Liu *Journal of Inequalities and Applications* (2016) 2016:9
DOI 10.1186/s13660-015-0947-2

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Refinements of the Erdös-Mordell inequality, Barrow's inequality, and Oppenheim's inequality

Jian Liu*

*Correspondence:
China99jian@163.com
East China Jiaotong University, Nanchang, Jiangxi 330013, China

Abstract

In this paper, we present some new refinements of the Erdös-Mordell inequality, Barrow's inequality, and Oppenheim's inequality. Based on verification by computer, several related interesting conjectures are put forward.

MSC: 51M16

Keywords: Erdös-Mordell inequality; Barrow's inequality; Oppenheim's inequality; triangle; interior point

1 Introduction

For any interior point P of the triangle ABC, let R_1, R_2, R_3 denote the distances from P to the vertices A, B, C, and let r_1, r_2, r_3 denote the distances from P to sides BC, CA, AB, respectively. In addition, we denote the cyclic sums over the triples (R_1,R_2,R_3) and (r_1,r_2,r_3) by $\sum$. Then the well-known Erdös-Mordell inequality states that

$$\sum R_1 \geq 2\sum r_1, \tag{1.1}$$

with equality if and only if $\triangle ABC$ is equilateral and P is its center.

Inequality (1.1) was conjectured by Erdös [1] in 1935. Mordell and Barrow [2] first proved it in 1937, and since then this inequality is known as the Erdös-Mordell inequality and has attracted the attention of many mathematicians who offered various new proofs, generalizations, variations, sharpness, and conjectures (see [3–33] and the references therein).

In [2], Barrow indeed proved the following stronger version of (1.1):

$$\sum R_1 \geq 2\sum w_1, \tag{1.2}$$

where w_1, w_2, w_3 are the lengths of the internal bisectors of $\angle BPC$, $\angle CPA$, $\angle APB$, respectively. For generalizations, sharpness, and extensions of Barrow's inequality, see [3, 4, 6, 10, 25, 31].

In 1961, Oppenheim [9] applied geometrical transformations to study Erdös-Mordell inequality and other inequalities connecting the segments R_1, R_2, R_3, r_1, r_2, r_3. At the end of this paper, he conjectured that the following inequality holds:

$$\sum R_2R_3 \geq \sum (r_3+r_1)(r_1+r_2). \tag{1.3}$$

Oppenheim [10] tried to prove this inequality in the same year. However, the author of this paper pointed out that there exist faults in Oppenheim's proof in a recent paper [28]. We also presented a new method to prove the above inequality in the paper. In fact, we proved the following refinement of (1.3):

$$\sum R_2R_3 \geq \sum h_a r_1 + \sum r_2r_3 \geq \sum (r_3 + r_1)(r_1 + r_2), \tag{1.4}$$

where h_a, h_b, h_c are the corresponding altitudes of $\triangle ABC$ and $\sum h_a r_1 = h_a r_1 + h_b r_2 + h_c r_3$.

The main purpose of this paper is to establish some new refinements of the Erdös-Mordell inequality, Barrow's inequality (1.2) and Oppenheim's inequality (1.3). We also propose some closely related interesting conjectures.

The remainder of this paper is organized as follows. In the next section, we first establish a new inequality involving an interior point of a triangle. Then we use this inequality, the Erdös-Mordell inequality and another well-known inequality to deduce a refinement of the Oppenheim inequality. In Section 3, we present a refinement with one parameter for Barrow's inequality (1.2). In Sections 4 and 5, some new refinements of the Erdös-Mordell inequality are established. Finally, in Section 6, we propose some interesting related conjectures as open problems.

In the following, we shall unceasingly use the above symbols. In addition, we also denote the lengths of the sides BC, CA, AB of the triangle ABC by a, b, c, respectively, and denote the corresponding medians of the triangle ABC by m_a, m_b, m_c. We denote cyclic sums over the triples (a,b,c) (including subscripts), (x,y,z), (u,v,w), (r_1,r_2,r_3), (R_1,R_2,R_3), and (A,B,C) by $\sum$, such as

$$\sum a(b-c)^2r_1^2 = a(b-c)^2r_1^2 + b(c-a)^2r_2^2 + c(a-b)^2r_3^2,$$

$$\sum w_1(R_2+R_3) = w_1(R_2+R_3) + w_2(R_3+R_1) + w_3(R_1+R_2),$$

$$\sum u(wy-vz)^2 = u(wy-vz)^2 + v(uz-wx)^2 + w(vx-uy)^2,$$

$$\sum (m_b+m_c)(r_1+R_1) = (m_b+m_c)(r_1+R_1) + (m_c+m_a)(r_2+R_2) + (m_a+m_b)(r_3+R_3).$$

2 A new refinement of Oppenheim's inequality

In this section, we first establish a new geometric inequality which may be of independent interest. We shall make use of this result and its weaker form (see inequality (4.3) below) several times in the sequel.

Lemma 2.1 *For any interior point P of the triangle ABC, we have*

$$\sum R_2R_3 \geq \frac{1}{2}\sum (2w_1 + w_2 + w_3)R_1, \tag{2.1}$$

with equality holding if and only if $\triangle ABC$ is equilateral and P is its center.

Proof First of all, we establish the following weighted trigonometric inequality for any triangle ABC:

$$\sum \frac{yz(2x+y+z)}{y+z}\cos A \leq \sum yz, \tag{2.2}$$

where A, B, C are the angles of $\triangle ABC$ and x, y, z are arbitrary positive numbers.

By the law of cosines, inequality (2.2) is equivalent to

$$\sum \frac{yz(2x+y+z)(b^2+c^2-a^2)}{bc(y+z)} \le 2\sum yz,$$

i.e.,

$$\begin{aligned} &2abc(y+z)(z+x)(x+y)\sum yz \\ &\quad - \sum yz(z+x)(x+y)(2x+y+z)a\bigl(b^2+c^2-a^2\bigr) \ge 0. \end{aligned} \tag{2.3}$$

Let $b+c-a=2u$, $c+a-b=2v$, $a+b-c=2w$, then $a=v+w$, $b=w+u$, $c=u+v$, where $u,v,w>0$ by the triangle inequality. Thus, we see that inequality (2.3) is equivalent to the following algebraic inequality:

$$\begin{aligned} &2(v+w)(w+u)(u+v)(y+z)(z+x)(x+y)\sum yz \\ &\quad - \sum yz(z+x)(x+y)(2x+y+z)(v+w)\bigl[(w+u)^2+(u+v)^2-(v+w)^2\bigr] \\ &\quad \ge 0. \end{aligned} \tag{2.4}$$

Expanding and rearranging gives the equivalent inequality:

$$4\sum x\sum ux^2(vy-wz)^2 + 4xyz\sum u(wy-vz)^2 \ge 0, \tag{2.5}$$

which is obviously true for the positive numbers x, y, z, u, v, w. From (2.5), it is easily seen that the equality in (2.4) holds if and only if $x=y=z$ and $u=v=w$. We further conclude that equality in (2.2) holds if and only if $\triangle ABC$ is equilateral and $x=y=z$.

