O(h^{2\mu+1+\alpha})$ with $q(x,t) = |x-t|$ (or $x-t$) for $\alpha \leq -1$ (or $\alpha < -1$ and α being non-integer), and the error power is $O(h^\eta)$ with $q(x,t) = x-t$ for α being integers less than -1, where $\eta = \min(2\mu, 2\mu+2+\alpha)$ and $\mu = 1, \ldots, m$. Since the derivatives of the density function $g(x)$ in the quadrature formulas can be eliminated by means of the extrapolation method, the formulas can easily be applied to solving corresponding hyper-singular boundary integral equations. The reliability and efficiency of the proposed formulas in this paper are demonstrated by some numerical examples.

MSC: 45E99; 65D30; 65D32; 41A55

Keywords: hyper-singular integral; Euler-Maclaurin expansions; quadrature formulas; Hadamard finite part

1 Introduction

We consider the following hyper-singular integral with an interval variable $t \in (a,b)$:

$$\big(I(g)\big)(t) = \int_a^b g(x)q^\alpha(x,t)\,dx, \tag{1.1}$$

where $q(x,t) = |x-t|$ (or $x-t$), and $\alpha \leq -1$ (or $\alpha < -1$) is a real number. Equation (1.1) denotes the Hadamard finite part [1, 2] of the hyper-singular integral. Hyper-singular integrals have been extensively used to elasticity problems [1, 3], for example, the calculation of stresses. Hyper-singular integral operators also attracted attention such as in [4] on modulation spaces. Especially, in the boundary element method, the hyper-singular integrals have attracted considerable attention such as in [1, 3, 5]. The authors of [6] apply boundary integral equations for the solution of the electrostatic field problem floating potentials in industrial applications.

So far, many numerical methods have been proposed to evaluate the hyper-singular integral (1.1) for $\alpha = -2$. According to the quadrature rules based on interpolation trigonometric polynomials, Kim and Choi gave two quadrature formulas for evaluating (1.1) with

$\alpha = -2$ in [7], in which the cosine transform of variables and trigonometric polynomial interpolation at the practical abscissa were used, where a three-term recurrence relation was used to evaluate the quadrature weights. In [8], Huang *et al.* got the Euler-Maclaurin expansions of (1.1) with $-2 \leq \alpha < -1$ by a modified trapezoidal formula. In [9], Kabir *et al.* used the piecewise quadratic polynomial technique to solve integral equations with logarithmic, Cauchy, and hyper-singular integrals. Hui and Shia presented a Gaussian quadrature formula for (1.1) with $q(x,t) = |x-t|$ for $\alpha = -1$, where the classical orthogonal polynomials such as the Legendre and Chebyshev polynomials were used in [10]. In [11], the hyper-singular integral equations were applied to solving the flat crack problem. On the basis of Euler-Maclaurin expansions in [12], Sidi and Israeli got the quadrature formulas and the error asymptotic expansions of the integral (1.1) with $q(x,t) = x - t$ for $\alpha = -1$. In 1998, Monegato and Lyness obtained the Euler-Maclaurin expansions of (1.1) by the Mellin transform, as $t = 0$ and $\alpha < -1$.

The quadrature formulas in [2] are not valid for solving hyper-singular integral equations and are only valid at the endpoint of the integrand interval. In this paper, by generalizing the results of Monegato and Lyness in 1998, we extend the formulas to any interior point of the integrand interval, and we present high accuracy quadrature formulas for hyper-singular integrals $\int_a^b g(x)q^\alpha(x,t)\,dx$, where $q(x,t) = |x-t|$ (or $x-t$), $t \in (a,b)$, and $\alpha \leq -1$ (or $\alpha < -1$). If $g(x)$ is $2m+1$ times differentiable on $[a,b]$, the asymptotic expansions of the error show that: (i) when $\alpha \leq -1$ (or α is a non-integer less than -1), the convergence order is $O(h^{2\mu+1+\alpha})$ with $q(x,t) = |x-t|$ (or $x-t$), where $\mu = 1, \ldots, m$; (ii) when α is an integer less than -1, the error power is $O(h^\eta)$ with $q(x,t) = x-t$, where $\eta = \min(2\mu, 2\mu+2+\alpha)$ and $\mu = 1, \ldots, m$. Since the derivatives of the density function $g(x)$ in the quadrature formulas can be removed by means of the extrapolation method, the formulas can easily be applied to solving the corresponding hyper-singular boundary integral equations. Quadrature formulas can also be used to solve singular integral equations in its corresponding forms. The hybrid Gauss-trapezoidal quadrature rule [13] is one of many quadrature formulas. There are also several methods to solve hyper-singular boundary integral equations beside quadrature rules, such as potential theory [14], the Green function approach [15], and so on. As far as this paper is concerned, we only deal with hyper-singular integrals. Moreover, numerical results display the significance of these formulas proposed, finally.

This paper is organized as follows: in Section 2, we introduce the Euler-Maclaurin expansions for hyper-singular integrals of (2.1) at the end points of the integrand interval; in Section 3, we present high accuracy quadrature formulas for hyper-singular integrals (1.1) with an interval variable, and also we get their Euler-Maclaurin expansions; in Section 4, some numerical examples are tested. A few conclusions are drawn in Section 5.

2 Euler-Maclaurin expansions for integrals of (2.1) at end points

In this section, we will recall some notations and extend the results of [2]. In [2], Monegato and Lyness presented the expansion of integrals whose integrand function is singular or hyper-singular at the end points of the integrand interval $[0,1]$. We extend the results to any interior point of the integrand interval $[a,b]$.

We discuss the following integrals:

$$\int_a^b f(x,t)\,dx, f(x,t) = (x-a)^\alpha (b-x)^\gamma g(x), \tag{2.1}$$

where $g(x) \in C^m[a,b]$ and $\omega = \min(\alpha,\gamma) \leq -1$. Note that

$$f.p.\int_a^b f(x,t)\,dx = f.p.\int_a^b (x-a)^\alpha (b-x)^\gamma g(x)\,dx, \tag{2.2}$$

where $f.p.$ denotes the *Hadamard finite part* [2] of the integral. When $\omega = -1$, (2.2) is a singular integral. When $\omega < -1$, (2.2) is a hyper-singular integral. By using the results of [2], we derive the Euler-Maclaurin expansion of (2.1).

