Zheng et al. *Journal of Inequalities and Applications* (2016) 2016:7
DOI 10.1186/s13660-015-0943-6

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

Some new generalized retarded inequalities for discontinuous functions and their applications

Zhaowen Zheng[1*], Xin Gao[1] and Jing Shao[2]

*Correspondence:
zhwzheng@126.com
[1]School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, P.R. China
Full list of author information is available at the end of the article

Abstract
In this paper, some new generalized retarded inequalities for discontinuous functions are discussed, which are effective in dealing with the qualitative theory of some impulsive differential equations and impulsive integral equations. Compared with some existing integral inequalities, these estimations can be used as tools in the study of differential-integral equations with impulsive conditions.

Keywords: retarded differential-integral equation; global existence; estimation; impulsive equation

1 Introduction

In analyzing the impulsive phenomenon of a physical system governed by certain differential and integral equations, one often needs some kinds of inequalities, such as Gronwall-like inequalities; these inequalities and their various linear and nonlinear generalizations are crucial in the discussion of the existence, uniqueness, boundedness, stability, and other qualitative properties of solutions of differential and integral equations (see [1–12] and references therein). In [1], Lipovan studied the inequality with delay ($b(t) \le t$, $b(t) \to \infty$ as $t \to \infty$)

$$u(t) \le c + \int_{t_0}^{t} f(s)w\bigl(u(s)\bigr)\,ds + \int_{b(t_0)}^{b(t)} g(s)w\bigl(u(s)\bigr)\,ds, \quad t_0 < t < t_1,$$

in [2], Agarwal *et al.* investigated the retarded Gronwall-like inequality

$$u(t) \le a(t) + \sum_{i=1}^{n} \int_{b_i(t_0)}^{b_i(t)} f_i(t,s)w_i\bigl(u(s)\bigr)\,ds,$$

in 2004, Borysenko [3] obtained the explicit bound to the unknown function of the following integral inequality with impulsive effect:

$$u(t) \le a(t) + \int_{t_0}^{t} f(s)u(s)\,ds + \sum_{t_0<t_i<t} \alpha_i u^r(t_i - 0),$$

in 2007, Iovane [4] studied the following integral inequalities:

$$u(t) \leq a(t) + \int_{t_0}^{t} f(s)u\big(\lambda(s)\big)\,ds + \sum_{t_0<t_i<t} \alpha_i u^r(t_i - 0),$$

$$u(t) \leq a(t) + q(t)\bigg[\int_{t_0}^{t} f(s)u\big(\alpha(s)\big)\,ds + \int_{t_0}^{t} f(s)\int_{t_0}^{s} g(t)u\big(\tau(t)\big)\,dt\,ds + \sum_{t_0<t_i<t} \alpha_i u^r(t_i - 0)\bigg],$$

in 2012, Wang and Li [5] gave the upper bound of solutions for the nonlinear inequality

$$v^p(t) \leq A_0(t) + \frac{p}{p-q}\int_{t_0}^{t} f(s)v^q\big(\tau(s)\big)\,ds + \sum_{t_0<t_i<t} \alpha_i v^q(t_i - 0),$$

in 2013, Yan [6] considered the following inequality:

$$u(t) \leq a(t) + \int_{t_0}^{t} f(t,s)u\big(\alpha(s)\big)\,ds + \int_{t_0}^{t} f(t,s)\bigg(\int_{t_0}^{s} g(s,\lambda)u\big(\tau(\lambda)\big)\,d\lambda\bigg)\,ds + q(t)\sum_{t_0<t_i<t} \alpha_i u^r(t_i - 0),$$

and gave an upper bound estimation. Because of the fundamental importance, over the years, many generalizations and analogous results have been established. However, the bounds given on such inequalities are not directly applicable in the study of some complicated retarded inequalities for discontinuous functions. It is desirable to establish new inequalities of the above type, which can be used more effectively in the study of certain classes of retarded nonlinear differential and integral equations. So in this paper, the following new integral inequalities are presented:

