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Mixed quasi-variational inequalities involving error bounds

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Abstract
In this paper, we define some new notions of gap functions for set-valued mixed quasi-variational inequalities under suitable conditions. Further, we obtain local/global bounds for the solution of set-valued mixed quasi-variational inequality problems in terms of the residual gap function, the regularized gap function, and the *D*-gap function. The results obtained in this paper are generalization and refinement of previously known results for some class of variational inequality problems.

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1 Introduction

The set-valued quasi-variational inequality problems containing the nonlinear term are definitely most notable one among the several variants of variational inequality problems. It is well known that all the projection type methods cannot be extended and generalized to suggest and analyze iterative methods for solving the mixed variational inequalities involving the nonlinear terms. To overcome this difficulty the resolvent operator methods can be used. In fact, if the nonlinear term involving the mixed variational inequalities is proper, convex, and lower-semicontinuous, then the mixed variational inequalities are equivalent to the fixed point problem and the resolvent equations. In this technique, the given operator is decomposed into the sum of (maximal) monotone operators, whose resolvents are easier to evaluate than the resolvent of the given original operator; see, for example [1–4] and the references therein.

Throughout this paper, let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let $T: H \to 2^H$ be the set-valued mapping and $A(\cdot,\cdot): H\times H \to H$ be a single-valued mapping. Let $f(\cdot,\cdot): H\times H \to \mathbb{R}\cup\{+\infty\}$ be a be a proper convex bifunction with respect to both arguments where $\operatorname{dom}(f)$ is closed. Let $K: H \to 2^H$ be a set-valued mapping such that $K(x)$ is closed convex set in H for any element x of H. Now, we consider the *set-valued mixed quasi-variational inequality problem*, denoted by SVMQVIP, which consists in finding $x \in H$ such that $u \in T(x)$, $x \in K(x)$ and

$$\langle A(u,u), y-x\rangle + f(x,y) - f(x,x) \geq 0, \quad \forall y \in K(x). \tag{1.1}$$

It can be shown that a wide class of set-valued odd order and non-symmetric free, obstacle, moving, equilibrium, and optimization problems arising in the pure and applied sciences can be studied via the set-valued quasi-variational inequality problems.

One of the classical approaches in the analysis of variational inequality problem is to transform it into an equivalent optimization problem via the notion of a gap function; see for example [5–25] and the references therein. This enables us to develop descent-like algorithms to solve the variational inequality problem. Besides these, gap functions also turned out to be very useful in designing new globally convergent algorithms, in analyzing the rate of convergence of some iterative methods, and in obtaining the error bounds, which provide a measure of the distance between solution set and an arbitrary point. Recently, many error bounds for various kinds of variational inequalities have been established; see for example [7–11, 13–22, 24, 25] and the references therein.

If $f(x,y) = f(y)$, $\forall x$ and $A(u,u) = T(x)$ and the mapping $x \to K(x)$ is a constant closed convex set K, then problem SVMQVIP (1.1) collapses to the *set-valued mixed variational inequality problem*, denoted by SVMVIP, which consists in finding $x \in H$ such that

$$\exists u \in T(x): \quad \langle u, y - x\rangle + f(y) - f(x) \geq 0, \quad \forall y \in K, \tag{1.2}$$

which was considered by Tang and Huang [21]. One introduced two regularized gap functions for the above SVMVIP (1.2) and studied their differentiable properties.

If T is single-valued, then problem SVMVIP (1.2) reduces to the *mixed variational inequality problem*, denoted by MVIP, which consists in finding $x \in H$ such that

$$\langle T(x), y - x\rangle + f(y) - f(x) \geq 0, \quad \forall y \in K, \tag{1.3}$$

which was studied by Solodov [20]. In this paper, he introduced three gap functions for MVIP (1.3) and by using these he obtained error bounds.

If the function $f(\cdot)$ is an indicator function of a closed set K in H, then problem MVIP (1.3) reduces to the *set-valued variational inequality problem*, denoted by SVVIP, which consists in finding $x \in K$ such that

$$\exists u \in T(x): \quad \langle u, y - x\rangle \geq 0, \quad \forall y \in K, \tag{1.4}$$

studied by Li and Mastroeni [15]. One obtained some existence results for global error bounds for gap function under strong monotonicity. Later, Aussel and Dutta [5] defined gap functions and by using it they obtained finiteness and error bounds properties for the above set-valued variational inequalities.

If T is single-valued and $K : H \to 2^H$ a set-valued mapping, such that $K(x)$ is a closed convex set in H, for each $x \in H$, then the above problem SVVIP (1.4) is equivalent to the *quasi-variational inequality problem*, denoted by QVIP, which consists in finding $x \in K(x)$ such that

$$\langle T(x), y - x\rangle \geq 0, \quad \forall y \in K(x), \tag{1.5}$$

which was studied by Gupta and Mehra [10] and Noor [17]. They derived local and global error bounds for the above quasi-variational inequality problems in terms of the regularized gap function and the D-gap function.

If T is single-valued, then problem SVVIP (1.4) reduces to the *variational inequality problem*, denoted by VIP, which consists in finding $x \in K$ such that

$$\langle T(x), y - x \rangle \geq 0, \quad \forall y \in K, \tag{1.6}$$

which was considered by many authors to derive gap functions and the corresponding error bounds; see for example [13, 16, 22–25].

Inspired and motivated by the recent research work above, we define some new notions of gap functions for set-valued mixed quasi-variational inequalities and obtain local/global bounds in terms of the residual, the regularized, and the D-gap function. Since this class is the most general and includes some previously studied classes of variational inequalities as special cases, our results cover and extend the previously known results. The results presented in this paper generalize and improve the work presented in [7, 10, 11, 14, 17, 20, 21].

This paper is organized as follows: In Section 2, we give some basic definitions and results which will be used in this paper. Further, we establish some conditions under which SVMQVIP (1.1) has a unique solution. Furthermore, by using the residue vector $R(x, \theta)$ we obtain the error bound for the solution of SVMQVIP (1.1). In Section 3, we introduce a regularized gap function for SVMQVIP (1.1) and derive the error bounds with and without using the Lipschitz continuity assumption. In Section 4, we introduce the D-gap function and derive global error bounds in terms of the D-gap function for the solution of SVMQVIP (1.1).

2 Preliminaries and basic facts

First of all, we recall the following well-known results and concepts.

Definition 2.1 A bifunction $f : H \times H \to \mathbb{R}$ is said to be *skew-symmetric* if $f(x,x) - f(x,y) - f(y,x) + f(y,y) \geq 0$, $\forall x, y \in H$.

Definition 2.2 Let κ be the domain of SVMQVIP (1.1). A function $p : \kappa \to \mathbb{R}$ is said to be a gap function for SVMQVIP (1.1), if it satisfies the following properties:

(i) $p(x) \geq 0$, $\forall x \in \kappa$;
(ii) $p(x^*) = 0$, $x^* \in \kappa$, if and only if x^* solves SVMQVIP (1.1).

For VIP (1.6), it is well known that $x \in H$ is a solution if, and only if,

$$0 = x - P_K\left[x - \theta T(x)\right],$$

where P_K is the orthogonal projector onto K and $\theta > 0$ is arbitrary. Hence, the norm of the right-hand side in this equation can serve as a gap function for VIP (1.6), which is commonly called natural residual vector.