Lemma 2.1 *Assume that $g(x)$ is m times differentiable on $[a,b]$ and $f(x) = (x-a)^\alpha(b-x)^\gamma g(x)$, with $\omega = \min(\alpha,\gamma) \leq -1$. n is the number of nodes in the rules, and $h = \frac{b-a}{n}$. Then the following expansions hold.*

(i) *If α and γ are non-integer, we have*

$$\begin{aligned} &h\sum_{k=0}^{n-1} f\bigl(a+h(\beta+k)\bigr) - \sum_{k=0}^{m} \frac{h^{k+1+\alpha}}{k!}\zeta(-k-\alpha,\beta)g_0^{(k)}(a) \\ &\quad - \sum_{k=0}^{m} \frac{(-1)^k h^{k+1+\gamma}}{k!}\zeta(-k-\gamma,1-\beta)g_1^{(k)}(b) \\ &\quad = f.p.\int_a^b f(x)\,dx + \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} h^{1-p}\bigl(\tilde{F}_0(p)+\tilde{F}_2(p)\bigr)\zeta(p,\beta)\,dp. \end{aligned} \tag{2.3}$$

(ii) *If $\alpha = -l-1$, $l = 1,2,\ldots,$ and γ is non-integer, the formula is*

$$\begin{aligned} &h\sum_{k=0}^{n-1} f\bigl(a+h(\beta+k)\bigr) - \sum_{k=0,k\neq l}^{m} \frac{h^{k+1+\alpha}}{k!}\zeta(-k-\alpha,\beta)g_0^{(k)}(a) \\ &\quad - \sum_{k=0}^{m} \frac{(-1)^k h^{k+1+\gamma}}{k!}\zeta(-k-\gamma,1-\beta)g_1^{(k)}(b) \\ &\quad + \frac{g_0^{(l)}(a)}{l!}\psi(\beta) - \frac{g_0^{(l)}(a)}{l!}\ln\frac{1}{h} \\ &\quad = f.p.\int_a^b f(x)\,dx + \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} h^{1-p}\bigl(\tilde{F}_0(p)+\tilde{F}_2(p)\bigr)\zeta(p,\beta)\,dp. \end{aligned} \tag{2.4}$$

(iii) *When $\gamma = -l-1$, $l = 1,2,\ldots,$ and α is a non-integer, the expansion can be written*

$$\begin{aligned} &h\sum_{k=0}^{n-1} f\bigl(a+h(\beta+k)\bigr) - \sum_{k=0}^{m} \frac{h^{k+1+\alpha}}{k!}\zeta(-k-\alpha,\beta)g_0^{(k)}(a) \\ &\quad - \sum_{k=0,k\neq l}^{m} \frac{(-1)^k h^{k+1+\gamma}}{k!}\zeta(-k-\gamma,1-\beta)g_1^{(k)}(b) \\ &\quad + (-1)^l\frac{g_1^{(l)}(b)}{l!}\psi(\beta) - (-1)^l\frac{g_1^{(l)}(b)}{l!}\ln\frac{1}{h} \\ &\quad = f.p.\int_a^b f(x)\,dx + \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} h^{1-p}\bigl(\tilde{F}_0(p)+\tilde{F}_2(p)\bigr)\zeta(p,\beta)\,dp. \end{aligned} \tag{2.5}$$

(iv) *When* $\alpha = -l-1$, $\gamma = -s-1$, l *and* s *are integers, we have the form*

$$\begin{aligned}
&h\sum_{k=0}^{n-1} f\bigl(a+h(\beta+k)\bigr) - \sum_{k=0,k\neq l}^{m} \frac{h^{k+1+\alpha}}{k!}\zeta(-k-\alpha,\beta)g_0^{(k)}(a) \\
&\quad - \sum_{k=0,k\neq s}^{m} \frac{(-1)^k h^{k+1+\gamma}}{k!}\zeta(-k-\gamma,1-\beta)g_1^{(k)}(b) + \frac{g_0^{(l)}(a)}{l!}\psi(\beta) \\
&\quad - \frac{g_0^{(l)}(a)}{l!}\ln\frac{1}{h} + (-1)^s\frac{g_1^{(s)}(b)}{s!}\psi(1-\beta) - (-1)^s\frac{g_1^{(s)}(b)}{s!}\ln\frac{1}{h} \\
&= f.p.\int_a^b f(x)\,dx + \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} h^{1-p}\bigl(\tilde{F}_0(p)+\tilde{F}_2(p)\bigr)\zeta(p,\beta)\,dp, \qquad (2.6)
\end{aligned}$$

where $0<\beta<1$, $\zeta(p,\beta)=\sum_{k=0}^{\infty}(k+\beta)^{-p}$ $(\mathrm{Re}(p)>1)$ *is the Riemann zeta function,* $\tilde{F}_i(p) = \int_0^\infty f_i(x)x^{p-1}\,dx$ $(i=0,2)$ *is the Mellin transform, and the other functions are defined by* $\psi(\beta)=\Gamma'(\beta)/\Gamma(\beta)$, $g_0(x)=(b-x)^\gamma g(x)$, $g_1(x)=(x-a)^\alpha g(x)$, $c'\in[-m-\omega-2,-m-\omega-1]$, $f_0(x)=f(x)\nu(x;1/3,2/3)$, $f_1(x)=f(x)(1-\nu(x;1/3,2/3))$, *and* $f_2(x)=f_1(1-x)$. *We also define* $\nu(x,k_1,k_2)$ $(k_1<k_2)$ *such that* $\nu(x,k_1,k_2)$ *belongs to* $C^\infty(-\infty,\infty)$ *and*

$$\nu(x,k_1,k_2)=\begin{cases}1 & \text{for } x\le k_1,\\ 0 & \text{for } x\ge k_2.\end{cases}$$

Proof Considering the hyper-singular integrals (2.1) and taking $y=1-(b-x)/(b-a)$, we obtain

$$\int_a^b (x-a)^\alpha(b-x)^\gamma g(x)\,dx = \int_0^1 y^\alpha(1-y)^\gamma\tilde{g}(y)\,dy = \int_0^1 \tilde{f}(y)\,dy,$$

where $\tilde{g}(y)=(b-a)^{1+\alpha+\gamma}g(a+(b-a)y)$ and $\tilde{f}(y)=y^\alpha(1-y)^\gamma\tilde{g}(y)=f(a+(b-a)y)$. According to the conclusions of [2], we obtain the results of Lemma 2.1. □

Obviously, the quadrature formulas can be derived by Lemma 2.1. To get the convergence order of the quadrature rules, we estimate the value of

$$\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} h^{1-p}\bigl(\tilde{F}_0(p)+\tilde{F}_2(p)\bigr)\zeta(p,\beta)\,dp$$

as Corollary 2.2.