We now make use of inequality (2.2) to deduce geometric inequality (2.1). Let $\angle BPC = 2\delta_1$, $\angle CPA = 2\delta_2$, and $\angle APB = 2\delta_3$, then we have the following known identity (*cf.* [4], p.317):

$$w_1 = \frac{2R_2R_3}{R_2+R_3}\cos\delta_1, \tag{2.6}$$

and two formulas are valid for w_2 and w_3. Since $0<\delta_1<\pi$, $0<\delta_2<\pi$, $0<\delta_3<\pi$, and $\delta_1+\delta_2+\delta_3=\pi$, we see that δ_1, δ_2, δ_3 are angles of a triangle. Hence, it follows from (2.2) and (2.6) that

$$\begin{aligned} \sum R_2R_3 &\ge \sum(2R_1+R_2+R_3)\frac{R_2R_3}{R_2+R_3}\cos\delta_1 \\ &= \frac{1}{2}\sum(2R_1+R_2+R_3)w_1 \\ &= \frac{1}{2}\sum(2w_1+w_2+w_3)R_1, \end{aligned}$$

which proves inequality (2.1). By the equality condition of (2.2), we conclude that equality in (2.1) holds if and only if $R_1=R_2=R_3$ and $\delta_1=\delta_2=\delta_3$, *i.e.*, ABC is equilateral and P is its center. This completes the proof of Lemma 2.1. □

It is well known that the following inequality is related to the Erdös-Mordell inequality. See [9] for instance.

Lemma 2.2 *For any interior point P of the triangle ABC, we have*

$$\sum r_1R_1 \geq 2\sum r_2r_3, \tag{2.7}$$

with equality holding if and only if P coincides with a vertex of $\triangle ABC$ or $\triangle ABC$ is equilateral and P is its center.

We now state and prove the following refinement of the Oppenheim inequality.

Theorem 2.1 *For any interior point P of the triangle ABC, the following inequalities hold*:

$$\sum R_2R_3 \geq \frac{1}{2}\sum R_1(2r_1 + r_2 + r_3) \geq \sum(r_3 + r_1)(r_1 + r_2). \tag{2.8}$$

Equalities in (2.8) *hold if and only if $\triangle ABC$ is equilateral and P is its center.*

Proof By Lemma 2.1 and the obvious facts that $w_1 \geq r_1$, $w_2 \geq r_2$, and $w_3 \geq r_3$, the first inequality of (2.8) follows immediately. In addition, by the Erdös-Mordell inequality (1.1) and inequality (2.7), we have

$$\begin{aligned}
&\frac{1}{2}\sum R_1(2r_1 + r_2 + r_3)\\
&\quad= \frac{1}{2}\sum R_1\sum r_1 + \frac{1}{2}\sum r_1R_1\\
&\quad\geq \left(\sum r_1\right)^2 + \sum r_2r_3 = \sum(r_3 + r_1)(r_1 + r_2),
\end{aligned}$$

which proves the second inequality of (2.8).

It is clear that the equality conditions in (2.8) are the same as in (1.1) and (2.1), *i.e.*, if and only if $\triangle ABC$ is equilateral and P is its center. The proof of Theorem 2.1 is completed. □

3 A refinement of Barrow's inequality

We first give the following compact inequality.

Lemma 3.1 *For any interior point P of the triangle ABC, we have*

$$\sum R_1^2 \geq \sum w_1(R_2 + R_3), \tag{3.1}$$

with equality holding if and only if $R_1 : R_2 : R_3 = \sin\frac{1}{2}\angle BPC : \sin\frac{1}{2}\angle CPA : \sin\frac{1}{2}\angle APB$.

In fact, inequality (3.1) is a simple consequence of the following well-known weighted inequality (see [4], p.317, Theorem 12):

$$\sum w_1\left(\frac{1}{R_2} + \frac{1}{R_3}\right)yz \leq \sum x^2, \tag{3.2}$$

where x, y, z are arbitrary real numbers and the equality holds if and only if $x : y : z = \sin\frac{1}{2}\angle BPC : \sin\frac{1}{2}\angle CPA : \sin\frac{1}{2}\angle APB$.

If we let $x = R_1$, $y = R_2$, and $z = R_3$ in (3.2), then inequality (3.1) follows immediately. To our surprise, inequality (3.1) has not been given in [4, 10] although it is compact.

Remark 3.1 Oppenheim [10] obtained a weighted generalization of Barrow's inequality (1.2), which is equivalent with (3.2) when x, y, z are positive.

For Barrow's inequality (1.2), we now give the following refinement with one parameter.

Theorem 3.1 *Let k be a real number such that $0 \le k \le 2$, then for any interior point P of the triangle ABC the following inequalities hold*:

$$\sum R_1 \ge \frac{2\sum(kR_1 + R_2 + R_3)(w_1 + R_1)}{(k+2)\sum R_1} \ge 2\sum w_1. \tag{3.3}$$

If $k = 0$, then the first equality in (3.3) *holds if and only if $R_1 : R_2 : R_3 = \sin\frac{1}{2}\angle BPC : \sin\frac{1}{2}\angle CPA : \sin\frac{1}{2}\angle APB$. If $0 < k \le 2$, then the equalities in* (3.3) *hold if and only if $\triangle ABC$ is equilateral and P is its center.*

Proof The first inequality of (3.3) is equivalent to

$$(k+2)\Bigl(\sum R_1\Bigr)^2 - 2\sum(kR_1 + R_2 + R_3)(w_1 + R_1) \ge 0.$$

Expanding and rearranging gives

$$2\Bigl[\sum R_1^2 - \sum w_1(R_2 + R_3)\Bigr] + k\Bigl[2\sum R_2R_3 - 2\sum w_1R_1 - \sum R_1^2\Bigr] \ge 0, \tag{3.4}$$

which is required to prove.

If $k = 0$, then the above inequality becomes inequality (3.1), and the equality condition in this case is obtained from Lemma 3.1. If $2 \ge k > 0$, by (3.1), to prove (3.4) we need to prove that

$$k\Bigl[\sum R_1^2 - \sum w_1(R_2 + R_3)\Bigr] + k\Bigl[2\sum R_2R_3 - 2\sum w_1R_1 - \sum R_1^2\Bigr] \ge 0,$$

i.e.,

$$2\sum R_2R_3 - \sum(2w_1 + w_2 + w_3)R_1 \ge 0, \tag{3.5}$$

which is equivalent with inequality (2.1) of Lemma 2.1. Hence, inequality (3.4) is proved when $0 < k \le 2$, while the equality condition in this case is the same as in (2.1).