We next derive a similar characterization for SVMQVIP (1.1). Recall that the proximal map [26, 27], $P_K^f : H \to \operatorname{dom}(f)$, is given by

$$P_K^f(z) = \arg \min_{y \in K(x)} \left\{ f(x,y) + \frac{1}{2\theta} \|y - z\|^2 \right\}, \quad z \in H, \theta > 0.$$

Note that the objective function above is proper strongly convex. Since dom(f) is closed, $P_K^f(\cdot)$ is well defined and single-valued. For $\theta > 0$, define

$$R(x,\theta) = x - P_{K(x)}^f\bigl[x - \theta A(u,u)\bigr], \quad x \in H.$$

We next show that $R(x,\theta)$ plays the role of the natural residual vector for SVMQVIP (1.1).

Lemma 2.1 *Let $\theta > 0$ be arbitrary. An element $x \in H$ solves SVMQVIP* (1.1) *if, and only if,* $R(x,\theta) = 0$.

Proof Let $R(x,\theta) = 0$, which implies that $x = P_{K(x)}^f[x - \theta A(u,u)]$. It is equivalent to

$$x = \arg\min_{y\in K(x)} \left\{ f(x,y) + \frac{1}{2\theta}\|y - z\|^2 \right\}.$$

By the optimality conditions (which are necessary and sufficient, by convexity), the latter is equivalent to

$$0 \in \partial f(x,y) + \frac{1}{\theta}\bigl(x - \bigl(x - \theta A(u,u)\bigr)\bigr) = \partial f(x,y) + A(u,u),$$

which implies $-A(u,u) \in \partial f(x,y)$. This in turn is equivalent, by the definition of the subgradient, to

$$f(x,y) \geq f(x,x) - \bigl\langle A(u,u), y - x\bigr\rangle, \quad \forall y \in K(x),$$

which implies x solves SVMQVIP (1.1). This completes the proof. □

Remark 2.1 It is easy to see the natural residual vector $R(x,\theta)$ is a gap function for SVMQVIP (1.1).

Definition 2.3 Let $A : H \times H \to H$ be a single-valued operator and $T : H \to 2^H$ be the set-valued operator, then $\forall x, x_0 \in H$, $u \in T(x)$, $u_0 \in T(x_0)$:

(a) A is said to be *strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\bigl\langle A(u,u) - A(u_0,u_0), x - x_0\bigr\rangle \geq \alpha\|x - x_0\|^2;$$

(b) A is said to be *cocoercive*, if there exists a constant $\mu > 0$ such that

$$\bigl\langle A(u,u) - A(u_0,u_0), x - x_0\bigr\rangle \geq \tau\bigl\|A(u,u) - A(u_0,u_0)\bigr\|^2;$$

(c) A is said to be *Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$\bigl\|A(u,u) - A(u_0,u_0)\bigr\| \leq \beta\|u - u_0\|;$$

(d) T is said to be *M-Lipschitz continuous*, if there exists a constant $\mu > 0$ such that

$$M\bigl(T(x), T(x_0)\bigr) \leq \mu\|x - x_0\|,$$

where $M(\cdot,\cdot)$ is the Hausdorff metric;

(e) $P^f_{K(x)}$ is said to be *nonexpansive*, if

$$\|P^f_{K(x)}(v) - P^f_{K(x)}(w)\| \leq \|v - w\|, \quad \forall x, v, w \in H.$$

Since the $P^f_{K(x)}(z)$ is a cocoercive map with modulus 1 on $z \in H$ (see Theorem 1.5.5(c) in [28]), for $\theta > 0$ and $x \in H$, define $e_x : H \to H$ as $e_x(z) = z - P^f_{K(x)}(z)$. We have the following result, to be used in the sequel.

Lemma 2.2 *$e_x(z)$ is a cocoercive map on H with modulus* 1.

Proof For all $z, w \in H$, we have

$$\langle P^f_{K(x)}(z) - P^f_{K(x)}(w), z - w\rangle \geq \|P^f_{K(x)}(z) - P^f_{K(x)}(w)\|^2,$$

implying

$$\langle z - P^f_{K(x)}(z) - (w - P^f_{K(x)}(w)), P^f_{K(x)}(z) - P^f_{K(x)}(w)\rangle \geq 0.$$

Therefore

$$\langle e_x(z) - e_x(w), z - e_x(z) - w + e_x(w)\rangle \geq 0,$$

yielding the required result. □

Next we study those conditions under which SVMQVIP (1.1) has a unique solution.

Theorem 2.1 *Let A be strongly monotone and Lipschitz continuous with constants $\alpha, \beta > 0$, respectively. Assume that f is skew-symmetric and T is M-Lipschitz continuous with constant $\mu > 0$. If there exists $k > 0$, such that for all $\theta < \frac{2\alpha}{\beta^2\mu^2}$ and*

$$\|P^f_{K(x)}(z) - P^f_{K(x_0)}(z)\| \leq k\|x - x_0\|, \quad \forall x, x_0, z \in H, \text{with } k < 1 - \sqrt{\theta^2\beta^2\mu^2 - 2\alpha\theta + 1},$$

then SVMQVIP (1.1) *has a unique solution.*

Proof (a) *Uniqueness*. Let $x_1 \neq x_2 \in H$ be two solutions of SVMQVIP (1.1). Then we have

$$\langle A(u_1, u_1), y - x_1\rangle + f(x_1, y) - f(x_1, x_1) \geq 0, \tag{2.1}$$

$$\langle A(u_2, u_2), y - x_2\rangle + f(x_2, y) - f(x_2, x_2) \geq 0. \tag{2.2}$$

Taking $y = x_2$ in (2.1) and $y = x_1$ in (2.2), adding the resultants and then using the skew-symmetry of f, we have

$$\langle A(u_1, u_1) - A(u_2, u_2), x_2 - x_1\rangle \geq 0.$$

Since A is strongly monotone with a constant $\alpha > 0$,

$$0 \leq \langle A(u_1, u_1) - A(u_2, u_2), x_2 - x_1\rangle \leq -\alpha\|x_1 - x_2\|^2,$$

which implies that $x_1 = x_2$, the uniqueness of the solution of SVMQVIP (1.1).

(b) *Existence*. From Lemma 2.1, it follows that SVMQVIP (1.1) is equivalent to the fixed point problem, given by

$$x = F(x) := P^f_{K(x)}\big[x - \theta A(u,u)\big] : H \to H. \tag{2.3}$$

In order to prove the existence of a solution of SVMQVIP (1.1), it is sufficient to show that $F(\cdot)$ has a fixed point. Thus, for all $x, x_0 \in H$, $x \neq x_0$, we have

$$\begin{aligned}\big\|F(x) - F(x_0)\big\| &= \big\|P^f_{K(x)}\big[x - \theta A(u,u)\big] - P^f_{K(x_0)}\big[x_0 - \theta A(u_0,u_0)\big]\big\| \\ &= \big\|P^f_{K(x)}\big[x - \theta A(u,u)\big] - P^f_{K(x_0)}\big[x - \theta A(u,u)\big] \\ &\quad + P^f_{K(x_0)}\big[x - \theta A(u,u)\big] - P^f_{K(x_0)}\big[x_0 - \theta A(u_0,u_0)\big]\big\| \\ &\leq \big\|P^f_{K(x)}\big[x - \theta A(u,u)\big] - P^f_{K(x_0)}\big[x - \theta A(u,u)\big]\big\| \\ &\quad + \big\|P^f_{K(x_0)}\big[x - \theta A(u,u)\big] - P^f_{K(x_0)}\big[x_0 - \theta A(u_0,u_0)\big]\big\| \\ &\leq k\|x - x_0\| + \big\|x - x_0 - \theta\big(A(u,u) - A(u_0,u_0)\big)\big\|. \end{aligned} \tag{2.4}$$