Corollary 2.2 *Under the assumptions of Lemma* 2.1,

$$\begin{aligned}
R(c',p) &= \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} h^{1-p}\bigl(\tilde{F}_0(p)+\tilde{F}_2(p)\bigr)\zeta(p,\beta)\,dp \\
&= o\bigl(h^{1-c'}\bigr) = o\bigl(h^{\mathrm{Re}(\alpha)+m+1}\bigr), \quad n\to\infty, \qquad (2.7)
\end{aligned}$$

where $-\mathrm{Re}(\alpha)-m-1<c'<-\mathrm{Re}(\alpha)-m$. *Furthermore* c' *belongs to* $(-\mathrm{Re}(\alpha)-2m-2,-\mathrm{Re}(\alpha)-2m-1)$ *as* $g(x)$ *is* $2m+1$ *times differentiable on* $[a,b]$ *and satisfies the conditions of Lemma* 2.1.

Proof Let $p = c' + is$, thus

$$\begin{aligned} R(c',p) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} h^{1-c'-is}\big(\tilde{F}_0(c'+is)+\tilde{F}_2(c'+is)\big)\zeta(c'+is,\beta)i\,ds \\ &= \frac{h^{1-c'}}{2\pi}\int_{-\infty}^{\infty} h^{-is}\big(\tilde{F}_0(c'+is)+\tilde{F}_2(c'+is)\big)\zeta(c'+is,\beta)\,ds. \end{aligned}$$

Based on the definition of $\tilde{F}_i(p)$, $i = 0,2$, we have

$$\left|\int_{-\infty}^{\infty} h^{-is}\big(\tilde{F}_0(c'+is)+\tilde{F}_2(c'+is)\big)\zeta(c'+is,\beta)\,ds\right| < c,$$

then $R(c',p) = o(h^{1-c'})$ as $n \to \infty$, where c is a constant number. □

3 Quadrature formulas of hyper-singular integrals and their Euler-Maclaurin expansions

In this section, we study the following integrals:

$$I(G) = f.p.\int_a^b G(x,t)\,dx = f.p.\int_a^b q^{\alpha}(x,t)g(x)\,dx, \tag{3.1}$$

$$I_1(G) = f.p.\int_a^b G_1(x,t)\,dx = f.p.\int_a^b |x-t|^{\alpha}\big(\ln|x-t|\big)^p g(x)\,dx, \quad \alpha < -1, \tag{3.2}$$

where $q(x,t) = |x-t|$ (or $x-t$) for $\alpha \le -1$ (or $\alpha < -1$), and p is a nonnegative integer, $g(x)$ is a smooth function on $[a,b]$. $G(x,t)$ and $G_1(x,t)$ are hyper-singular functions about interval variable t as $\alpha < -1$.

We divide the interval $[a,b]$ into n equal parts, that is, $h = (b-a)/n$. Let $x_j = a + jh$ ($j = 0,1,\dots,n$) and the singular point t satisfies $t \in \{x_j : 1 \le j \le n-1\}$, and we also take $\beta = 1/2$. In terms of Lemma 2.1 and the classic Euler-Maclaurin expansions on modified trapezoidal formulas, we derive the following formulas of the integral (3.1) with $q(x,t) = |x-t|$.

Theorem 3.1 *Suppose $g(x)$ is $2m+1$ times differentiable on $[a,b]$, $G(x,t) = g(x)|x-t|^{\alpha}$ with $\alpha \le -1$, and $t \in \{x_j : 1 \le j \le n-1\}$. Then the modified rule is*

$$Q(h) = \sum_{j=0}^{n-1} hG\left(a + \left(j + \frac{1}{2}\right)h\right) - 2h^{1+\alpha}\zeta\left(-\alpha, \frac{1}{2}\right)g(t), \tag{3.3}$$

at the same time the following assertions hold.

(i) *If α is a non-integer, the error expansion can be written*

$$\begin{aligned} E_n(h) &= I(G) - Q(h) \\ &= \sum_{k=1}^{m+1} \frac{h^{2k}B_{2k}(\frac{1}{2})}{(2k)!}\big[G^{(2k-1)}(a) - G^{(2k-1)}(b)\big] \\ &\quad - \sum_{k=1}^{m} \frac{2h^{2k+1+\alpha}}{(2k)!}\zeta\left(-2k-\alpha, \frac{1}{2}\right)g^{(2k)}(t) + O\big(h^{2m+\alpha+2}\big), \end{aligned} \tag{3.4}$$

where $B_{2\mu}$ is for the Bernoulli numbers.

(ii) *When* $\alpha = -2l - 1$ *with* $l \in N$, *the error expansion is given by*

$$E_n(h) = \sum_{k=1}^{m+1} \frac{h^{2k}}{(2k)!} B_{2k}\left(\frac{1}{2}\right)\left[G^{(2k-1)}(a) - G^{(2k-1)}(b)\right] - 2\sum_{k=1,k\neq l}^{m} \frac{h^{2(k-l)}}{(2k)!}\zeta\left(2(l-k)+1, \frac{1}{2}\right)g^{(2k)}(t) + 2\frac{g^{(2l)}(t)}{(2l)!}\psi\left(\frac{1}{2}\right) - 2\frac{g^{(2l)}(t)}{(2l)!}\ln\frac{1}{h} + O\left(h^{2m+2+\alpha}\right). \tag{3.5}$$

(iii) *As* $\alpha = -2l$ *with* $l \in N^+$, *we obtain the error expansion of the form*

$$E_n(h) = \sum_{k=1}^{m+1} \frac{h^{2k}}{(2k)!} B_{2k}\left(\frac{1}{2}\right)\left[G^{(2k-1)}(a) - G^{(2k-1)}(b)\right] - 2\sum_{k=1}^{\min(m,l-1)} \frac{h^{2(k-l)+1}}{(2k)!}\zeta\left(2(l-k), \frac{1}{2}\right)g^{(2k)}(t) + O\left(h^{2m+2+\alpha}\right), \tag{3.6}$$

where $E_n(h) = I(G) - Q(h)$ *and* $\psi(1/2) = -0.577215 - 2\ln 2$.