The second inequality of (3.3) is equivalent to

$$\sum(kR_1 + R_2 + R_3)(w_1 + R_1) - (k+2)\sum w_1\sum R_1 \ge 0,$$

which can be rewritten as

$$2k\Bigl[\sum R_1^2 - \sum w_1(R_2 + R_3)\Bigr] + 2\sum R_2R_3 - \sum R_1(2w_1 + w_2 + w_3) \ge 0. \tag{3.6}$$

By inequalities (2.1) and (3.1), we see that the above inequality holds for $k \geq 0$. Also, by the equality conditions of (2.1) and (3.1), it is easy to conclud that if $k \geq 0$ then the equality in (3.6) holds if and only if $\triangle ABC$ is equilateral and P is its center. This completes the proof of Theorem 3.1. □

The following particular case ($k = 0$) of Theorem 3.1 is of interest.

Corollary 3.1 *For any interior point P of $\triangle ABC$, we have*

$$\sum R_1 \geq \frac{\sum (R_2 + R_3)(w_1 + R_1)}{\sum R_1} \geq 2 \sum w_1. \tag{3.7}$$

This double inequality can be regarded as an associated result of (2.1) and (3.1).

4 Refinements of the Erdös-Mordell inequality I

There are few refinements of the Erdös-Mordell inequality in the literature. In [27], the author gave the following result:

$$\sum R_1 \geq \frac{1}{2} \sum \sqrt{a^2 + 4r_1^2} \geq \sum \left(\frac{c}{b} + \frac{b}{c}\right) r_1 \geq 2\sqrt{\sum h_a r_1} \geq 2 \sum r_1. \tag{4.1}$$

In this section and next section, we shall give some new refinements of the Erdös-Mordell inequality. First, we point out that we have the following result, which is a counterpart of Theorem 3.1.

Theorem 4.1 *Let k be real number such that $0 \leq k \leq 2$, then for any interior point P of the triangle ABC the following inequalities hold*:

$$\sum R_1 \geq \frac{2 \sum (kR_1 + R_2 + R_3)(r_1 + R_1)}{(k+2) \sum R_1} \geq 2 \sum r_1. \tag{4.2}$$

Equalities in (4.2) *hold if and only if $\triangle ABC$ is equilateral and P is its center.*

Proof Noting the fact that $w_1 \geq r_1$ *etc.*, the following two inequalities follow from (2.1) and (3.1), respectively:

$$2 \sum R_2 R_3 \geq \sum (2r_1 + r_2 + r_3) R_1, \tag{4.3}$$

$$\sum R_1^2 \geq \sum r_1 (R_2 + R_3). \tag{4.4}$$

According to these two inequalities, by the same arguments used in the proof of Theorem 3.1, we deduce that the double inequality (4.2) holds.

Since the equalities of $w_1 \geq r_1$, $w_2 \geq r_2$, and $w_3 \geq r_3$ are all valid if and only if P is the circumcenter of $\triangle ABC$, thus by the equality conditions of (2.1) and (3.1), it is easy to conclude that both equalities of (4.3) and (4.4) hold if and only if $\triangle ABC$ is equilateral and P is its center. Further, we know that the statement in Theorem 4.1 for the equalities is right. This completes the proof of Theorem 4.1. □

In particular, for $k = 0$ in Theorem 4.1, we obtain the following.

Corollary 4.1 *For any interior point P of $\triangle ABC$, we have*

$$\sum R_1 \ge \frac{\sum (R_2 + R_3)(r_1 + R_1)}{\sum R_1} \ge 2\sum r_1. \tag{4.5}$$

In a recent paper [33], Theorem 1, the author established the following sharpened version of the Erdös-Mordell inequality:

$$\sum \frac{(r_2 + r_3)^2}{R_1} \le \sum R_1, \tag{4.6}$$

with equality holding if and only if $\triangle ABC$ is equilateral and P is its center or $\triangle ABC$ is a right isosceles triangle and P is its circumcenter.

Next, we apply inequality (4.6) to prove another refinement with one parameter for the Erdös-Mordell inequality.

Theorem 4.2 *Let P be an interior point P of the triangle ABC (P may lie on the boundary except the vertices of ABC) and let $k \ge 1$ be a real number, then*

$$\sum R_1 \ge \frac{1}{(k+1)^2}\sum \frac{(kR_1 + r_2 + r_3)^2}{R_1} \ge 2\sum r_1. \tag{4.7}$$

Equalities in (4.7) *hold if and only if $\triangle ABC$ is equilateral and P is its center.*

Proof We have

$$\begin{aligned}
&\sum \frac{(kR_1 + r_2 + r_3)^2}{R_1} \\
&\quad = \frac{k^2R_1^2 + 2kR_1(r_2 + r_3) + (r_2 + r_3)^2}{R_1} \\
&\quad = k^2\sum R_1 + 4k\sum r_1 + \sum \frac{(r_2 + r_3)^2}{R_1} \\
&\quad \le k^2\sum R_1 + 2k\sum R_1 + \sum R_1 \\
&\quad = (k+1)^2\sum R_1,
\end{aligned}$$

where we used the Erdös-Mordell inequality (1.1) and inequality (4.6). Thus, the first inequality in (4.7) is proved (which in fact holds for $k \ge 0$).

On the other hand, by $k \ge 1$ and the Cauchy-Schwarz inequality we have

$$\sum \frac{(kR_1 + r_2 + r_3)^2}{R_1} \ge \frac{[\sum (kR_1 + r_2 + r_3)]^2}{\sum R_1} = \frac{(k\sum R_1 + 2\sum r_1)^2}{\sum R_1}.$$

Thus, to prove the second inequality of (4.7), we only need to prove that

$$\Bigl(k\sum R_1 + 2\sum r_1\Bigr)^2 \ge 2(k+1)^2\sum R_1\sum r_1. \tag{4.8}$$

By the hypothesis $k \geq 1$ and the Erdös-Mordell inequality (1.1), we obtain

$$\begin{aligned}&\left(k\sum R_1 + 2\sum r_1\right)^2 - 2(k+1)^2\sum R_1\sum r_1\\&\quad = \left(\sum R_1 - 2\sum r_1\right)\left(k^2\sum R_1 - 2\sum r_1\right)\\&\quad \geq \left(\sum R_1 - 2\sum r_1\right)^2 \geq 0.\end{aligned}$$

Thus, inequality (4.8) and the second inequality in (4.7) are proved. According to the equality conditions of (1.1), (4.6) and the Cauchy-Schwarz inequality, we see that the equalities in (4.7) hold if and only if $\triangle ABC$ is equilateral and P is its center. This completes the proof of the Theorem 4.2. □

For $k = 0$ in Theorem 4.2, we get the following.