We know that

$$\begin{aligned}&\big\|x - x_0 - \theta\big(A(u,u) - A(u_0,u_0)\big)\big\|^2 \\ &\quad = \|x - x_0\|^2 - 2\theta\big\langle A(u,u) - A(u_0,u_0), x - x_0\big\rangle + \theta^2\big\|A(u,u) - A(u_0,u_0)\big\|^2.\end{aligned}$$

By using the strong monotonicity and the Lipschitz continuity of A with constants $\alpha > 0$ and $\beta > 0$, respectively, we get

$$\begin{aligned}&\big\|x - x_0 - \theta\big(A(u,u) - A(u_0,u_0)\big)\big\|^2 \\ &\quad \leq \|x - x_0\|^2 - 2\theta\alpha\|x - x_0\|^2 + \theta^2\beta^2\|u - u_0\|^2.\end{aligned}$$

Further, using the M-Lipschitz continuity of T with constant $\mu > 0$, we get

$$\big\|x - x_0 - \theta\big(A(u,u) - A(u_0,u_0)\big)\big\|^2 \leq \big(\theta^2\beta^2\mu^2 - 2\alpha\theta + 1\big)\|x - x_0\|^2. \tag{2.5}$$

From (2.4) and (2.5), we get

$$\big\|F(x) - F(x_0)\big\| \leq \big(k + \sqrt{\theta^2\beta^2\mu^2 - 2\alpha\theta + 1}\big)\|x - x_0\| = m\|x - x_0\|,$$

where $m = k + \sqrt{\theta^2\beta^2\mu^2 - 2\alpha\theta + 1}$. From the assumption on k, it follows that $m < 1$, so the mapping $F(x)$ defined by (2.3) has a fixed point $x \in K(x)$ such that $u \in T(x)$ satisfying SVMQVIP (1.1). This completes the proof. □

Now by using normal residual vector $R(x,\theta)$, we derive the error bounds for the solution of SVMQVIP (1.1).

Theorem 2.2 *Assume that x_0 be a solution of SVMQVIP* (1.1). *Let the operator A be strongly monotone and Lipschitz continuous with constants $\alpha, \beta > 0$, respectively. Let T*

be a M-Lipschitz continuous with constant $\mu > 0$ and f is skew-symmetric. If there exists $k < \frac{\alpha}{\beta\mu}$ such that, for any $\theta > \frac{k}{\alpha-\beta\mu k}$,

$$\|P^f_{K(x)}(w) - P^f_{K(x_0)}(w)\| \leq k\|x - x_0\|, \quad \forall x, x_0, w \in H,$$

then, for any $x \in H$ and $\theta > \frac{k}{\alpha-\beta\mu k}$, we have

$$\|x - x_0\| \leq \frac{1 + \theta\beta\mu}{\alpha\theta - (1 + \theta\beta\mu)k}\|R(x,\theta)\|.$$

Proof Let $x_0 \in H$ be a solution of SVMQVIP (1.1), then

$$\langle A(u_0, u_0), y - x_0\rangle + f(x_0, y) - f(x_0, x_0) \geq 0, \quad \forall y \in K(x_0).$$

Substituting $y = P^f_{K(x_0)}[x - \theta A(u, u)]$ in the above inequality, we have

$$\langle A(u_0, u_0), P^f_{K(x_0)}[x - \theta A(u, u)] - x_0\rangle + f(x_0, P^f_{K(x_0)}[x - \theta A(u, u)]) - f(x_0, x_0) \geq 0. \quad (2.6)$$

For any fixed $x \in H$, and $\theta > 0$, we observe that

$$x - \theta A(u, u) \in (I + \theta\partial f)(I + \theta\partial f)^{-1}(x - \theta A(u, u)) = (I + \theta\partial f)P^f_K[x - \theta A(u, u)],$$

which is equivalent to

$$-A(u, u) + \frac{1}{\theta}[x - P^f_K[x - \theta A(u, u)]] \in \partial f(P^f_K[x - \theta A(u, u)]).$$

By the definition of a sub-differential, we have

$$\Big\langle A(u, u) - \frac{1}{\theta}(x - P^f_{K(x_0)}[x - \theta A(u, u)]), y - P^f_{K(x_0)}[x - \theta A(u, u)]\Big\rangle$$
$$+ f(P^f_{K(x_0)}[x - \theta A(u, u)], y) - f(P^f_{K(x_0)}[x - \theta A(u, u)], P^f_{K(x_0)}[x - \theta A(u, u)]) \geq 0.$$

Taking $y = x_0$ in the above we get

$$\Big\langle A(u, u) - \frac{1}{\theta}(x - P^f_{K(x_0)}[x - \theta A(u, u)]), x_0 - P^f_{K(x_0)}[x - \theta A(u, u)]\Big\rangle$$
$$+ f(P^f_{K(x_0)}[x - \theta A(u, u)], x_0) - f(P^f_{K(x_0)}[x - \theta A(u, u)], P^f_{K(x_0)}[x - \theta A(u, u)]) \geq 0.$$

This implies that

$$\Big\langle -A(u, u) + \frac{1}{\theta}(x - P^f_{K(x_0)}[x - \theta A(u, u)]), P^f_{K(x_0)}[x - \theta A(u, u)] - x_0\Big\rangle$$
$$+ f(P^f_{K(x_0)}[x - \theta A(u, u)], x_0)$$
$$- f(P^f_{K(x_0)}[x - \theta A(u, u)], P^f_{K(x_0)}[x - \theta A(u, u)]) \geq 0. \quad (2.7)$$

Adding (2.6) and (2.7), we get

$$\begin{aligned}&\left\langle A(u_0,u_0)-A(u,u)+\frac{1}{\theta}\big(x-P^f_{K(x_0)}\big[x-\theta A(u,u)\big]\big),P^f_{K(x_0)}\big[x-\theta A(u,u)\big]-x_0\right\rangle\\&\quad+f\big(x_0,P^f_{K(x_0)}\big[x-\theta A(u,u)\big]\big)-f(x_0,x_0)+f\big(P^f_{K(x_0)}\big[x-\theta A(u,u)\big],x_0\big)\\&\quad-f\big(P^f_{K(x_0)}\big[x-\theta A(u,u)\big],P^f_{K(x_0)}\big[x-\theta A(u,u)\big]\big)\geq 0.\end{aligned}$$

Since f is skew-symmetric,

$$\left\langle A(u_0,u_0)-A(u,u)+\frac{1}{\theta}\big(x-P^f_{K(x_0)}\big[x-\theta A(u,u)\big]\big),P^f_{K(x_0)}\big[x-\theta A(u,u)\big]-x_0\right\rangle\geq 0.$$