Proof Take $t = x_i$, then t is an interior point of division of the interval (a, b). The integral of (3.1) can be decomposed into two parts

$$f.p.\int_a^b G(x,t)\,dx = f.p.\int_a^b g(x)|x-t|^\alpha\,dx = f.p.\int_a^t g(x)(t-x)^\alpha\,dx + f.p.\int_t^b g(x)(x-t)^\alpha\,dx. \tag{3.7}$$

We consider the first item of the theorem at first. By equations (2.3) of Lemma 2.1 and (2.7) of Corollary 2.2, we have

$$\sum_{j=0}^{i-1} hG\left(a + \left(j+\frac{1}{2}\right)h\right) = f.p.\int_a^t G(x,t)\,dx + \sum_{k=1}^{m} \frac{h^{2k}}{(2k-1)!}\zeta\left(-2k+1, \frac{1}{2}\right)G^{(2k-1)}(a) + \sum_{k=0}^{2m+1} \frac{(-1)^k h^{k+1+\alpha}}{k!}\zeta\left(-k-\alpha, \frac{1}{2}\right)g^{(k)}(t) + O\left(h^{2m+2+\alpha}\right) \tag{3.8}$$

and

$$\sum_{j=i}^{n-1} hG\left(a + \left(j+\frac{1}{2}\right)h\right) = f.p.\int_t^b G(x,t)\,dx - \sum_{k=1}^{m+1} \frac{h^{2k}}{(2k-1)!}\zeta\left(-2k+1, \frac{1}{2}\right)G^{(2k-1)}(b) + \sum_{k=0}^{2m+1} \frac{h^{k+1+\alpha}}{k!}\zeta\left(-k-\alpha, \frac{1}{2}\right)g^{(k)}(t) + O\left(h^{2m+2+\alpha}\right). \tag{3.9}$$

Combining (3.8) with (3.9), we obtain

$$\begin{aligned}\sum_{j=0}^{n-1} hG\left(a+\left(j+\frac{1}{2}\right)h\right) &= f.p.\int_a^b G(x,t)\,dx \\ &\quad + \sum_{k=1}^{m+1} \frac{h^{2k}}{(2k-1)!}\zeta\left(-2k+1,\frac{1}{2}\right)\left[G^{(2k-1)}(a)-G^{(2k-1)}(b)\right] \\ &\quad + \sum_{k=0}^{m} 2\frac{h^{2k+1+\alpha}}{(2k)!}\zeta\left(-2k-\alpha,\frac{1}{2}\right)g^{(2k)}(t)+O\left(h^{2m+2+\alpha}\right). \quad (3.10)\end{aligned}$$

Since

$$\zeta\left(-2k,\frac{1}{2}\right)=0, \qquad \zeta\left(-2k-1,\frac{1}{2}\right)=-\frac{B_{2k+2}(\frac{1}{2})}{2k+2}, \quad k=0,1,2,\ldots,$$

we have the forms of (3.3) and (3.4).

Next, we will derive the conclusions of (ii) and (iii). Using the rules of (2.4), (2.5), and (2.7), we get the corresponding formulas:

$$\begin{aligned}\sum_{j=0}^{i-1} hG\left(a+\left(j+\frac{1}{2}\right)h\right) &= f.p.\int_a^t G(x,t)\,dx - \sum_{k=1}^{m+1}\frac{h^{2k}}{(2k)!}B_{2k}\left(\frac{1}{2}\right)G^{(2k-1)}(a) \\ &\quad - (-1)^{-\alpha-1}\frac{g^{(-\alpha-1)}(t)}{(-\alpha-1)!}\psi\left(\frac{1}{2}\right)+(-1)^{-\alpha-1}\frac{g^{(-\alpha-1)}(t)}{(-\alpha-1)!}\ln\frac{1}{h} \\ &\quad + \sum_{k=0,k\neq-\alpha-1}^{2m+1}\frac{(-1)^k h^{k+1+\alpha}}{k!}\zeta\left(-k-\alpha,\frac{1}{2}\right)g^{(k)}(t) \\ &\quad + O\left(h^{2m+2+\alpha}\right) \quad (3.11)\end{aligned}$$

and

$$\begin{aligned}\sum_{j=i}^{n-1} hG\left(a+\left(j+\frac{1}{2}\right)h\right) &= f.p.\int_t^b G(x,t)\,dx + \sum_{k=1}^{m+1}\frac{h^{2k}}{(2k)!}B_{2k}\left(\frac{1}{2}\right)G^{(2k-1)}(b) \\ &\quad - \frac{g^{(-\alpha-1)}(t)}{(-\alpha-1)!}\psi\left(\frac{1}{2}\right)+\frac{g^{(-\alpha-1)}(t)}{(-\alpha-1)!}\ln\frac{1}{h} \\ &\quad + \sum_{k=0,k\neq-\alpha-1}^{2m+1}\frac{h^{k+1+\alpha}}{k!}\zeta\left(-k-\alpha,\frac{1}{2}\right)g^{(k)}(t) \\ &\quad + O\left(h^{2m+2+\alpha}\right). \quad (3.12)\end{aligned}$$

Combining (3.11) with (3.12), we have

$$\begin{aligned}\sum_{j=0}^{n-1} hG\left(a+\left(j+\frac{1}{2}\right)h\right) &= f.p.\int_a^b G(x,t)\,dx \\ &\quad + \sum_{k=1}^{m+1}\frac{h^{2k}}{(2k)!}B_{2k}\left(\frac{1}{2}\right)\left[G^{(2k-1)}(b)-G^{(2k-1)}(a)\right]\end{aligned}$$

$$
\begin{aligned}
&- \big((-1)^{-\alpha-1}+1\big)\left[\frac{g^{(-\alpha-1)}(t)}{(-\alpha-1)!}\psi\left(\frac{1}{2}\right)-\frac{g^{(-\alpha-1)}(t)}{(-\alpha-1)!}\ln\frac{1}{h}\right] \\
&+ \sum_{k=0,k\neq-\alpha-1}^{2m+1}\big((-1)^k+1\big)\frac{h^{k+\alpha+1}}{k!}\zeta\left(-k-\alpha,\frac{1}{2}\right)g^{(k)}(t) \\
&+ O\big(h^{2m+2+\alpha}\big). \qquad (3.13)
\end{aligned}
$$

It is easy to obtain the formula of the form

$$
\begin{aligned}
\sum_{j=0}^{n-1} hG\left(a+\left(j+\frac{1}{2}\right)h\right) ={}& f.p.\int_a^b G(x,t)\,dx - 2\frac{g^{(2l)}(t)}{(2l)!}\psi\left(\frac{1}{2}\right) \\
&+ 2\frac{g^{(2l)}(t)}{(2l)!}\ln\frac{1}{h}+\sum_{k=1}^{m+1}\frac{h^{2k}}{(2k)!}B_{2k}\left(\frac{1}{2}\right)\big[G^{(2k-1)}(b)-G^{(2k-1)}(a)\big] \\
&+ 2\sum_{k=0,k\neq l}^{m}\frac{h^{2(k-l)}}{(2k)!}\zeta\left(2(l-k)+1,\frac{1}{2}\right)g^{(2k)}(t) \qquad (3.14)
\end{aligned}
$$

from (3.13) for $\alpha = -2l-1$. Hence, equations (3.3) and (3.5) hold.