$$u(t) \leq a(t) + \sum_{i=1}^{N}\int_{t_0}^{t} g_i(s)u\big(\phi_i(s)\big)\,ds + \sum_{j=1}^{L}\int_{t_0}^{t} b_j(s)\int_{t_0}^{s} c_j(\theta)u\big(w_j(\theta)\big)\,d\theta\,ds, \tag{1}$$

$$u^p(t) \leq a_0(t) + \frac{p}{p-q}\sum_{i=1}^{N}\int_{t_0}^{t} g_i(s)u^q\big(\phi_i(s)\big)\,ds + \sum_{j=1}^{L}\int_{t_0}^{t} b_j(s)\int_{t_0}^{s} c_j(\theta)u^q\big(w_j(\theta)\big)\,d\theta\,ds + \sum_{t_0<t_i<t} \beta_i u^q(t_i - 0), \tag{2}$$

$$u^p(t) \leq a_0(t) + q_1(t)\sum_{i=1}^{N}\int_{t_0}^{t} g_i(t,s)u^q\big(\phi_i(s)\big)\,ds + \sum_{j=1}^{L}\int_{t_0}^{t} b_j(t,s)\int_{t_0}^{s} c_j(s,\theta)u^q\big(w_j(\theta)\big)\,d\theta\,ds + q_2(t)\sum_{t_0<t_i<t} \beta_i u^q(t_i - 0). \tag{3}$$

We give the explicit upper bounds estimation of unknown function of these new inequalities, some applications of these inequalities in impulsive differential equations are also involved.

2 Main results

We consider the inequality (1) first.

Theorem 2.1 *Suppose that for $t_0 \in \mathbb{R}$ and $t_0 \le t < \infty$, the functions $u(t)$, $a(t)$, and $g_i(t)$, $b_j(t)$, $c_j(t)$ $(1 \le i \le N, 1 \le j \le L)$ are positive and continuous functions on $[t_0, \infty)$, and $c_j(t)$ are nondecreasing functions on $[t_0, \infty)$. Moreover, $\phi_i(t)$, $w_j(t)$ are continuous functions on $[t_0, \infty)$ and $t_0 \le \phi_i(t) \le t$, $t_0 \le w_j(t) \le t$ for $1 \le i \le N$ and $1 \le j \le L$. Then the inequality* (1) *implies that*

$$u(t) \le a(t) + \exp\left(\int_{t_0}^{t} Q_1(s)\,ds\right)\left(\int_{t_0}^{t} Y(s)\exp\left(-\int_{t_0}^{s} Q_1(\tau)\,d\tau\right)ds\right), \tag{4}$$

where

$$Q_1(t) = L + \sum_{i=1}^{N} g_i(t) + \sum_{j=1}^{L} b_j(t)c_j^1(t),$$

and $c_j^1(t) = \max\{c_j(t), 1\}$.

Proof Let $a(t) + z_1(t)$ denote the function on the right-hand side of inequality (1). Obviously $z_1(t)$ is a positive and increasing function, and it satisfies $u(t) \le a(t) + z_1(t)$,

$$\begin{aligned}\frac{dz_1(t)}{dt} &= \sum_{i=1}^{N} g_i(t)u\bigl(\phi_i(t)\bigr) + \sum_{j=1}^{L} b_j(t)\int_{t_0}^{t} c_j(\theta)u\bigl(w_j(\theta)\bigr)\,d\theta \\ &\le \sum_{i=1}^{N} g_i(t)\bigl(a\bigl(\phi_i(t)\bigr) + z_1\bigl(\phi_i(t)\bigr)\bigr) \\ &\quad + \sum_{j=1}^{L} b_j(t)\int_{t_0}^{t} c_j(\theta)\bigl(a\bigl(w_j(\theta)\bigr) + z_1\bigl(w_j(\theta)\bigr)\bigr)\,d\theta.\end{aligned} \tag{5}$$