This also can be written as

$$\begin{aligned}&\theta\big\langle A(u_0,u_0)-A(u,u),P^f_{K(x_0)}\big[x-\theta A(u,u)\big]-x\big\rangle+\theta\big\langle A(u_0,u_0)-A(u,u),x-x_0\big\rangle\\&\quad+\big\langle x-P^f_{K(x_0)}\big[x-\theta A(u,u)\big],P^f_{K(x_0)}\big[x-\theta A(u,u)\big]-x\big\rangle\\&\quad+\big\langle x-P^f_{K(x_0)}\big[x-\theta A(u,u)\big],x-x_0\big\rangle\geq 0,\end{aligned}$$

which implies that

$$\begin{aligned}&\theta\big\langle A(u_0,u_0)-A(u,u),P^f_{K(x_0)}\big[x-\theta A(u,u)\big]-x\big\rangle+\big\langle x-P^f_{K(x_0)}\big[x-\theta A(u,u)\big],x-x_0\big\rangle\\&\quad\geq\theta\big\langle A(u_0,u_0)-A(u,u),x_0-x\big\rangle\\&\qquad+\big\langle x-P^f_{K(x_0)}\big[x-\theta A(u,u)\big],x-P^f_{K(x_0)}\big[x-\theta A(u,u)\big]\big\rangle.\end{aligned}$$

By using the strong monotonicity of A, we get

$$\begin{aligned}&\theta\big\langle A(u_0,u_0)-A(u,u),P^f_{K(x_0)}\big[x-\theta A(u,u)\big]-x\big\rangle+\big\langle x-P^f_{K(x_0)}\big[x-\theta A(u,u)\big],x-x_0\big\rangle\\&\quad\geq\alpha\theta\|x_0-x\|^2+\big\|R(x,\theta)\big\|^2.\end{aligned}$$

Also the above inequality can be written as

$$\begin{aligned}&\theta\big\langle A(u_0,u_0)-A(u,u),P^f_{K(x_0)}\big[x-\theta A(u,u)\big]\\&\qquad-x-P^f_{K(x)}\big[x-\theta A(u,u)\big]+P^f_{K(x)}\big[x-\theta A(u,u)\big]\big\rangle\\&\qquad+\big\langle x-P^f_{K(x_0)}\big[x-\theta A(u,u)\big]-P^f_{K(x)}\big[x-\theta A(u,u)\big]+P^f_{K(x)}\big[x-\theta A(u,u)\big],x-x_0\big\rangle\\&\quad\geq\alpha\theta\|x_0-x\|^2+\big\|R(x,\theta)\big\|^2.\end{aligned}$$

By using the Cauchy-Schwarz inequality along with the triangular inequality, we have

$$\begin{aligned}&\theta\big\|A(u_0,u_0)-A(u,u)\big\|\cdot\big\|P^f_{K(x_0)}\big[x-\theta A(u,u)\big]-P^f_{K(x)}\big[x-\theta A(u,u)\big]\big\|\\&\qquad+\theta\big\|A(u_0,u_0)-A(u,u)\big\|\cdot\big\|P^f_{K(x)}\big[x-\theta A(u,u)\big]-x\big\|\\&\qquad+\big\|x-P^f_{K(x)}\big[x-\theta A(u,u)\big]\big\|\cdot\|x-x_0\|\end{aligned}$$

$$+ \left\| P^f_{K(x)}\left[x - \theta A(u,u)\right] - P^f_{K(x_0)}\left[x - \theta A(u,u)\right] \right\| \cdot \|x - x_0\|$$
$$\geq \alpha\theta \|x_0 - x\|^2 + \left\| R(x,\theta) \right\|^2.$$

Now using the Lipschitz continuity of the operator A and assumption on $P^f_{K(x)}(\cdot)$, we have

$$\theta\beta \|u_0 - u\| \cdot k\|x_0 - x\| + \theta\beta \|u_0 - u\| \cdot \left\| R(x,\theta) \right\| + \left\| R(x,\theta) \right\| \cdot \|x - x_0\| + k\|x - x_0\|^2$$
$$\geq \alpha\theta \|x_0 - x\|^2 + \left\| R(x,\theta) \right\|^2.$$

Now using the M-Lipschitz continuity of T, we have

$$k\theta\beta\mu \|x_0 - x\|^2 + \theta\beta\mu \|x_0 - x\| \cdot \left\| R(x,\theta) \right\| + \left\| R(x,\theta) \right\| \cdot \|x - x_0\| + k\|x - x_0\|^2$$
$$\geq \alpha\theta \|x_0 - x\|^2 + \left\| R(x,\theta) \right\|^2.$$

Therefore, we have

$$\|x - x_0\| \leq \frac{1 + \theta\beta\mu}{\alpha\theta - (1 + \theta\beta\mu)k} \left\| R(x,\theta) \right\|, \quad \forall x \in H,$$

where $\theta > \frac{k}{\alpha - \beta\mu k}$. This completes the proof. □

Remark 2.2 Theorem 2.2 generalizes Theorem 1 of [7].

Besides the above result on the error bound, we provide another error bound for SVMQVIP (1.1).

Theorem 2.3 *Assume that x_0 is the solution of SVMQVIP* (1.1). *Let the operator A be strongly monotone and Lipschitz continuous with constants $\alpha, \beta > 0$, respectively. Let T be M-Lipschitz continuous with constant $\mu > 0$ and f be skew-symmetric. If there exists $k > 0$, such that, for any $\theta < \frac{2\alpha}{\beta^2\mu^2}$ and*

$$\left\| P^f_{K(x)}(z) - P^f_{K(x_0)}(z) \right\| \leq k\|x - x_0\|, \quad \forall x, x_0, z \in H, \text{with } k < 1 - \sqrt{\theta^2\beta^2\mu^2 - 2\alpha\theta + 1},$$

then, for any $x \in H$, we have

$$\|x - x_0\| \leq \frac{4}{4\alpha\theta - \beta^2\mu^2\theta^2 - 4k} \left\| R(x,\theta) \right\|.$$

Proof For any $x, x_0 \in H$, let us consider $v = x - \theta A(u,u)$ and $w = x_0 - \theta A(u_0,u_0)$. Now

$$\left\langle R(x,\theta) - R(x_0,\theta), x - x_0 \right\rangle$$
$$= \left\langle x - P^f_{K(x)}\left[x - \theta A(u,u)\right] - x_0 + P^f_{K(x_0)}\left[x_0 - \theta A(u_0,u_0)\right], x - x_0 \right\rangle$$
$$= \left\langle x - \theta A(u,u) - P^f_{K(x)}\left[x - \theta A(u,u)\right] - \left(x_0 - \theta A(u_0,u_0)\right)\right.$$
$$\left. + P^f_{K(x_0)}\left[x_0 - \theta A(u_0,u_0)\right], x - x_0 \right\rangle + \theta\left\langle A(u,u) - A(u_0,u_0), x - x_0 \right\rangle$$
$$= \left\langle v - P^f_{K(x)}\left[x - \theta A(u,u)\right] - w + P^f_{K(x_0)}\left[x_0 - \theta A(u_0,u_0)\right], x - x_0 \right\rangle$$