As $\alpha = -2l$ ($l \geq 1$), we have

$$
\begin{aligned}
\sum_{j=0}^{n-1} hG\left(a+\left(j+\frac{1}{2}\right)h\right) ={}& f.p.\int_a^b G(x,t)\,dx \\
&+ \sum_{k=1}^{m+1}\frac{h^{2k}}{(2k)!}B_{2k}\left(\frac{1}{2}\right)\big[G^{(2k-1)}(b)-G^{(2k-1)}(a)\big] \\
&+ 2\sum_{k=0}^{\min(m,l-1)}\frac{h^{2(k-l)+1}}{(2k)!}\zeta\left(2(l-k),\frac{1}{2}\right)g^{(2k)}(t) \\
&+ O\big(h^{2m+2+\alpha}\big) \qquad (3.15)
\end{aligned}
$$

from the rules of (3.13).

This completes the proof of theorem. □

Clearly, the convergence order of the quadrature form of (3.3) is $O(h^{2\mu+1+\alpha})$ ($\mu = 1,2,\ldots,m$).

Corollary 3.2 *Under the assumption of Theorem* 3.1 *and letting* $\alpha = -1$, *the quadrature rule is given by*

$$
Q(h) = \sum_{j=0}^{n-1} hG\left(a+\left(j+\frac{1}{2}\right)h\right) + 2g(t)\psi\left(\frac{1}{2}\right) - 2g(t)\ln\frac{1}{h}, \qquad (3.16)
$$

and the form of the asymptotic error expansion is

$$
\begin{aligned}
E_n(h) ={}& \sum_{k=1}^{m}\frac{h^{2k}}{(2k)!}B_{2k}\left(\frac{1}{2}\right)\big[G^{(2k-1)}(a)-G^{(2k-1)}(b)\big] \\
&- 2\sum_{k=1}^{m}\frac{h^{2k}}{(2k)!}\zeta\left(1-2k,\frac{1}{2}\right)g^{(2k)}(t) + O\big(h^{2m+1}\big). \qquad (3.17)
\end{aligned}
$$

Proof Let $l = 0$ of (3.3) and (3.5), then the results hold. □

We have discussed the case of the kernel function $q(x,t) = |x - t|$ above, while some modelings of the phenomena naturally require numerical schemes of the hyper-singular integral (3.1) with $q(x,t) = x - t$. In Theorem 3.3, we will lay out the quadrature formulas of (3.1) with $q(x,t) = x - t$ and demonstrate them.

Theorem 3.3 *Let $g(x)$ be C^{2m+1} function on $[a,b]$ and $G(x,t) = g(x)(x-t)^\alpha$ for $\alpha < -1$. At the same time, we take $t = x_i$, $1 \le i \le n-1$, then the following assertions hold.*

(i) *When $\alpha = -2l - 1$ for $l \in N^+$, the expansion is*

$$\begin{aligned} Q(h) &= \sum_{j=0}^{n-1} hG\left(a + \left(j + \frac{1}{2}\right)h\right) \\ &= f.p.\int_a^b G(x,t)\,dx + \sum_{k=1}^{m+1} \frac{h^{2k}}{(2k)!} B_{2k}\left(\frac{1}{2}\right)\left[G^{(2k-1)}(b) - G^{(2k-1)}(a)\right] \\ &\quad + 2\sum_{k=0}^{\min(m,l-1)} \frac{h^{2(k-l)+1}}{(2k+1)!}\zeta\left(2(l-k), \frac{1}{2}\right) g^{(2k+1)}(t) + O\left(h^{2m+2+\alpha}\right). \end{aligned} \tag{3.18}$$

(ii) *When $\alpha = -2l - 1$ with $l = 0$, the form of the rule can be written*

$$\begin{aligned} \sum_{j=0}^{n-1} hG\left(a + \left(j + \frac{1}{2}\right)h\right) &= \sum_{k=1}^{m+1} \frac{h^{2k}}{(2k)!} B_{2k}\left(\frac{1}{2}\right)\left[G^{(2k-1)}(b) - G^{(2k-1)}(a)\right] \\ &\quad + f.p.\int_a^b G(x,t)\,dx + O\left(h^{2m+2+\alpha}\right). \end{aligned} \tag{3.19}$$

(iii) *If α is a negative even (or non-integer), the results are the same as the formulas of Theorem 3.1(iii) (or (i)).*

Proof Setting $t = x_i$, we divide the integral into two parts, as in the following forms:

$$\begin{aligned} f.p.\int_a^b G(x,t)\,dx &= f.p.\int_a^b g(x)(x-t)^\alpha\,dx \\ &= f.p.\int_a^t (-1)^\alpha g(x)(t-x)^\alpha\,dx + f.p.\int_t^b g(x)(x-t)^\alpha\,dx. \end{aligned}$$

The proof of the theorem is similar to Theorem 3.1. The only one difference of the proof is that $(-1)^\alpha g(x)$ in the above formulas can be regarded as $g(x)$. Therefore, by using (2.5), (2.4) of Lemma 2.1, and (2.7) of Corollary 2.2, respectively, we obtain

$$\begin{aligned} &\sum_{j=0}^{i-1} hG\left(a + \left(j + \frac{1}{2}\right)h\right) \\ &\quad = f.p.\int_a^t G(x,t)\,dx - \sum_{k=1}^{m+1} \frac{h^{2k}}{(2k)!} B_{2k}\left(\frac{1}{2}\right) G^{(2k-1)}(a) \\ &\qquad - (-1)^{-\alpha-1}\left[\frac{((-1)^\alpha g(t))^{(-\alpha-1)}}{(-\alpha-1)!}\psi\left(\frac{1}{2}\right) - \frac{((-1)^\alpha g(t))^{(-\alpha-1)}}{(-\alpha-1)!}\ln\frac{1}{h}\right] \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=0,k\neq-\alpha-1}^{2m+1}\frac{(-1)^k h^{k+1+\alpha}}{k!}\zeta\left(-k-\alpha,\frac{1}{2}\right)\left((-1)^\alpha g(t)\right)^{(k)}\\
&+O\left(h^{2m+2+\alpha}\right)
\end{aligned}
\tag{3.20}
$$