Let

$$Y(t) = \sum_{i=1}^{N} g_i(t)a\bigl(\phi_i(t)\bigr) + \sum_{j=1}^{L} b_j(t)\int_{t_0}^{t} c_j(\theta)a\bigl(w_j(\theta)\bigr)\,d\theta, \tag{6}$$

$$z_2(t) = \sum_{i=1}^{N} g_i(t)z_1(t) + \sum_{j=1}^{L} b_j(t)\int_{t_0}^{t} c_j(\theta)z_1\bigl(w_j(\theta)\bigr)\,d\theta. \tag{7}$$

Obviously, $\frac{dz_1(t)}{dt} \le Y(t) + z_2(t)$. Let $c_j^1(t) = \max\{c_j(t), 1\}$. We can obtain

$$z_2(t) \le \left(\sum_{i=1}^{N} g_i(t) + \sum_{j=1}^{L} b_j(t)c_j^1(t)\right)\left(z_1(t) + \sum_{j=1}^{L}\int_{t_0}^{t} z_1\bigl(w_j(\theta)\bigr)\,d\theta\right). \tag{8}$$

Let

$$z_3(t) = z_1(t) + \sum_{j=1}^{L}\int_{t_0}^{t} z_1\bigl(w_j(\theta)\bigr)\,d\theta,$$

we note that $z_3(t)$ is a positive and nondecreasing function on I with $z_3(t_0) = 0$, and $z_1(t) \leq z_3(t)$, which satisfies

$$\begin{aligned}
\frac{dz_3(t)}{dt} &= \frac{dz_1(t)}{dt} + \sum_{j=1}^{L} z_1\bigl(w_j(t)\bigr) \\
&\leq Y(t) + \left(\sum_{i=1}^{N} g_i(t) + \sum_{j=1}^{L} b_j(t)c_j^1(t)\right) z_3(t) + \sum_{j=1}^{L} z_3\bigl(w_j(t)\bigr) \\
&\leq \left(L + \sum_{i=1}^{N} g_i(t) + \sum_{j=1}^{L} b_j(t)c_j^1(t)\right) z_3(t) + Y(t) \\
&= Q_1(t)z_3(t) + Y(t). \qquad (9)
\end{aligned}$$

Consider the initial value problem of the differential equation

$$\begin{cases} \frac{dz_4(t)}{dt} = Q_1(t)z_4(t) + Y(t), \\ z_4(t_0) = 0. \end{cases} \qquad (10)$$

The solution of equation (10) is

$$z_4(t) = \exp\left(\int_{t_0}^{t} Q_1(s)\,ds\right)\left(\int_{t_0}^{t} Y(s)\exp\left(-\int_{t_0}^{s} Q_1(\tau)\,d\tau\right)ds\right). \qquad (11)$$

Then by comparison of the differential inequality, we have $z_3(t) \leq z_4(t)$, so

$$u(t) \leq a(t) + \exp\left(\int_{t_0}^{t} Q_1(s)\,ds\right)\left(\int_{t_0}^{t} Y(s)\exp\left(-\int_{t_0}^{s} Q_1(\tau)\,d\tau\right)ds\right). \qquad (12)$$

This completes the proof. □

Now, we consider the inequality (2).

Theorem 2.2 *Suppose that $g_i(t)$, $b_j(t)$, $\phi_i(t)$, $w_j(t)$ are defined as those in Theorem* 2.1, *$p > q > 0$, $t_0 < t_1 < t_2 < \cdots$, $\beta_i \geq 0$, $a_0(t)$ is continuous and nondecreasing function on $[t_0, t_1)$ and $a_0(t) \geq 1$. $u(t)$ is a piecewise continuous nonnegative function on $[t_0, \infty)$ with only the first discontinuous points t_i, $i = 1, 2, \ldots$. Then, for all $t \in I_k$ and $I_k = [t_{k-1}, t_k]$, we obtain*