Corollary 4.2 *Let P be an interior point of the triangle ABC (P may lie on the boundary except the vertices of ABC), then*

$$\sum R_1 \geq \frac{1}{4}\sum\frac{(R_1 + r_2 + r_3)^2}{R_1} \geq 2\sum r_1. \tag{4.9}$$

In [28], for proving Oppenheim's inequality (1.3), we presented the following inequality:

$$R_2 + R_3 \geq 2r_1 + \frac{(r_2 + r_3)^2}{R_1}, \tag{4.10}$$

with equality holds if and only if $b = c$ and P is its circumcenter of triangle ABC. We have also pointed out that the Erdös-Mordell inequality (1.1) can easily be obtained from (4.10) in [33]. Next, we shall use (4.10), the previous inequalities (2.7) and (4.4) to establish the following refinement of the Erdös-Mordell inequality.

Theorem 4.3 *For any interior point P of the triangle ABC, the following inequalities hold*:

$$\sum R_1 \geq \sqrt{\sum\left[R_1^2 + 2r_1R_1 + (r_2 + r_3)^2\right]} \geq 2\sum r_1. \tag{4.11}$$

Equalities in (4.11) *hold if and only if $\triangle ABC$ is equilateral and P is its center.*

Proof By inequality (4.10), we have

$$\sum R_1(R_2 + R_3) \geq 2\sum R_1r_1 + \sum(r_2 + r_3)^2,$$

so that

$$2\sum R_2R_3 \geq 2\sum r_1R_1 + \sum(r_2 + r_3)^2. \tag{4.12}$$

Adding $\sum R_1^2$ to both sides of (4.12), we get

$$\left(\sum R_1\right)^2 \geq \sum R_1^2 + 2\sum r_1R_1 + \sum(r_2 + r_3)^2 = \sum\left[R_1^2 + 2r_1R_1 + (r_2 + r_3)^2\right],$$

which shows that the first inequality in (4.11) is true.

On the other hand, by inequalities (4.4), (2.7) and the Erdös-Mordell inequality we have

$$\begin{aligned}
&\sum\left[R_1^2 + 2r_1R_1 + (r_2 + r_3)^2\right] \\
&\quad = \sum R_1^2 + 2\sum r_1R_1 + \sum(r_2 + r_3)^2 \\
&\quad \geq \sum r_1(R_2 + R_3) + \sum r_1R_1 + 2\sum r_2r_3 + \sum(r_2 + r_3)^2 \\
&\quad = \sum r_1 \sum R_1 + 2\sum r_2r_3 + \sum(r_2 + r_3)^2 \\
&\quad \geq 2\Big(\sum r_1\Big)^2 + 2\sum r_2r_3 + \sum(r_2 + r_3)^2 \\
&\quad = 4\Big(\sum r_1\Big)^2.
\end{aligned}$$

Thus, the second inequality in (4.11) holds.

In view of the equality conditions of (4.10), (2.7), and (1.1), we immediately conclude that the equalities in (4.11) hold if and only if $\triangle ABC$ is equilateral and P is its center. This completes the proof of Theorem 4.3. □

In the proof of the final theorem given in this section, we shall use the following well-known result (for proofs, see *e.g.*, [5, 14, 19]).

Lemma 4.1 *For any interior point P of the triangle ABC, we have*

$$\begin{cases} aR_1 \geq br_3 + cr_2, \\ bR_2 \geq cr_1 + ar_3, \\ cR_3 \geq ar_2 + br_1. \end{cases} \tag{4.13}$$

Equalities in (4.13) *successively hold if and only if P lies on the line AO, BO, and CO, where O is the circumcenter of the triangle ABC.*

Theorem 4.4 *For any interior point P of the triangle ABC, the following inequalities hold*:

$$\sum R_1 \geq \sqrt{\sum(R_2 + R_3)(r_1 + R_1)} \geq \sqrt{\frac{4}{3}\sum(r_2 + R_2)(r_3 + R_3)} \geq 2\sum r_1. \tag{4.14}$$

Equalities in (4.14) *hold if and only if $\triangle ABC$ is equilateral and P is its center.*

Proof The first inequality in (4.14) is equivalent to the proved inequality (4.4), since

$$\Big(\sum R_1\Big)^2 - \sum(R_2 + R_3)(r_1 + R_1) = \sum R_1^2 - \sum r_1(R_2 + R_3).$$

By the previous inequality, (4.3), we have

$$\begin{aligned}
&3\sum(R_2 + R_3)(r_1 + R_1) - 4\sum(r_2 + R_2)(r_3 + R_3) \\
&\quad = 2\sum R_2R_3 - \sum r_1(R_2 + R_3) - 4\sum r_2r_3 \\
&\quad \geq \sum(2r_1 + r_2 + r_3)R_1 - \sum r_1(R_2 + R_3) - 4\sum r_2r_3
\end{aligned}$$

$$
\begin{aligned}
&= \sum (r_2 + r_3)R_1 + 2\sum r_1R_1 - \sum r_1(R_2 + R_3) - 4\sum r_2r_3 \\
&= 2\Big(\sum r_1R_1 - 2\sum r_2r_3\Big) \\
&\geq 0,
\end{aligned}
$$

where we used Lemma 2.2 in the last step. Thus, the second inequality in (4.14) is proved.

The third inequality in (4.14) is equivalent to

$$\sum (r_2 + R_2)(r_3 + R_3) \geq 3\Big(\sum r_1\Big)^2. \tag{4.15}$$

By Lemma 4.1 it is sufficient to prove that

$$\sum \left(r_2 + \frac{cr_1 + ar_3}{b}\right)\left(r_3 + \frac{ar_2 + br_1}{c}\right) \geq 3\Big(\sum r_1\Big)^2,$$

i.e.,

$$\sum a(cr_1 + ar_3 + br_2)(ar_2 + br_1 + cr_3) - 3abc\Big(\sum r_1\Big)^2 \geq 0,$$

which is equivalent to

$$\sum a(b-c)^2r_1^2 + \sum \big(a^3 + ab^2 + ac^2 + bc^2 + b^2c - 5abc\big)r_2r_3 \geq 0. \tag{4.16}$$

By the arithmetic-geometric means inequality, we have

$$a^3 + ab^2 + ac^2 + bc^2 + b^2c - 5abc \geq 0,$$

with equality holding if and only if $a = b = c$. Hence, inequalities (4.16) and then (4.15) are proved, while the equality in (4.15) occurs only when $\triangle ABC$ is equilateral and P is its center. This completes the proof of Theorem 4.4. □

5 Refinements of the Erdös-Mordell inequality II

The refinements of the Erdös-Mordell inequality, given in the last section, involve six segments $R_1, R_2, R_3, r_1, r_2, r_3$ but not the geometric elements of $\triangle ABC$. In this section, we use an unified method based on two lemmas (Lemmas 5.1 and 5.2 below) to establish three new refinements of the Erdös-Mordell inequality, which also cover the sides, the altitudes, and the medians of the triangle ABC besides the above six segments.