and

$$
\begin{aligned}
\sum_{j=i}^{n-1}hG\left(a+\left(j+\frac{1}{2}\right)h\right)={}&f.p.\int_t^b G(x,t)\,dx+\sum_{k=1}^{m+1}\frac{h^{2k}}{(2k)!}B_{2k}\left(\frac{1}{2}\right)G^{(2k-1)}(b)\\
&-\frac{g^{(-\alpha-1)}(t)}{(-\alpha-1)!}\psi\left(\frac{1}{2}\right)+\frac{g^{(-\alpha-1)}(t)}{(-\alpha-1)!}\ln\frac{1}{h}\\
&+\sum_{k=0,k\neq-\alpha-1}^{2m+1}\frac{h^{k+1+\alpha}}{k!}\zeta\left(-k-\alpha,\frac{1}{2}\right)g^{(k)}(t)\\
&+O\left(h^{2m+2+\alpha}\right).
\end{aligned}
\tag{3.21}
$$

Adding the rules of (3.20) and (3.21), we have the following form:

$$
\begin{aligned}
\sum_{j=0}^{n-1}hG\left(a+\left(j+\frac{1}{2}\right)h\right)={}&f.p.\int_a^b G(x,t)\,dx\\
&+\sum_{k=1}^{m+1}\frac{h^{2k}}{(2k)!}B_{2k}\left(\frac{1}{2}\right)\left[G^{(2k-1)}(b)-G^{(2k-1)}(a)\right]\\
&-\left((-1)^{-\alpha-1}(-1)^\alpha+1\right)\left[\frac{g^{(-\alpha-1)}(t)}{(-\alpha-1)!}\psi\left(\frac{1}{2}\right)-\frac{g^{(-\alpha-1)}(t)}{(-\alpha-1)!}\ln\frac{1}{h}\right]\\
&+\sum_{k=0,k\neq-\alpha-1}^{2m+1}\left((-1)^{k+\alpha}+1\right)\frac{h^{k+\alpha+1}}{k!}\zeta\left(-k-\alpha,\frac{1}{2}\right)g^{(k)}(t)\\
&+O\left(h^{2m+2+\alpha}\right).
\end{aligned}
\tag{3.22}
$$

We complete the proof of Theorem 3.3 from (3.22). □

The quadrature formulas and error expansions of (3.1) have been given in the front part of this section. While we study some boundary integral equations arising in many problems, we notice that these are not only required for calculating the integrals of (3.1), but we also need to discuss the integrals with logarithmic functions just like (3.2) to solve the equations. From equations (2.4), (2.5), and (2.6), one can obtain the Euler-Maclaurin expansions for hyper-singular integrals with logarithmic functions. We deal with them in another paper.

Remark (i) We note the terms

$$
2\sum_{k=0}^{\min(m,l-1)}\frac{h^{2(k-l)+1}}{(2k+1)!}\zeta\left(2(l-k),\frac{1}{2}\right)g^{(2k+1)}(t)
$$

of equations (3.6) and (3.18) depending on $h^{2(k-l)+1}$ and $g^{(2k+1)}(t)$, where $2(k-l)+1\leq-1$. The accuracy order of quadrature formulas can be improved by utilizing the Richardson extrapolation method to take the terms away. Letting $\overline{Q}(h) = Q_l$, $l = \frac{-1-\alpha}{2}$, $l \in N^+$, the modified quadrature rule is

$$Q_0(h) = Q(h), \qquad Q_i(h) = \frac{2^{2(l-k)-1}Q_{i-1}(h) - Q_{i-1}(\frac{h}{2})}{2^{2(l-k)-1} - 1}, \tag{3.23}$$

where $k = 0, 1, \dots, l-1$, $i = 1, \dots, l$, and the order of error of $Q_l(h)$ is $O(h^2)$.

(ii) Considering equations (3.4), (3.5), (3.17), and (3.19), we can obtain better numerical results by the Richardson extrapolation or the Romberg extrapolation method just like the above item (i).

Now, taking Example 4.3 for example with $l = 1$, we will obtain the quadrature formula and its error expansion. The quadrature rule of (3.18) is $Q(h) = \sum_{j=0}^{n-1} hG(a + (j + 1/2)h)$. Then the kth extrapolation is given by

$$\begin{cases} \overline{Q}^{(0)}(h) = 2Q(h) - Q(\frac{h}{2}), \\ \overline{Q}^{(k)}(h) = [2^{2k}\overline{Q}^{(k-1)}(\frac{h}{2}) - \overline{Q}^{(k-1)}(h)]/(2^{2k} - 1), \quad 1 \le k \le m+1, \end{cases} \tag{3.24}$$

and the corresponding asymptotic expansion of error is

$$E_n^{(k)}(h) = \sum_{\mu=k+1}^{m+1} c_\mu^{(k)} h^{2\mu} + O\bigl(h^{2m+2+\alpha}\bigr), \tag{3.25}$$

where $E_n^{(k)}(h) = I(g) - \overline{Q}^{(k)}(h)$, $c_\mu^{(0)} = \frac{B_{2\mu}(\frac{1}{2})(2-2^{-2\mu})[G^{(2\mu-1)}(a)-G^{(2\mu-1)}(b)]}{(2\mu)!}$ ($\mu = 1, \dots, m+1$), and $c_\mu^{(k)} = \frac{2^{2k-2\mu}-1}{2^{2k}-1}c_\mu^{(k-1)}$. Clearly, $\overline{Q}^{(k)}(h)$ has a high convergence order of $O(h^{2(k+1)})$, where $k = 1, \dots, m+1$.

As is seen from (3.4), (3.5), (3.6), (3.17), (3.18), and (3.19), their error expansions contain the term $[G^{(2k-1)}(b) - G^{(2k-1)}(a)]$. If $G(x)$ is a periodic function, those related terms will vanish. Then the quadrature formulas with higher orders of accuracy can be achieved by a Romberg extrapolation. Furthermore, if $G(x,t)$ is not a periodic function, we can obtain better numerical results by using a Richardson extrapolation or a Romberg extrapolation method in a different way just like in our remark.

4 Numerical experiments

In this section, we will display several numerical experiments which are associated with the implementation of the quadrature formulas proposed in this paper. The numerical results for non-periodic hyper-singular integrals are given. Let $h = \frac{b-a}{n}$ be the step length used in the quadrature, where n is the number of nodes. $h^{2k} - ex$ ($k = 0, 1, 2, 3$) are the absolute errors of the kth extrapolation, and $r_{2n}^{(k)} = \frac{h^{2k}-ex}{(\frac{h}{2})^{2k}-ex}$.