$$u(t) \leq \tau_k^{q^{-1}}(t), \qquad (13)$$

where

$$\begin{aligned}
\tau_k(t) &= \left\{a_{k-1}(t)\exp\left(\int_{t_{k-1}}^{t} Q_2(s)\,ds\right)\right\}^{\frac{q}{p}} \\
&\quad\times \left[1 + L\left(1 - \frac{q}{p}\right)\bigl(1 + a_{k-1}(t)\bigr)^{\frac{p}{p-q}}\right. \\
&\quad\times \left.\int_{t_{k-1}}^{t} \exp\left(-\int_{t_{k-1}}^{s}\left(1 - \frac{q}{p}\right)Q_2(\tau)\,d\tau\right)ds\right]^{\frac{q}{p-q}}, \qquad (14)
\end{aligned}$$

$$a_k(t) = a_0(t) + \frac{p}{p-q}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\sum_{i=1}^{N} g_i(s)\tau_j\big(\phi_i(s)\big)\,ds + \sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\sum_{m=1}^{L} b_m(s)\int_{t_0}^{s} c_m(\theta)\tau_j\big(w_m(\theta)\big)\,d\theta\,ds + \sum_{j=1}^{k}\beta_j\tau_j(t_j), \quad t \in I_k.$$

Proof Let $V = u^q$. The inequality (2) is equivalent to

$$V^{\frac{p}{q}}(t) \le a_0(t) + \frac{p}{p-q}\sum_{i=1}^{N}\int_{t_0}^{t} g_i(s)V\big(\phi_i(s)\big)\,ds + \sum_{j=1}^{L}\int_{t_0}^{t} b_j(s)\int_{t_0}^{s} c_j(\theta)V\big(w_j(\theta)\big)\,d\theta\,ds + \sum_{t_0<t_i<t}\beta_i u^q(t_i-0), \quad \forall t \in [0,\infty). \tag{15}$$

Let $I_i = [t_{i-1}, t_i]$, $i = 1, 2, \ldots$. First, we consider the following inequality on I_1:

$$V^{\frac{p}{q}}(t) \le a_0(t) + \frac{p}{p-q}\sum_{i=1}^{N}\int_{t_0}^{t} g_i(s)V\big(\phi_i(s)\big)\,ds + \sum_{j=1}^{L}\int_{t_0}^{t} b_j(s)\int_{t_0}^{s} c_j(\theta)V\big(w_j(\theta)\big)\,d\theta\,ds. \tag{16}$$

For $T \in I_1$ and $t \in [t_0, T)$, let

$$Y_1(t) = a_0(T) + \frac{p}{p-q}\sum_{i=1}^{N}\int_{t_0}^{t} g_i(s)V\big(\phi_i(s)\big)\,ds + \sum_{j=1}^{L}\int_{t_0}^{t} b_j(s)\int_{t_0}^{s} c_j(\theta)V\big(w_j(\theta)\big)\,d\theta\,ds. \tag{17}$$

Obviously $Y_1(t) \ge 1$, and $V^{\frac{p}{q}}(t) \le Y_1(t)$, so $V(t) \le Y_1^{\frac{q}{p}}(t)$, and

$$\begin{aligned}\frac{dY_1(t)}{dt} &= \frac{p}{p-q}\sum_{i=1}^{N} g_i(t)V\big(\phi_i(t)\big) + \sum_{j=1}^{L} b_j(t)\int_{t_0}^{t} c_j(\theta)V\big(w_j(\theta)\big)\,d\theta \\ &\le \frac{p}{p-q}\sum_{i=1}^{N} g_i(t)Y_1^{\frac{q}{p}}(t) + \sum_{j=1}^{L} b_j(t)\int_{t_0}^{t} c_j(\theta)Y_1^{\frac{q}{p}}\big(w_j(\theta)\big)\,d\theta \\ &\le \left(\frac{p}{p-q}\sum_{i=1}^{N} g_i(t) + \sum_{j=1}^{L} b_j(t)c_j^1(t)\right)\left(Y_1(t) + \sum_{j=1}^{L}\int_{t_0}^{t} Y_1^{\frac{q}{p}}\big(w_j(\theta)\big)\,d\theta\right).\end{aligned} \tag{18}$$