$$
\begin{aligned}
&+\theta\langle A(u,u)-A(u_0,u_0),x-x_0\rangle\\
&=\langle v-P^f_{K(x)}[x-\theta A(u,u)]-w+P^f_{K(x)}[x_0-\theta A(u_0,u_0)],x-x_0\rangle\\
&\quad+\langle P^f_{K(x_0)}[x_0-\theta A(u_0,u_0)]-P^f_{K(x)}[x_0-\theta A(u_0,u_0)],x-x_0\rangle\\
&\quad+\theta\langle A(u,u)-A(u_0,u_0),x-x_0\rangle\\
&=\langle v-P^f_{K(x)}[x-\theta A(u,u)]-\bigl(w-P^f_{K(x)}[x_0-\theta A(u_0,u_0)]\bigr),\\
&\quad v+\theta A(u,u)-w-\theta A(u_0,u_0)\rangle\\
&\quad+\langle P^f_{K(x_0)}[x_0-\theta A(u_0,u_0)]-P^f_{K(x)}[x_0-\theta A(u_0,u_0)],x-x_0\rangle\\
&\quad+\theta\langle A(u,u)-A(u_0,u_0),x-x_0\rangle\\
&=\langle v-P^f_{K(x)}[x-\theta A(u,u)]-\bigl(w-P^f_{K(x)}[x_0-\theta A(u_0,u_0)]\bigr),v-w\rangle\\
&\quad+\theta\langle v-P^f_{K(x)}[x-\theta A(u,u)]-\bigl(w-P^f_{K(x)}[x_0-\theta A(u_0,u_0)]\bigr),A(u,u)-A(u_0,u_0)\rangle\\
&\quad+\langle P^f_{K(x_0)}[x_0-\theta A(u_0,u_0)]-P^f_{K(x)}[x_0-\theta A(u_0,u_0)],x-x_0\rangle\\
&\quad+\theta\langle A(u,u)-A(u_0,u_0),x-x_0\rangle.
\end{aligned}
$$

By using the cocoercive property of $P^f_{K(x)}(\cdot)$, given in Lemma 2.2,

$$
\begin{aligned}
&\langle R(x,\theta)-R(x_0,\theta),x-x_0\rangle\\
&\quad\geq\bigl\|v-P^f_{K(x)}[x-\theta A(u,u)]-\bigl(w-P^f_{K(x)}[x_0-\theta A(u_0,u_0)]\bigr)\bigr\|^2\\
&\qquad+\theta\langle v-P^f_{K(x)}[x-\theta A(u,u)]-\bigl(w-P^f_{K(x)}[x_0-\theta A(u_0,u_0)]\bigr),A(u,u)-A(u_0,u_0)\rangle\\
&\qquad-\bigl\|P^f_{K(x)}[x_0-\theta A(u_0,u_0)]-P^f_{K(x_0)}[x_0-\theta A(u_0,u_0)]\bigr\|\cdot\|x-x_0\|\\
&\qquad+\theta\langle A(u,u)-A(u_0,u_0),x-x_0\rangle\\
&\quad\geq\bigl\|v-P^f_{K(x)}[x-\theta A(u,u)]-\bigl(w-P^f_{K(x)}[x_0-\theta A(u_0,u_0)]\bigr)\bigr\|^2\\
&\qquad+\theta\langle v-P^f_{K(x)}[x-\theta A(u,u)]-\bigl(w-P^f_{K(x)}[x_0-\theta A(u_0,u_0)]\bigr),A(u,u)-A(u_0,u_0)\rangle\\
&\qquad-k\|x-x_0\|^2+\theta\langle A(u,u)-A(u_0,u_0),x-x_0\rangle.
\end{aligned}
\tag{2.8}
$$

Since

$$
\begin{aligned}
&\bigl\|v-P^f_{K(x)}[x-\theta A(u,u)]-\bigl(w-P^f_{K(x)}[x_0-\theta A(u_0,u_0)]\bigr)\bigr\|^2\\
&\qquad+2\left(\frac{\theta}{2}\right)\langle v-P^f_{K(x)}[x-\theta A(u,u)]\\
&\qquad-\bigl(w-P^f_{K(x)}[x_0-\theta A(u_0,u_0)]\bigr),A(u,u)-A(u_0,u_0)\rangle\\
&\quad\geq-\left(\frac{\theta}{2}\right)^2\bigl\|A(u,u)-A(u_0,u_0)\bigr\|^2,
\end{aligned}
\tag{2.9}
$$

from (2.8) and (2.9) we have

$$
\begin{aligned}
&\langle R(x,\theta)-R(x_0,\theta),x-x_0\rangle\\
&\quad\geq-\frac{\theta^2}{4}\bigl\|A(u,u)-A(u_0,u_0)\bigr\|^2-k\|x-x_0\|^2+\theta\langle A(u,u)-A(u_0,u_0),x-x_0\rangle.
\end{aligned}
$$

By using the strong monotonicity and the Lipschitz continuity of A we have

$$\langle R(x,\theta) - R(x_0,\theta), x - x_0\rangle \geq -\frac{\beta^2\theta^2}{4}\|u - u_0\|^2 - k\|x - x_0\|^2 + \alpha\theta\|x - x_0\|^2.$$

Finally, using the M-Lipschitz continuity of T, we get

$$\langle R(x,\theta) - R(x_0,\theta), x - x_0\rangle \geq \left(\alpha\theta - \frac{\beta^2\mu^2\theta^2}{4} - k\right)\|x - x_0\|^2,$$

implies

$$\langle R(x,\theta) - R(x_0,\theta), x - x_0\rangle \geq \xi(\theta)\|x - x_0\|^2, \quad \text{where } \xi(\theta) = \alpha\theta - \frac{\beta^2\mu^2\theta^2}{4} - k > 0.$$

By a classical argument

$$\|R(x,\theta) - R(x_0,\theta)\| \geq \xi(\theta)\|x - x_0\|, \quad \forall x, x_0 \in H.$$

Since x_0 is solution of SVMQVIP (1.1), $R(x_0,\theta) = 0$ and hence

$$\|x - x_0\| \leq \frac{4}{4\alpha\theta - \beta^2\mu^2\theta^2 - 4k}\|R(x,\theta)\|.$$

This completes the proof. □

Remark 2.3 Theorem 2.3 generalizes Lemma 1 of [7] and Lemma 2 of [10].

3 Regularized gap functions for SVMQVIP (1.1)

In this section by using an approach due to Fukushima [8], we construct another gap function associated with problem SVMQVIP (1.1), which can be viewed as a regularized gap function. For $\theta > 0$, the functions G_θ is defined by

$$G_\theta(x) = \max_{y\in K(x)}\left\{\langle A(u,u), x - y\rangle - f(x,y) + f(x,x) - \frac{1}{2\theta}\|x - y\|^2\right\}, \tag{3.1}$$

which is finite-valued everywhere and is differentiable whenever all operators involved in $G_\theta(x)$, are differentiable.

Lemma 3.1 *For any $\theta > 0$, $G_\theta(x)$ can be written as*

$$\begin{aligned} G_\theta(x) = {}& \langle A(u,u), x - P^f_{K(x)}[x - \theta A(u,u)]\rangle - f(x, P^f_{K(x)}[x - \theta A(u,u)]) \\ & + f(x,x) - \frac{1}{2\theta}\|x - P^f_{K(x)}[x - \theta A(u,u)]\|^2, \quad \forall x \in H. \end{aligned} \tag{3.2}$$

Proof If $x \notin \operatorname{dom} f$ then equation (3.2) is correct, because $f \equiv \infty$ while the other terms are all finite (recall that $P^f_{K(x)}(z) \in \operatorname{dom} f$ for any $z \in H$).