The result of Theorem 4.1 prompts us to consider the following general refinements of the Erdös-Mordell inequality in the form:

$$\sum R_1 \geq \frac{2\sum k_1(r_1 + R_1)}{\sum k_1} \geq 2\sum r_1, \tag{5.1}$$

where k_1, k_2, k_3 are positive real numbers.

Before offering the results of this kind of double inequalities, we first prove two related lemmas.

Lemma 5.1 *If the positive real numbers* k_1, k_2, k_3 *form a triangle and satisfy*

$$\begin{cases} k_1(b-c)^2 + (k_2 - k_3)(b^2 - c^2) \geq 0, \\ k_2(c-a)^2 + (k_3 - k_1)(c^2 - a^2) \geq 0, \\ k_3(a-b)^2 + (k_1 - k_2)(a^2 - b^2) \geq 0, \end{cases} \tag{5.2}$$

then for any interior point P of the triangle ABC the following inequality holds:

$$\sum R_1 \geq \frac{2\sum k_1(r_1 + R_1)}{\sum k_1}. \tag{5.3}$$

Equality in (5.3) *holds if and only if the equalities in* (5.2) *are all valid and P is the circumcenter of the triangle ABC*.

Proof Note that

$$\sum k_1 \sum R_1 - 2\sum k_1(r_1 + R_1) = \sum (k_2 + k_3 - k_1)R_1 - 2\sum k_1 r_1.$$

Inequality (5.3) is equivalent to

$$\sum (k_2 + k_3 - k_1)R_1 \geq 2\sum k_1 r_1, \tag{5.4}$$

which is obviously a generalized form of the Erdös-Mordell inequality. Since k_1, k_2, k_3 form a triangle, we have $k_2 + k_3 - k_1 > 0$ *etc.* Thus, by Lemma 4.1, to prove (5.4) we only need to prove that

$$\sum (k_2 + k_3 - k_1)\frac{br_3 + cr_2}{a} - 2\sum k_1 r_1 \geq 0,$$

i.e.,

$$\sum bc(k_2 + k_3 - k_1)(br_3 + cr_2) - 2abc\sum k_1 r_1 \geq 0.$$

Expending and rearranging gives the equivalent inequality

$$\sum a\left[k_1(b-c)^2 + (k_2 - k_3)\left(b^2 - c^2\right)\right]r_1 \geq 0, \tag{5.5}$$

which is clearly true if (5.2) is valid. From Lemma 4.1 and (4.16), one sees that the equality in (4.15) occurs if and only if the equalities in (5.2) are all valid and P is the circumcenter of the triangle ABC. This completes the proof of Lemma 5.1. □

Lemma 5.2 *If the positive real numbers* k_1, k_2, k_3 *satisfy*

$$\begin{cases} (b-c)(bk_3 - ck_2) \geq 0, \\ (c-a)(ck_1 - ak_3) \geq 0, \\ (a-b)(ak_2 - bk_1) \geq 0, \end{cases} \tag{5.6}$$

then for any interior point P of the triangle ABC the following inequality holds:

$$\frac{\sum k_1(r_1 + R_1)}{\sum k_1} \geq \sum r_1. \tag{5.7}$$

Equality in (5.7) *holds if and only if equalities in* (5.6) *are all valid and P is the circumcenter of the triangle ABC*.

Proof It is easy to know that inequality (5.7) is equivalent to

$$\sum k_1R_1 - \sum(k_2 + k_3)r_1 \geq 0, \tag{5.8}$$

which is obviously a generalized form of the Erdös-Mordell inequality. Since $k_1, k_2, k_3 > 0$, by Lemma 4.1, to prove the above inequality we need to prove that

$$\sum k_1 \frac{br_3 + cr_2}{a} - \sum(k_2 + k_3)r_1 \geq 0.$$

Multiplying both sides by abc and rearranging gives the equivalent inequality

$$\sum a(b - c)(bk_3 - ck_2)r_1 \geq 0, \tag{5.9}$$

which is clearly true if (5.6) is valid. By Lemma 4.1 and (5.6), we conclude that the equality conditions mentioned in the lemma is true. This completes the proof of Lemma 5.2. □

Lemma 5.1 and Lemma 5.2 show that if the positive real numbers k_1, k_2, k_3 form a triangle and satisfy (5.2) and (5.6), then the refinement (5.1) of the Erdös-Mordell inequality holds. Next, we shall establish three refinement of the Erdös-Mordell inequality by applying this conclusion.

Theorem 5.1 *For any interior point P of the triangle ABC and arbitrary non-negative real numbers* λ *and* μ (λ, μ *not both zero*) *the following inequalities hold*:

$$\sum R_1 \geq \frac{2\sum(\lambda a + \mu b + \mu c)(r_1 + R_1)}{(\lambda + 2\mu)\sum a} \geq 2\sum r_1. \tag{5.10}$$

If $\lambda = 0$, *then the first equality in* (5.10) *holds if and only if P is the circumcenter of ABC*; *If* $\mu = 0$, *then the second equality in* (5.10) *holds if and only if P is the circumcenter of ABC*; *in other cases, the equalities hold if and only if* $\triangle ABC$ *is equilateral and P is its center.*

Proof We first prove the first inequality in (5.10) by means of Lemma 5.1. Let $\lambda a + \mu b + \mu c = k_1$, $\lambda b + \mu c + \mu a = k_2$, and $\lambda c + \mu a + \mu b = k_3$, where $\lambda \geq 0$, $\mu \geq 0$ and λ, μ not both zero. Then

$$k_2 + k_3 - k_1 = \lambda(b + c - a) + 2\mu a > 0.$$

Similarly, we have $k_3 + k_1 - k_2 > 0$ and $k_1 + k_2 - k_3 > 0$. Thus, k_1, k_2, k_3 form a triangle, and $\sum k_1 = (\lambda + 2\mu)\sum a$. A short calculation gives

$$k_1(b - c)^2 + (k_2 - k_3)\bigl(b^2 - c^2\bigr) = \lambda(a + b + c)(b - c)^2 \geq 0,$$

and two similar relations hold. We hence by Lemma 5.1 conclude that the first inequality in (5.10) holds.