Let

$$Y_2(t) = Y_1(t) + \sum_{j=1}^{L}\int_{t_0}^{t} Y_1^{\frac{q}{p}}\big(w_j(\theta)\big)\,d\theta.$$

Then $Y_1(t) \le Y_2(t)$, and differentiating $Y_2(t)$ implies

$$\begin{aligned}\frac{dY_2(t)}{dt} &= \frac{dY_1(t)}{dt} + \sum_{j=1}^{L} Y_1^{\frac{q}{p}}\big(w_j(t)\big) \\ &\le \left(\frac{p}{p-q}\sum_{i=1}^{N} g_i(t) + \sum_{j=1}^{L} b_j(t)c_j^1(t)\right) Y_2(t) + LY_2^{\frac{q}{p}}(t).\end{aligned} \tag{19}$$

Let

$$Q_2(t) = \frac{p}{p-q}\sum_{i=1}^{N} g_i(t) + \sum_{j=1}^{L} b_j(t)c_j^1(t).$$

Considering $\frac{dY_3(t)}{dt} = Q_2(t)Y_3(t) + LY_3^{\frac{q}{p}}(t)$, we obtain $Y_3^{\frac{-q}{p}}(t)\frac{dY_3(t)}{dt} = Q_2(t)Y_3^{1-\frac{q}{p}}(t) + L$. Denote $R(t) = Y_3^{\frac{p-q}{p}}(t)$, we have $Y_3(t) = R^{\frac{p}{p-q}}(t)$. Furthermore,

$$\begin{cases} \frac{dR(t)}{dt} = (Q_2(t)R(t) + L)(1 - \frac{q}{p}), \\ R(t_0) = a_0^{\frac{p-q}{p}}(T). \end{cases} \tag{20}$$

Then we get

$$\begin{aligned} R(t) = {}& \exp\left[\left(1 - \frac{q}{p}\right)\int_{t_0}^{t} Q_2(s)\,ds\right] \\ & \times \left[a_0^{\frac{p-q}{p}}(T) + L\left(1 - \frac{q}{p}\right)\int_{t_0}^{t} \exp\left(-\int_{t_0}^{s}\left(\frac{p-q}{p}\right)Q_2(\tau)\,d\tau\right)ds\right].\end{aligned} \tag{21}$$

Then

$$\begin{aligned} Y_3(t) = {}& a_0(T)\exp\left(\int_{t_0}^{t} Q_2(s)\,ds\right) \\ & \times \left[1 + L\left(1 - \frac{q}{p}\right)a_0^{\frac{p}{p-q}}(T)\int_{t_0}^{t} \exp\left(-\int_{t_0}^{s}\left(1 - \frac{q}{p}\right)Q_2(\tau)\,d\tau\right)ds\right]^{\frac{p}{p-q}}.\end{aligned} \tag{22}$$

By comparison of the differential inequality, we have $Y_2(t) \le Y_3(t)$. Moreover, $V^{\frac{p}{q}}(t) \le Y_2(t)$ implies $V(t) \le Y_3^{\frac{q}{p}}(t)$, and this inequality is equivalent to

$$\begin{aligned} V(t) \le {}& \left[a_0(T)\exp\left(\int_{t_0}^{t} Q_2(s)\,ds\right)\right]^{\frac{q}{p}} \\ & \times \left[1 + L\left(1 - \frac{q}{p}\right)a_0^{\frac{p}{p-q}}(T)\int_{t_0}^{t} \exp\left(-\int_{t_0}^{s}\left(1 - \frac{q}{p}\right)Q_2(\tau)\,d\tau\right)ds\right]^{\frac{q}{p-q}}.\end{aligned} \tag{23}$$