Consider now any $x \in \operatorname{dom} f$. Denote by $t(y)$ the function being maximized in (3.1). Let z be the (unique, by concavity of $t(y)$) element at which the maximum is realized in (3.1).

Then z is uniquely characterized by the optimality condition

$$0 \in \partial\big(-t(z)\big) = A(u,u) + \partial f(\cdot,z) + \frac{1}{\theta}(z-x) = \partial f(\cdot,z) + \frac{1}{\theta}\big[z-\big(x-\theta A(u,u)\big)\big].$$

That is to say, $z = \arg\min_{y\in K(x)}\{f(x,y) + \frac{1}{2\theta}\|y-(x-\theta A(u,u))\|^2\} = P^f_{K(x)}[x-\theta A(u,u)]$, where the second equation follows from the definition of the proximal mapping $P^f_{K(x)}(\cdot)$. This completes the proof. □

Next, we show the function $G_\theta(x)$ for $\theta > 0$ given (3.1) is a gap function for SVMQVIP (1.1).

Theorem 3.1 *If $\theta > 0$, then we have*

$$G_\theta(x) \geq \frac{1}{2\theta}\big\|R(x,\theta)\big\|^2, \quad \forall x \in H.$$

In particular, $G_\theta(x) = 0$, if and only if x is a solution of SVMQVIP (1.1).

Proof Fix any fixed $x \in H$, and $\theta > 0$. Observe that

$$x-\theta A(u,u) \in (I+\theta\partial f)(I+\theta\partial f)^{-1}\big(x-\theta A(u,u)\big) = (I+\theta\partial f)P^f_{K(x)}\big[x-\theta A(u,u)\big],$$

which is equivalent to

$$-A(u,u) + \frac{1}{\theta}\big[x-P^f_{K(x)}\big[x-\theta A(u,u)\big]\big] \in \partial f\big(P^f_{K(x)}\big[x-\theta A(u,u)\big]\big).$$

By the definition of a sub-differential, we have

$$\begin{aligned}&\Big\langle A(u,u) - \frac{1}{\theta}\big(x-P^f_{K(x)}\big[x-\theta A(u,u)\big]\big), y-P^f_{K(x)}\big[x-\theta A(u,u)\big]\Big\rangle\\ &\quad + f\big(P^f_{K(x)}\big[x-\theta A(u,u)\big],y\big) - f\big(P^f_{K(x)}\big[x-\theta A(u,u)\big],P^f_{K(x)}\big[x-\theta A(u,u)\big]\big) \geq 0.\end{aligned}$$

Taking $y = x$ in the above inequality, we get

$$\begin{aligned}&\Big\langle A(u,u) - \frac{1}{\theta}\big(x-P^f_{K(x)}\big[x-\theta A(u,u)\big]\big), x-P^f_{K(x)}\big[x-\theta A(u,u)\big]\Big\rangle\\ &\quad + f\big(P^f_{K(x)}\big[x-\theta A(u,u)\big],x\big) - f\big(P^f_{K(x)}\big[x-\theta A(u,u)\big],P^f_{K(x)}\big[x-\theta A(u,u)\big]\big) \geq 0,\\ &\big\langle A(u,u), x-P^f_{K(x)}\big[x-\theta A(u,u)\big]\big\rangle + f\big(P^f_{K(x)}\big[x-\theta A(u,u)\big],x\big)\\ &\quad - f\big(P^f_{K(x)}\big[x-\theta A(u,u)\big],P^f_{K(x)}\big[x-\theta A(u,u)\big]\big) \geq \frac{1}{\theta}\big\langle R(x,\theta),R(x,\theta)\big\rangle. \qquad (3.3)\end{aligned}$$

Combining (3.2) with (3.3) and by using the skew-symmetry of f, we get

$$G_\theta(x) \geq \frac{1}{\theta}\big\langle R(x,\theta),R(x,\theta)\big\rangle - \frac{1}{2\theta}\big\|R(x,\theta)\big\|^2 = \frac{1}{2\theta}\big\|R(x,\theta)\big\|^2.$$

Clearly, we have $G_\theta(x) \geq 0$, for all $x \in H$.

Now from the above conclusion, if $G_\theta(x) = 0$, then $R(x,\theta) = 0$. Hence by Lemma 2.1, we see that $x \in H$ is a solution of SVMQVIP (1.1). Conversely, if $x \in H$ is a solution of SVMQVIP (1.1), then $x = P^f_{K(x)}[x - \theta A(u,u)]$, consequently, from (3.2) we have $G_\theta(x) = 0$. This completes the proof. □

As a consequence of Theorem 2.2 and Theorem 3.1, we have the following result on error bound in terms of $G_\theta(x)$ for SVMQVIP (1.1).

Corollary 3.1 *Assume that x_0 be a solution of SVMQVIP* (1.1). *Let the operator A be strongly monotone and Lipschitz continuous with constants $\alpha, \beta > 0$, respectively. Let T be a M-Lipschitz continuous with constant $\mu > 0$ and f is skew-symmetric. If there exists $k < \frac{\alpha}{\beta\mu}$ such that for any $\theta > \frac{k}{\alpha - \beta\mu k}$ and*

$$\left\| P^f_{K(x)}(w) - P^f_{K(x_0)}(w) \right\| \leq k\|x - x_0\|, \quad \forall x, x_0, w \in H,$$

then, for any $x \in H$ and $\theta > \frac{k}{\alpha - \beta\mu k}$, we have

$$\|x - x_0\| \leq \frac{1 + \theta\beta\mu}{\alpha\theta - (1 + \theta\beta\mu)k} \sqrt{2\theta}\sqrt{G_\theta(x)}, \quad \forall x \in H.$$

Also by using Theorem 2.3 and Theorem 3.1, we obtain another error bound for SVMQVIP (1.1).

Corollary 3.2 *Assume that x_0 is the solution of SVMQVIP* (1.1). *Let the operator A be strongly monotone and Lipschitz continuous with constants $\alpha, \beta > 0$, respectively. Let T be a M-Lipschitz continuous with constant $\mu > 0$ and f is skew-symmetric. If there exists $k > 0$, such that for any $\theta < \frac{2\alpha}{\beta^2\mu^2}$ and*

$$\left\| P^f_{K(x)}(z) - P^f_{K(x_0)}(z) \right\| \leq k\|x - x_0\|, \quad \forall x, x_0, z \in H, \text{with } k < 1 - \sqrt{\theta^2\beta^2\mu^2 - 2\alpha\theta + 1},$$

then, for any $x \in H$, we have

$$\|x - x_0\| \leq \frac{4}{4\alpha\theta - \beta^2\mu^2\theta^2 - 4k} \sqrt{2\theta}\sqrt{G_\theta(x)}, \quad \forall x \in H.$$

Remark 3.1 Corollary 3.1 and Corollary 3.2 generalize the corresponding results of [7, 10, 17, 20].

Now, we derive the error bound for SVMQVIP (1.1) without using the Lipschitz continuity of the A and T.