Example 4.1 Calculate the hyper-singular integral

$$\bigl(I(g)\bigr)(t) = \int_0^1 \frac{g(x)}{(x-t)^2}\,dx, \qquad g(x) = (2x-1)^3, \quad t \in (0,1), \tag{4.1}$$

where the exact solution is

$$\bigl(I(g)\bigr)(t) = 8(2t-1) + 6(2t-1)^2 \ln\bigl[(1-t)/t\bigr] - (2t-1)^3/\bigl(t(1-t)\bigr).$$

Table 1 The numerical results for $t = 0.25$

$ex\backslash e\backslash n$	2^3	2^4	2^5	2^6	2^7	2^8
$h^0 - ex$	2.331e–2	6.077e–3	1.573e–3	3.854e–4	9.643e–5	2.411e–5
$r_n^{(0)}$		$2^{1.9395}$	$2^{1.9833}$	$2^{1.9957}$	$2^{1.9989}$	$2^{1.9997}$
$h^2 - ex$		3.329e–4	2.363e–5	1.533e–6	9.696e–8	6.065e–9
$r_n^{(1)}$			$2^{3.8163}$	$2^{3.9463}$	$2^{3.9859}$	$2^{3.9964}$
$h^4 - ex$			3.014e–6	5.972e–8	1.004e–9	1.597e–11
$r_n^{(2)}$				$2^{5.6571}$	$2^{5.8942}$	$2^{5.9745}$
$h^6 - ex$				1.284e–8	7.213e–11	2.800e–13

Table 2 The numerical results for $t = 0.984375$

$ex\backslash e\backslash n$	2^8	2^9	2^{10}	2^{11}
$h^0 - ex$	6.894e–3	1.737e–3	4.352e–4	1.088e–4
$r_n^{(0)}$		$2^{1.989}$	$2^{1.997}$	$2^{1.999}$
$h^2 - ex$		1.832e–5	1.178e–6	7.430e–8
$r_n^{(1)}$			$2^{3.959}$	$2^{3.987}$
$h^4 - ex$			3.611e–8	6.577e–10
$r_n^{(2)}$				$2^{5.779}$
$h^6 - ex$				9.495e–11

Since $g(x) = (2x - 1)^3$ and $(I(g))(t)$ are non-periodic functions on $(0, 1)$, we use equations (3.3). The errors of approximation solution are listed in Table 1 for $t = 0.25$ by (3.3) and (3.6). The numerical results in the table show that

$$r_n^{(k)} \approx 2^{2k+2}, \quad k = 0, 1, 2,$$

which accord with the error expansion of (3.6) perfectly. Furthermore, the numerical results in the table also display the fact that a higher convergence order can be got by the extrapolation method.

Example 4.2 Calculate the hyper-singular integral

$$\bigl(I(g)\bigr)(t) = \int_{-1}^{1} \frac{g(x)}{|x - t|}\, dx, \qquad g(x) = e^x, \tag{4.2}$$

where the exact solution is

$$\bigl(I(g)\bigr)(t) = e^t \sum_{k=1}^{\infty} \frac{1}{k!k}\bigl[(-1 - t)^k + (1 - t)^k\bigr] + 2e^t\bigl(\ln|1 + t| + \ln|1 - t|\bigr).$$

The numerical results are listed in Table 2 based on the quadrature formulas (3.16) and the Romberg extrapolation at $t = 0.984375$.

Note that the convergence order of numerical solutions can be improved by an extrapolation method. From the numerical results in Table 2, we have $r_n^k \approx 2^{2k+2}$ $(k = 0, 1, 2)$, which agrees with the rules of (3.16) and (3.17).

Example 4.3 Calculate the hyper-singular integral

$$\bigl(I(g)\bigr)(t) = \int_{0}^{1} \frac{g(x)}{(x - t)^3}\, dx, \qquad g(x) = x^6, \tag{4.3}$$

Table 3 The numerical results for $t = 0.25$

h	2^3	2^4	2^5	2^6	2^7	2^8
$Q(h)$	1.3382	1.8031	2.7290	4.5797	8.2808	15.6830
$\overline{Q}^{(0)}(h)$		0.87321	0.87726	0.87827	0.87852	0.87859
$\lvert\overline{Q}^{(0)}(h) - I\rvert$		5.3987e−3	1.3502e−3	3.3757e−4	8.4394e−5	2.1099e−5
$r_n^{(1)}$		$2^{1.9995}$	$2^{1.9999}$	$2^{2.0000}$	$2^{2.0000}$	$2^{2.0000}$
$\overline{Q}^{(1)}(h)$			0.878607	0.878608	0.878608	0.878608
$\lvert\overline{Q}^{(1)}(h) - I\rvert$			6.5498e−7	3.8287e−8	2.3688e−9	1.4785e−10
$r_n^{(2)}$			$2^{4.0965}$	$2^{4.0146}$	$2^{4.0020}$	$2^{3.9961}$

Table 4 The numerical results for $t = 0.25$

$ex\backslash e\backslash n$	2^3	2^4	2^5	2^6	2^7	2^8	2^9
$h^0 - ex$	3.600e−2	1.240e−2	4.286e−3	1.490e−3	5.202e−4	1.823e−4	6.405e−5
$r_n^{(0)}$		$2^{1.5379}$	$2^{1.5326}$	$2^{1.5246}$	$2^{1.5178}$	$2^{1.5128}$	$2^{1.5091}$
$h^2 - ex$		5.107e−4	1.514e−4	3.955e−5	9.998e−6	2.506e−6	6.271e−7
$r_n^{(1)}$			$2^{1.7538}$	$2^{1.9368}$	$2^{1.9840}$	$2^{1.9960}$	$2^{1.9990}$
$h^4 - ex$			3.168e−5	2.262e−6	1.469e−7	9.272e−9	5.810e−10
$r_n^{(2)}$				$2^{3.8081}$	$2^{3.9447}$	$2^{3.9855}$	$2^{3.9963}$

where the exact solution is

$$\bigl(I(g)\bigr)(t) = \frac{60t^5 - 90t^4 + 20t^3 + 5t^2 + 2t + 1}{4(t-1)^2} + 15t^4 \ln\left(\frac{1-t}{t}\right), \quad t \in (0,1).$$

We get the quadrature formulas $Q_h = h\sum_{k=0}^{n-1} g((\frac{1}{2}+k)h)/((\frac{1}{2}+k)h - t)^3$ from rules (3.18) and extrapolations (3.23) for this example. We have the numerical results listed in Table 3 for (4.3) by using the rules (3.18) and (3.23) at $t = 0.25$ with $\alpha = -3$ and I is the Hadamard part of the example.