We now use Lemma 5.2 to prove the second inequality in (5.10). Under the above assumptions, we easily get

$$(b-c)(bk_3-ck_2)=\mu(a+b+c)(b-c)^2\geq 0.$$

Similarly, we have $(c-a)(ck_1-ak_3)\geq 0$ and $(a-b)(ak_2-bk_1)\geq 0$. Thus, by Lemma 5.2, the second inequality in (5.10) is proved.

It is easy to see that the first inequality in (5.10) when $\lambda=0$ and the second inequality in (5.10) when $\mu=0$ are both equivalent to

$$\sum aR_1\geq\sum(b+c)r_1, \tag{5.11}$$

which follows from (4.13). By Lemma 4.1, we see that the equalities (5.10) in the above two cases holds if and only if P is the circumcenter of $\triangle ABC$. For other cases, by Lemma 5.1 and Lemma 5.2 we easily conclude that the equalities in (5.10) are valid if and only if $a=b=c$ and P is the circumcenter of $\triangle ABC$, *i.e.*, $\triangle ABC$ is equilateral and P is its center. This completes the proof of Theorem 5.1. □

In (5.10), for $\mu=0$, we obtain the following.

Corollary 5.1 *For any interior point P of the triangle ABC, we have*

$$\sum R_1\geq\frac{2\sum a(r_1+R_1)}{\sum a}\geq 2\sum r_1. \tag{5.12}$$

Remark 5.1 By applying Lemma 5.1 and Lemma 5.2, it is easy to prove the following exponential generalization of (5.12):

$$\sum R_1\geq\frac{2\sum a^k(r_1+R_1)}{\sum a^k}\geq 2\sum r_1, \tag{5.13}$$

where k is a real number such that $0<k\leq 1$.

In (5.10), for $\lambda=0$, we obtain

Corollary 5.2 *For any interior point P of the triangle ABC, we have*

$$\sum R_1\geq\frac{\sum(b+c)(r_1+R_1)}{\sum a}\geq 2\sum r_1. \tag{5.14}$$

Since $r_1+R_1\geq h_a=r\sum a/a$, where r is the inradius of $\triangle ABC$. Thus, by the first inequality of (5.12), we obtain a lower bound of $\sum R_1$, namely

Corollary 5.3 *For any interior point P of the triangle ABC, we have*

$$\sum R_1\geq r\sum\frac{b+c}{a}. \tag{5.15}$$

Motivated by Theorem 5.1, we find that if the lengths of the sides a, b, c are replaced by the corresponding altitudes h_a, h_b, h_c of $\triangle ABC$ in (5.10), respectively, then the inequalities hold for $\mu > 0$ and $2\mu \ge \lambda \ge 0$, *i.e.*, the following double inequality holds:

$$\sum R_1 \ge \frac{2\sum(\lambda h_a + \mu h_b + \mu h_c)(r_1 + R_1)}{(\lambda + 2\mu)\sum h_a} \ge 2\sum r_1, \tag{5.16}$$

which is required to prove. If we put $\frac{\lambda}{\mu} = k$, then (5.16) becomes the double inequality (5.17) below.

Theorem 5.2 *Let k be a real number such that $0 \le k \le 2$, then for any interior point P of the triangle ABC the following inequalities hold*:

$$\sum R_1 \ge \frac{2\sum(kh_a + h_b + h_c)(r_1 + R_1)}{(k + 2)\sum h_a} \ge 2\sum r_1. \tag{5.17}$$

Equalities in (5.17) *hold if and only if $\triangle ABC$ is equilateral and P is its center.*

Proof We put

$$kh_a + h_b + h_c = k_1, \qquad kh_b + h_c + h_a = k_2, \qquad kh_c + h_a + h_b = k_3.$$

Then by the given condition $0 \le k \le 2$ we have

$$k_2 + k_3 - k_1 = (2 - k)h_a + k(h_b + h_c) > 0.$$

Similarly, we get $k_3 + k_1 - k_2 > 0$ and $k_1 + k_2 - k_3 > 0$. Thus, k_1, k_2, k_3 form a triangle and $\sum k_1 = (k + 2)\sum h_a$. Also, it is easy to check the following identity:

$$\begin{aligned} &k_1(b - c)^2 + (k_2 - k_3)\bigl(b^2 - c^2\bigr) \\ &\quad = 2(b - c)(bh_c - ch_b) + k(b - c)(bh_a + bh_b + ch_b - bh_c - ch_a - ch_c). \end{aligned}$$

Thus, by $2 \ge k \ge 0$, $(b - c)(bh_c - ch_b) \ge 0$, and $bh_b = ch_c$, we have

$$\begin{aligned} &k_1(b - c)^2 + (k_2 - k_3)\bigl(b^2 - c^2\bigr) \\ &\quad \ge k(b - c)(bh_c - ch_b) + k(b - c)(bh_a + ch_b - bh_c - ch_a) \\ &\quad = kh_a(b - c)^2 \ge 0. \end{aligned}$$

Two inequalities similar to $k_1(b - c)^2 + (k_2 - k_3)(b^2 - c^2) \ge 0$ of course hold. Thus, the first inequality in (5.17) is proved by Lemma 5.1.

On the other hand, it is easy to get the following identity:

$$(b - c)(bk_3 - ck_2) = (h_a + kh_b + kh_c)(b - c)^2.$$

Hence, $(b - c)(bk_3 - ck_2) \ge 0$ is true and its two analogs hold for $k \ge 0$. Thus, by Lemma 5.2, the second inequality in (5.17) is valid for $k \ge 0$.