Theorem 3.2 *Let A is strongly monotone with constant $\alpha > 0$ and f is skew-symmetric. If x_0 is a solution of SVMQVIP* (1.1), *then*

$$\|x - x_0\| \leq \frac{1}{\sqrt{(\alpha - \frac{1}{2\theta})}} \sqrt{G_\theta(x)}, \quad \forall x \in H, \theta > \frac{1}{2\alpha}.$$

Proof From (3.1), it can be written as

$$G_\theta(x) \geq \langle A(u,u), x - x_0\rangle - f(x,x_0) + f(x,x) - \frac{1}{2\theta}\|x - x_0\|^2.$$

By using the strong monotonicity of A, we have

$$\begin{aligned} G_\theta(x) \geq{}& \langle A(u_0,u_0), x - x_0\rangle + \alpha\|x - x_0\|^2 - f(x,x_0) \\ &+ f(x,x) - \frac{1}{2\theta}\|x - x_0\|^2. \end{aligned} \tag{3.4}$$

Since x_0 is a solution of SVMQVIP (1.1),

$$\langle A(u_0,u_0), y - x_0\rangle + f(x_0,y) - f(x_0,x_0) \geq 0, \quad \forall y \in K(x_0).$$

Taking $y = x$ in the above inequality

$$\langle A(u_0,u_0), x - x_0\rangle + f(x_0,x) - f(x_0,x_0) \geq 0. \tag{3.5}$$

Combining (3.4) with (3.5) and using the skew-symmetry of f, we get

$$G_\theta(x) \geq \alpha\|x - x_0\|^2 - \frac{1}{2\theta}\|x - x_0\|^2 = \left(\alpha - \frac{1}{2\theta}\right)\|x - x_0\|^2,$$

which implies

$$\|x - x_0\| \leq \frac{1}{\sqrt{(\alpha - \frac{1}{2\theta})}}\sqrt{G_\theta(x)}.$$

This completes the proof. □

Remark 3.2 Theorem 3.2 generalizes Theorem 5 of [20] and Theorem 3.4 of [17].

4 Global error bounds for SVMQVIP (1.1)

In this section, we consider another gap function associated with SVMQVIP (1.1), which can be viewed as a difference of two regularized gap functions with distinct parameters, known as the D-gap function, which was introduced and studied by [19, 20, 22] for solving variational inequalities and complementarity problems.

For each $x \in H$, the difference of two regularized gap functions $G_\theta(x) - G_\psi(x)$, where $\theta > \psi > 0$ for SVMQVIP (1.1) will not be well defined for $x \notin \operatorname{dom}(f)$, as both quantities are not finite. Nevertheless, we shall define the D-gap function by taking a formal difference of equations (3.1) for the two parameters $\theta > \psi > 0$.

The D-gap function associated with SVMQVIP (1.1) is given by

$$\begin{aligned} D_{\theta,\psi}(x) = \max_{y\in K(x)}\biggl\{&\langle A(u,u), x - y\rangle - f(x,y) + f(x,x) \\ &+ \frac{1}{2\psi}\|x - y\|^2 - \frac{1}{2\theta}\|x - y\|^2\biggr\}, \quad x \in H, \theta > \psi > 0. \end{aligned} \tag{4.1}$$

The D-gap function defined by (4.1) can be written as

$$\begin{aligned} D_{\theta,\psi}(x) = {} & \langle A(u,u), P^f_{K(x)}[x - \psi A(u,u)] - P^f_{K(x)}[x - \theta A(u,u)]\rangle \\ & - f(P^f_{K(x)}[x - \psi A(u,u)], P^f_{K(x)}[x - \theta A(u,u)]) \\ & + f(P^f_{K(x)}[x - \psi A(u,u)], P^f_{K(x)}[x - \psi A(u,u)]) \\ & + \frac{1}{2\psi}\|x - P^f_{K(x)}[x - \psi A(u,u)]\|^2 - \frac{1}{2\theta}\|x - P^f_{K(x)}[x - \theta A(u,u)]\|^2. \end{aligned}$$

Further, it can be written as

$$\begin{aligned} D_{\theta,\psi}(x) = {} & \langle A(u,u), R(x,\theta) - R(x,\psi)\rangle - f(P^f_{K(x)}[x - \psi A(u,u)], P^f_{K(x)}[x - \theta A(u,u)]) \\ & + f(P^f_{K(x)}[x - \psi A(u,u)], P^f_{K(x)}[x - \psi A(u,u)]) \\ & + \frac{1}{2\psi}\|R(x,\psi)\|^2 - \frac{1}{2\theta}\|R(x,\theta)\|^2. \end{aligned} \tag{4.2}$$

Next, we derive global error bounds for SVMQVIP (1.1).

Theorem 4.1 $\forall x \in H$, $\theta > \psi > 0$, *we have*

$$\frac{1}{2}\left(\frac{1}{\psi} - \frac{1}{\theta}\right)\|R(x,\psi)\|^2 \leq \|D_{\theta,\psi}(x)\| \leq \frac{1}{2}\left(\frac{1}{\psi} - \frac{1}{\theta}\right)\|R(x,\theta)\|^2.$$

In particular $D_{\theta,\psi}(x) = 0$, *if and only if*, $x \in H$ *solves SVMQVIP* (1.1).

Proof By the definition of a sub-differential, we have

$$\begin{aligned} & \left\langle A(u,u) - \frac{1}{\theta}(x - P^f_{K(x)}[x - \theta A(u,u)]), y - P^f_{K(x)}[x - \theta A(u,u)]\right\rangle \\ & \quad + f(P^f_{K(x)}[x - \theta A(u,u)], y) - f(P^f_{K(x)}[x - \theta A(u,u)], P^f_{K(x)}[x - \theta A(u,u)]) \geq 0. \end{aligned}$$

Taking $y = P^f_{K(x)}[x - \psi A(u,u)]$ in the above inequality, we get

$$\begin{aligned} & \left\langle A(u,u) - \frac{1}{\theta}(x - P^f_{K(x)}[x - \theta A(u,u)]), P^f_{K(x)}[x - \psi A(u,u)] - P^f_{K(x)}[x - \theta A(u,u)]\right\rangle \\ & \quad + f(P^f_{K(x)}[x - \theta A(u,u)], P^f_{K(x)}[x - \psi A(u,u)]) \\ & \quad - f(P^f_{K(x)}[x - \theta A(u,u)], P^f_{K(x)}[x - \theta A(u,u)]) \geq 0, \end{aligned}$$

which implies that

$$\begin{aligned} & \langle A(u,u), R(x,\theta) - R(x,\psi)\rangle \\ & \quad \geq \frac{1}{\theta}\langle R(x,\theta), R(x,\theta) - R(x,\psi)\rangle - f(P^f_{K(x)}[x - \theta A(u,u)], P^f_{K(x)}[x - \psi A(u,u)]) \\ & \qquad + f(P^f_{K(x)}[x - \theta A(u,u)], P^f_{K(x)}[x - \theta A(u,u)]). \end{aligned} \tag{4.3}$$

Combining (4.2) with (4.3) and using the skew-symmetry of f, we get

$$\begin{aligned} D_{\theta,\psi}(x) &\geq \frac{1}{\theta}\langle R(x,\theta), R(x,\theta)-R(x,\psi)\rangle + \frac{1}{2\psi}\|R(x,\psi)\|^2 - \frac{1}{2\theta}\|R(x,\theta)\|^2 \\ &= \frac{1}{2}\left(\frac{1}{\psi}-\frac{1}{\theta}\right)\|R(x,\psi)\|^2 + \frac{1}{\theta}\langle R(x,\theta), R(x,\theta)-R(x,\psi)\rangle \\ &\quad - \frac{1}{2\theta}\|R(x,\theta)-R(x,\psi)\|^2 - \frac{1}{\theta}\langle R(x,\psi), R(x,\theta)-R(x,\psi)\rangle \\ &= \frac{1}{2}\left(\frac{1}{\psi}-\frac{1}{\theta}\right)\|R(x,\psi)\|^2 + \frac{1}{2\theta}\|R(x,\theta)-R(x,\psi)\|^2 \\ &\geq \frac{1}{2}\left(\frac{1}{\psi}-\frac{1}{\theta}\right)\|R(x,\psi)\|^2, \end{aligned} \tag{4.4}$$

which implies the left-most inequality in the assertion.