Clearly, the numerical results in Table 3 imply that

$$r_n^{(k)} \approx 2^{2k}, \quad k = 1, 2,$$

which meet equation (3.25).

Example 4.4 Calculate the hyper-singular integral with the fractional order singularity

$$\bigl(I(g)\bigr)(t) = \int_0^1 \frac{g(x)}{\sqrt{|x-t|^3}}\,dx, \qquad g(x) = (2x-1)^3, \tag{4.4}$$

where the exact solution is

$$\bigl(I(g)\bigr)(t) = -0.4\left(\frac{128y^3 - 160y^2 + 60y - 5}{\sqrt{y}} + \frac{128y^3 - 224y^2 + 124y - 23}{\sqrt{1-y}}\right).$$

The numerical results for (4.4) at $t = 0.25$ are listed in Table 4 by (3.3) and a Richardson extrapolation. Since $g^{(i)}(x) = 0$ for $i = 4, 5, \ldots$, the error analysis of equation (3.4) is

$$E_n(h) = \sum_{k=1}^{m+1} a_k h^{2k} + b_1 h^{1.5} + O\bigl(h^{2m-1}\bigr), \tag{4.5}$$

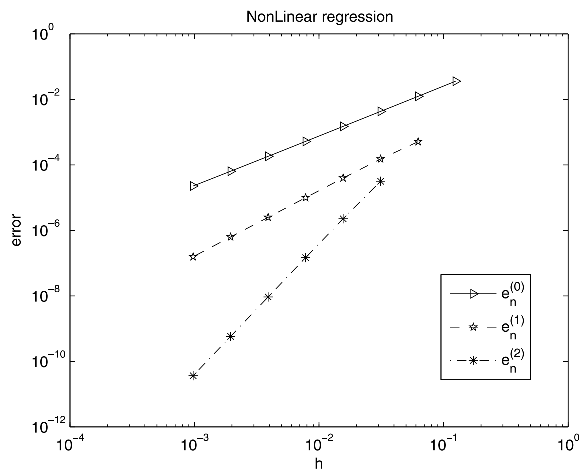

Figure 1 Nonlinear regression analysis at $t = 0.25$. Note that the order of convergence matches the error analysis and the order is clearly improved by using an extrapolation.

Table 5 The numerical results of QF, EQF, and RR with $t = 0.25$

	$ex\backslash e\backslash n$	2^3	2^4	2^5	2^6	2^7	2^8
RR	$h^0 - ex$		3.5318e–3	8.8306e–4	2.2077e–4	5.5193e–5	1.3798e–5
QF	$h^0 - ex$	7.8652e–2	2.0782e–2	5.2765e–3	1.3244e–3	3.3144e–4	8.2880e–5
EQF	$h^2 - ex$		1.4922e–3	1.0794e–4	7.0429e–6	4.4520e–7	2.7905e–8
	$h^4 - ex$			1.5657e–5	3.1634e–7	5.3531e–9	8.5167e–11
	$r_n^{(0)}$		$2^{1.9201}$	$2^{1.9777}$	$2^{1.9942}$	$2^{1.9985}$	$2^{1.9996}$
	$r_n^{(1)}$			$2^{3.7891}$	$2^{3.9379}$	$2^{3.9836}$	$2^{3.9959}$
	$r_n^{(2)}$				$2^{5.6292}$	$2^{5.8849}$	$2^{5.9740}$

where a_k and b_1 are constants, which are independent of h. The numerical results of Table 4 also indicate that

$$r_n^{(0)} \simeq 2^{1.5}, \qquad r_n^{(1)} \simeq 2^2, \qquad r_n^{(2)} \simeq 2^4,$$

which coincide with (4.5) perfectly.

The nonlinear regression analysis shows that $e_n^{(0)} = 0.834h^{1.51}$ after the fractional order extrapolation, while $e_n^{(1)} = 0.127h^{1.99}$ and $e_n^{(2)} = 31.2h^{3.98}$ are after the integer order extrapolation. We show this nonlinear regression analysis result graphically on Figure 1.

Example 4.5 Calculate the hyper-singular integral of Example 4.1 with $g(x) = x^4 + 1$ and the exact value of this finite-part integral is

$$\bigl(I(g)\bigr)(t) = 4t^2 + 2t + \frac{4}{3} + \frac{t+1}{t(t-1)} + 4t^3 \ln\frac{1-t}{t}.$$

We use equations (QF), (3.3), and (3.6) and the corresponding extrapolation of the quadrature equation (EQF) to evaluate the hyper-singular integral, and obtain the numerical results in Table 5. The rectangle rules (RR) in [16] and the quadrature formula (QF) both have the second order accuracy.

The numerical results in the table display the fact that EQF has a high accuracy compared with the method of RR and QF. Furthermore,

$$r_n^{(k)} \approx 2^{2k+2}, \quad k = 0, 1, 2,$$

which accord with the error expansion of (3.6) perfectly. The rate $\log_2(r_n^k)$ shows that EQF has fourth and sixth order accuracy as $k = 1$ and $k = 2$, respectively.

In a consideration of the non-periodic functions $g(x)$ of all the numerical examples above, we can periodize the functions by a $\sin^p$ transformation [12] to take away some terms of the error expansions. By utilizing the extrapolation method, we can get numerical solutions with higher convergence order from (3.3), (3.16), and (3.23), respectively.

5 Conclusion

From the above results in this paper, we draw conclusions as follows: According to the quadrature formulas to calculate hyper-singular integrals, the algorithms of modified trapezoidal formulas have a low cost for real world problems compared with some other methods, such as the Gaussian method [10, 17, 18] and the Newton-Cotes method [19–22]. The rules can be calculated in a fairly straightforward way, without the need to calculate any weight. The accuracy order of the algorithms is very high. Finally, the numerical experiments match with the error analyses. These excellent numerical results show the significance of the quadrature formulas proposed in this paper.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the work was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Acknowledgements
The authors would like to thank the editor and referees for useful comments and suggestions. The authors of this paper were supported by the National Science Foundation of China (11371079).

Received: 1 July 2015 Accepted: 21 December 2015 Published online: 04 January 2016

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