Summarizing, we deduce that the double inequality (5.17) holds for $0 \leq k \leq 2$. In addition, by Lemma 5.1 and Lemma 5.2, we easily conclude that the equalities in (5.17) occur hold only when $\triangle ABC$ is equilateral and P is its center. The proof of Theorem 5.2 is completed. □

In particular, for $k = 0$ in Theorem 5.2, we obtain

Corollary 5.4 *For any interior point P of the triangle ABC, we have*

$$\sum R_1 \geq \frac{\sum (h_b + h_c)(r_1 + R_1)}{\sum h_a} \geq 2 \sum r_1. \tag{5.18}$$

Finally, we give a dual result of Theorem 5.2, which shows that if we replace the altitudes h_a, h_b, h_c by the corresponding medians m_a, m_b, m_c in (5.17) then the inequalities still hold, *i.e.*, we have the following conclusion:

Theorem 5.3 *Let k be a real number such that $0 \leq k \leq 2$, then for any interior point P of $\triangle ABC$ the following inequalities hold*:

$$\sum R_1 \geq \frac{2 \sum (km_a + m_b + m_c)(r_1 + R_1)}{(k+2) \sum m_a} \geq 2 \sum r_1. \tag{5.19}$$

Equalities in (5.19) *hold if and only if $\triangle ABC$ is equilateral and P is its center.*

Proof In order to prove double inequality (5.19), we first prove the following two inequalities involving the medians and the sides of the triangle, *i.e.*,

$$(b - c)(bm_c - cm_b) \geq 0, \tag{5.20}$$

and

$$(b - c)\big[b(m_a + m_b) - c(m_c + m_a)\big] \geq 0. \tag{5.21}$$

Inequality (5.20) can be obtained directly from the following identity:

$$2(b - c)(bm_c - cm_b)(bm_c + cm_b) = (b + c)\big(a^2 + b^2 + c^2\big)(b - c)^2, \tag{5.22}$$

which is easily checked by using the known formulas $4m_b^2 = 2(c^2 + a^2) - b^2$ and $4m_c^2 = 2(a^2 + b^2) - c^2$. For inequality (5.21), expanding and rearranging gives its equivalent form:

$$m_a\big(b^2 + c^2\big) + m_b b^2 + m_c c^2 - bc(2m_a + m_b + m_c) \geq 0, \tag{5.23}$$

which is needed to prove. According to the median-dual transformation (*cf.* [4], pp.109-111), we know that the above inequality is equivalent to

$$a\big(m_b^2 + m_c^2\big) + bm_b^2 + cm_c^2 - m_b m_c(2a + b + c) \geq 0. \tag{5.24}$$

Using the formulas for the medians m_b and m_c again, it is easy to verify the following identity:

$$\begin{aligned}&\left[a\left(m_b^2+m_c^2\right)+bm_b^2+cm_c^2\right]^2-(2a+b+c)^2(m_bm_c)^2\\&\quad=\frac{3}{16}(b+c)(b+c-a)(b-c)^2\left(a^2+b^2+c^2+3ab+3ac\right),\end{aligned}\tag{5.25}$$

which shows that (5.24) holds true. Therefore, inequalities (5.23) and (5.21) are proved. Also, we easily known that both equalities of (5.20) and (5.21) occur if and only if $b=c$.

We now apply Lemma 5.1 and Lemma 5.2 to prove the double inequality (5.19). Let

$$km_a+(m_b+m_c)=k_1,\qquad km_b+(m_c+m_a)=k_2,\qquad km_c+(m_a+m_b)=k_3,$$

then it is not difficult to show that k_1, k_2, k_3 can be form a triangle if $0\le k\le 2$ and we have $\sum k_1=(k+2)\sum m_a$. By Lemma 5.1, to prove the first inequality in (5.19), it remains to show that $k_1(b-c)^2+(k_2-k_3)(b^2-c^2)\ge 0$. Taking into account that $2\ge k\ge 0$, by (5.20) and (5.21) we have

$$\begin{aligned}&k_1(b-c)^2+(k_2-k_3)\left(b^2-c^2\right)\\&\quad=2(b-c)(bm_c-cm_b)+k(b-c)(bm_a+bm_b+cm_b-bm_c-cm_a-cm_c)\\&\quad\ge k(b-c)(bm_c-cm_b)+k(b-c)(bm_a+bm_b+cm_b-bm_c-cm_a-cm_c)\\&\quad=k(b-c)\left[b(m_a+m_b)-c(m_c+m_a)\right]\ge 0,\end{aligned}$$

as claimed. Hence, the first inequality in (5.19) is proved.

The second inequality in (5.19) can also easily be proved by Lemma 5.2. With the above assumptions, it is easy to obtain the following identity:

$$\begin{aligned}&(b-c)(bk_3-ck_2)\\&\quad=(b-c)\left[b(m_a+m_b)-c(m_a+m_c)\right]+k(b-c)(bm_c-cm_b),\end{aligned}$$

which together with (5.20) and (5.21) shows that $(b-c)(bk_3-ck_2)\ge 0$ holds for $k\ge 0$. Thus, we finish the proof of the second inequality in (5.19).

The equality conditions of (5.19) follow easily from Lemma 5.1 and 5.2. This completes the proof of Theorem 5.3. □

For $k=0$ in (5.19), we get the following counterpart of (5.18).

Corollary 5.5 *For any interior point P of the triangle ABC, we have*

$$\sum R_1\ge\frac{\sum(m_b+m_c)(r_1+R_1)}{\sum m_a}\ge 2\sum r_1.\tag{5.26}$$

6 Open problems

For the inequalities established in this paper, we can propose a lot of new problems. We next introduce some related conjectures as open problems, which have been checked by computer.

The first inequalities of (1.4) and (2.8) prompt the author to propose the following conjecture.

Conjecture 6.1 *For any interior point P of $\triangle ABC$, we have*

$$\sum R_1(2r_1 + r_2 + r_3) \geq 2\sum h_a r_1 + 2\sum r_2 r_3. \tag{6.1}$$

If inequality (6.1) holds true, then we have the following refinement of the Oppenheim inequality (1.3):

$$\begin{aligned}\sum R_2 R_3 &\geq \frac{1}{2}\sum R_1(2r_1 + r_2 + r_3) \geq \sum h_a r_1 + \sum r_2 r_3 \\ &\geq \sum (r_3 + r_1)(r_1 + r_2).\end{aligned} \tag{6.2}$$

in which the last inequality is proved by the author in [28].

From the refinement (4.1) of the Erös-Mordell inequality, we see that the following inequality holds:

$$\sum \sqrt{a^2 + 4r_1^2} \geq 4\sqrt{\sum h_a r_1}, \tag{6.3}$$

which is difficult to prove directly. Here, we conjecture that the above inequality can be strengthened as follows.

Conjecture 6.2 *For any interior point P of $\triangle ABC$, we have*

$$\sum \sqrt{a^2 + 4r_1^2} \geq 4\sqrt{\sum m_a r_1}. \tag{6.4}$$

The following conjecture is similar to Theorem 4.1.

Conjecture 6.3 *Let k be a real number such that $0.48 \leq k \leq 1.36$, then for any interior point P of $\triangle ABC$ the following inequalities hold*:

$$\sum R_1 \geq \frac{2\sum (kr_1 + r_2 + r_3)(R_1 + r_1)}{(k+2)\sum r_1} \geq 2\sum r_1. \tag{6.5}$$

Another conjecture with one parameter, which comes from considering generalizations of the previous double inequality (5.12), is as follows.