On the other hand,

$$-A(u,u) + \frac{1}{\psi}R(x,\psi) \in \partial f\left(P^f_{K(x)}\left[x-\psi A(u,u)\right]\right),$$

which implies that

$$\begin{aligned} &\langle A(u,u), R(x,\theta)-R(x,\psi)\rangle \\ &\quad\leq \frac{1}{\psi}\langle R(x,\psi), R(x,\theta)-R(x,\psi)\rangle - f\left(P^f_{K(x)}\left[x-\theta A(u,u)\right], P^f_{K(x)}\left[x-\psi A(u,u)\right]\right) \\ &\qquad + f\left(P^f_{K(x)}\left[x-\theta A(u,u)\right], P^f_{K(x)}\left[x-\theta A(u,u)\right]\right). \end{aligned} \tag{4.5}$$

Similarly to the analysis above, we then obtain

$$\begin{aligned} D_{\theta,\psi}(x) &\leq \frac{1}{\psi}\langle R(x,\psi), R(x,\theta)-R(x,\psi)\rangle + \frac{1}{2\psi}\|R(x,\psi)\|^2 - \frac{1}{2\theta}\|R(x,\theta)\|^2 \\ &= \frac{1}{2}\left(\frac{1}{\psi}-\frac{1}{\theta}\right)\|R(x,\theta)\|^2 - \frac{1}{2\psi}\|R(x,\theta)-R(x,\psi)\|^2 \\ &\leq \frac{1}{2}\left(\frac{1}{\psi}-\frac{1}{\theta}\right)\|R(x,\theta)\|^2, \end{aligned} \tag{4.6}$$

which implies the right-most inequality in the assertion. Combining (4.4) and (4.6), we obtain the required result. The last assertion now follows from Lemma 2.1. □

As a consequence of Theorem 2.2 and Theorem 4.1, we obtain the following result on the global error bound for SVMQVIP (1.1).

Corollary 4.1 *Assume that x_0 be a solution of SVMQVIP* (1.1). *Let the operator A be strongly monotone and Lipschitz continuous with constants $\alpha, \beta > 0$, respectively. Let T be a M-Lipschitz continuous with constant $\mu > 0$ and f is skew-symmetric. If there exists $k < \frac{\alpha}{\beta\mu}$ such that for any $\theta > \frac{k}{\alpha-\beta\mu k}$ and*

$$\left\|P^f_{K(x)}(w) - P^f_{K(x_0)}(w)\right\| \leq k\|x-x_0\|, \quad \forall x, x_0, w \in H,$$

then, for any $x \in H$ and $\theta > \frac{k}{\alpha - \beta\mu k}$, we have

$$\|x - x_0\| \le \frac{1 + \theta\beta\mu}{\alpha\theta - (1 + \theta\beta\mu)k} \sqrt{\frac{2\theta\psi}{\theta - \psi}} \sqrt{D_{\theta,\psi}(x)}, \quad \forall x \in H.$$

Also by using Theorem 2.3 and Theorem 4.1, we obtain another global error bound for SVMQVIP (1.1).

Corollary 4.2 *Assume that x_0 is the solution of SVMQVIP* (1.1). *Let the operator A be strongly monotone and Lipschitz continuous with constants $\alpha, \beta > 0$, respectively. Let T be a M-Lipschitz continuous with constant $\mu > 0$ and f is skew-symmetric. If there exists $k > 0$, such that for any $\theta < \frac{2\alpha}{\beta^2\mu^2}$ and*

$$\left\| P^f_{K(x)}(z) - P^f_{K(x_0)}(z) \right\| \le k\|x - x_0\|, \quad \forall x, x_0, z \in H, \text{with } k < 1 - \sqrt{\theta^2\beta^2\mu^2 - 2\alpha\theta + 1},$$

then, for any $x \in H$, we have

$$\|x - x_0\| \le \frac{4}{4\alpha\theta - \beta^2\mu^2\theta^2 - 4k} \sqrt{\frac{2\theta\psi}{\theta - \psi}} \sqrt{D_{\theta,\psi}(x)}, \quad \forall x \in H.$$

Remark 4.1 Corollary 4.1 and Corollary 4.2 generalize the corresponding results of [7, 10, 17, 20].

Now, we derive the global error bound for SVMQVIP (1.1) without using the Lipschitz continuity of A and T.

Theorem 4.2 *Let x_0 is a solution of SVMQVIP* (1.1). *Suppose that A is strongly monotone with constant $\alpha > 0$ and f is skew-symmetric, then*

$$\|x - x_0\| \le \frac{1}{\sqrt{[\alpha + \frac{1}{2}(\frac{1}{\psi} - \frac{1}{\theta})]}} \sqrt{D_{\theta,\psi}(x)}, \quad \forall x \in H, \alpha > \frac{1}{2}\left(\frac{1}{\theta} - \frac{1}{\psi}\right).$$

Proof From (4.1), it can be written as

$$D_{\theta,\psi}(x) \ge \langle A(u,u), x - x_0 \rangle - f(x, x_0) + f(x, x) + \frac{1}{2\psi}\|x - x_0\|^2 - \frac{1}{2\theta}\|x - x_0\|^2.$$

By using the strong monotonicity of A, we have

$$\begin{aligned} D_{\theta,\psi}(x) \ge{}& \langle A(u_0,u_0), x - x_0 \rangle + \alpha\|x - x_0\|^2 - f(x, x_0) + f(x, x) \\ &+ \frac{1}{2\psi}\|x - x_0\|^2 - \frac{1}{2\theta}\|x - x_0\|^2. \end{aligned} \tag{4.7}$$

Since x_0 is a solution of SVMQVIP (1.1),

$$\langle A(u_0,u_0), y - x_0 \rangle + f(x_0, y) - f(x_0, x_0) \ge 0, \quad \forall y \in K(x_0).$$

Taking $y = x$ in the above inequality

$$\langle A(u_0, u_0), x - x_0 \rangle + f(x_0, x) - f(x_0, x_0) \geq 0. \tag{4.8}$$

Combining (4.7) with (4.8) and using the skew-symmetry of f, we get

$$D_{\theta,\psi}(x) \geq \alpha \|x - x_0\|^2 + \frac{1}{2\psi}\|x - x_0\|^2 - \frac{1}{2\theta}\|x - x_0\|^2 = \left(\alpha + \frac{1}{2\psi} - \frac{1}{2\theta}\right)\|x - x_0\|^2,$$

which implies

$$\|x - x_0\| \leq \frac{1}{\sqrt{[\alpha + \frac{1}{2}(\frac{1}{\psi} - \frac{1}{\theta})]}}\sqrt{D_{\theta,\psi}(x)}.$$

This completes the proof. □

Remark 4.2 Theorem 4.2 generalizes Theorem 7 of [20] and Theorem 3.6 of [17].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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