Conjecture 6.4 *Let k be a real number such that $0 < k < 4$, then for any interior point P of $\triangle ABC$ the following inequalities hold*:

$$\sum R_1 \geq \frac{2\sum (R_1 + r_1)\sqrt{a^2 + kr_1^2}}{\sum \sqrt{a^2 + kr_1^2}} \geq 2\sum r_1. \tag{6.6}$$

Many years ago, the author found the following interesting 'r-w' phenomenon: If an inequality involving the segments r_1, r_2, r_3 and other geometric elements holds for any interior P of $\triangle ABC$, then the inequality from replacing r_1, r_2, r_3 by w_1, w_2, w_3 in the original inequality, respectively, often still holds for either any or acute triangle ABC. Based on this phenomenon and with the verification by computer, we can propose some dual inequalities for the results presented in this paper. Here, we only give one example, which is a dual conjecture of Theorem 5.1 as follows.

Conjecture 6.5 *For any interior point P of $\triangle ABC$ and non-negative real numbers λ and μ (λ, μ not both zero), the following inequalities hold*:

$$\sum R_1 \geq \frac{2\sum(\lambda a + \mu b + \mu c)(w_1 + R_1)}{(\lambda + 2\mu)\sum a} \geq 2\sum w_1. \tag{6.7}$$

When $\lambda = 0$, the first inequality in (6.7) actually is equivalent to the following beautiful inequality:

$$\sum aR_1 \geq \sum(b + c)w_1, \tag{6.8}$$

which is clearly sharper than the previous inequality (5.11) but has not been proved at present.

Competing interests
The author declares that he has no competing interests.

Received: 18 July 2015 Accepted: 16 December 2015 Published online: 04 January 2016

References

1. Erdös, P: Problem 3740. Am. Math. Mon. **42**, 396-397 (1935)
2. Mordell, LJ, Barrow, DF: Solution of problem 3740. Am. Math. Mon. **44**, 252-254 (1937)
3. Bottema, O, Djordjević, RZ, Janić, RR, Mitrinović, DS, Vasić, PM: Geometric Inequalities. Wolters-Noordhoff, Groningen (1969)
4. Mitrinović, DS, Pečarić, JE, Volenec, V: Recent Advances in Geometric Inequalities. Kluwer Academic, Dordrecht (1989)
5. Kazarinoff, DK: A simple proof of the Erdös-Mordell inequality for triangles. Mich. Math. J. **4**, 97-98 (1957)
6. Ozeki, N: On Paul Erdös-Mordell inequality for the triangle. J. Coll. Arts Sci., Chiba Univ. A. **2**, 247-250 (1957)
7. Bankoff, L: An elementary proof of the Erdös-Mordell theorem. Am. Math. Mon. **65**, 521 (1958)
8. Lenhard, HC: Verallgemeinerung und Verschärfung der Erdoös-Mordell schen Ungleichung fürpolygone. Arch. Math. **12**, 311-314 (1961)
9. Oppenheim, A: The Erdös-Mordell inequality and other inequalities for a triangle. Am. Math. Mon. **68**, 226-230 (1961)
10. Oppenheim, A: New inequalities for a triangle and an internal point. Ann. Mat. Pura Appl. **53**, 157-163 (1961)
11. Mitrinović, DS, Pečarić, JE: On the Erdös Mordell's inequality for a polygon. J. Coll. Arts Sci., Chiba Univ. A. **B-19**, 3-6 (1986)
12. Avez, A: A short proof of a theorem of Erdös and Mordell. Am. Math. Mon. **100**, 60-62 (1993)
13. Pech, P: Erdös-Mordell inequality for space *n*-gons. Math. Pannon. **5**, 3-6 (1994)
14. Komornik, V: A short proof of the Erdös-Mordell theorem. Am. Math. Mon. **104**, 57-60 (1997)
15. Lee, H: Another proof of the Erdös-Mordell theorem. Forum Geom. **1**, 7-8 (2001)
16. Abi-Khuzam, FF: A trigonometric inequality and its geometric applications. Math. Inequal. Appl. **3**, 437-442 (2003)
17. Satnoianu, RA: Erdös-Mordell type inequality in a triangle. Am. Math. Mon. **110**, 727-729 (2003)
18. Si, L, He, BW, Leng, GS: Erdös-Mordell inequality on a sphere in R^3. J. Shanghai Univ. Nat. Sci. **10**, 56-58 (2004)
19. Dergiades, N: Signed distances and the Erdös-Mordell inequality. Forum Geom. **4**, 67-68 (2004)
20. Janous, W: Further inequalities of Erdös-Mordell type. Forum Geom. **4**, 203-206 (2004)
21. Gueron, S, Shafrir, I: A weighted Erdös-Mordell inequality for polygons. Am. Math. Mon. **112**, 257-263 (2005)
22. Liu, J: A sharpening of the Erdös-Mordell inequality and its applications. J. Chongqing Norm. Univ. Nat. Sci. Ed. **22**, 12-14 (2005) (in Chinese)
23. Alsina, C, Nelsen, RB: A visual proof of the Erdös-Mordell inequality. Forum Geom. **7**, 99-102 (2007)
24. Wu, SH, Debnath, L: Generalization of the Wolstenholme cyclic inequality and its application. Comput. Math. Appl. **53**(1), 104-114 (2007)
25. Pambuccian, V: The Erdös-Mordell inequality is equivalent to non-positive curvature. J. Geom. **88**, 134-139 (2008)
26. Bombardelli, M, Wu, SH: Reverse inequalities of Erdös-Mordell type. Math. Inequal. Appl. **12**(2), 403-411 (2009)
27. Liu, J: A new proof of the Erdös-Mordell inequality. Int. Electron. J. Geom. **4**(2), 114-119 (2011)
28. Liu, J: On a geometric inequality of Oppenheim. J. Sci. Arts **18**(1), 5-12 (2012)
29. Sakurai, A: Vector analysis proof of Erdös' inequality for triangles. Am. Math. Mon. **8**, 682-684 (2012)
30. Malešević, B, Petrović, M, Popkonstantinović, B: On the extension of the Erdös-Mordell type inequalities. Math. Inequal. Appl. **17**, 269-281 (2014)
31. Malešević, B, Petrović, M: Barrow's inequality and signed angle bisectors. J. Math. Inequal. **8**(3), 537-544 (2014)
32. Liu, J: A refinement of an equivalent form of a Gerretsen inequality. J. Geom. **106**(3), 605-615 (2015)
33. Liu, J: Sharpened versions of the Erdös-Mordell inequality. J. Inequal. Appl. **2015**, 206 (2015)