Letting $t = T$, where T is a positive constant chosen arbitrarily, we get

$$\begin{aligned} V(T) \le {}& \left[a_0(T)\exp\left(\int_{t_0}^{T} Q_2(s)\,ds\right)\right]^{\frac{q}{p}} \\ & \times \left[1 + L\left(1 - \frac{q}{p}\right)a_0^{\frac{p}{p-q}}(T)\int_{t_0}^{T} \exp\left(-\int_{t_0}^{s}\left(1 - \frac{q}{p}\right)Q_2(\tau)\,d\tau\right)ds\right]^{\frac{q}{p-q}}.\end{aligned} \tag{24}$$

Obviously,

$$\begin{aligned} V(t) &\le \left[a_0(t)\exp\left(\int_{t_0}^{t} Q_2(s)\,ds\right)\right]^{\frac{q}{p}} \\ &\quad\times\left[1 + L\left(1-\frac{q}{p}\right)a_0^{\frac{p}{p-q}}(t)\int_{t_0}^{t}\exp\left(-\int_{t_0}^{s}\left(1-\frac{q}{p}\right)Q_2(\tau)\,d\tau\right)ds\right]^{\frac{q}{p-q}} \\ &= \tau_1(t). \end{aligned} \tag{25}$$

For all $t \in I_2$, we can obtain the following estimation by (15) and (25):

$$\begin{aligned} V^{\frac{p}{q}}(t) &\le a_0(t) + \frac{p}{p-q}\sum_{i=1}^{N}\int_{t_0}^{t} g_i(s)V\bigl(\phi_i(s)\bigr)\,ds \\ &\quad + \sum_{j=1}^{L}\int_{t_0}^{t} b_j(s)\int_{t_0}^{s} c_j(\theta)V\bigl(w_j(\theta)\bigr)\,d\theta\,ds + \beta_1 V(t_1-0) \\ &\le a_0(t) + \frac{p}{p-q}\sum_{i=1}^{N}\int_{t_0}^{t_1} g_i(s)\tau_1\bigl(\phi_i(s)\bigr)\,ds + \sum_{j=1}^{L}\int_{t_0}^{t_1} b_j(s)\int_{t_0}^{s} c_j(\theta)\tau\bigl(w_j(\theta)\bigr)\,d\theta\,ds \\ &\quad + \beta_1\tau_1(t_1) + \frac{p}{p-q}\sum_{i=1}^{N}\int_{t_1}^{t} g_i(s)V\bigl(\phi_i(s)\bigr)\,ds \\ &\quad + \sum_{j=1}^{L}\int_{t_1}^{t} b_j(s)\int_{t_0}^{s} c_j(\theta)V\bigl(w_j(\theta)\bigr)\,d\theta\,ds \\ &= a_1(t) + \frac{p}{p-q}\sum_{i=1}^{N}\int_{t_1}^{t} g_i(s)V\bigl(\phi_i(s)\bigr)\,ds \\ &\quad + \sum_{j=1}^{L}\int_{t_1}^{t} b_j(s)\int_{t_0}^{s} c_j(\theta)V\bigl(w_j(\theta)\bigr)\,d\theta\,ds. \end{aligned} \tag{26}$$

Since it has the same style as (16), we can use the same ways to obtain the estimation as (25). Therefore

$$\begin{aligned} V(t) &\le \left[a_1(t)\exp\left(\int_{t_1}^{t} Q_2(s)\,ds\right)\right]^{\frac{q}{p}} \\ &\quad\times\left[1 + L\left(1-\frac{q}{p}\right)a_1^{\frac{p}{p-q}}(t)\int_{t_1}^{t}\exp\left(-\int_{t_1}^{s}\left(1-\frac{q}{p}\right)Q_2(\tau)\,d\tau\right)ds\right]^{\frac{q}{p-q}}. \end{aligned} \tag{27}$$

Let $\tau_2(t)$ denote the function of the right-hand side of (26), which is a positive and non-decreasing function on I_2. Using mathematical induction, $\forall k \in Z$, when $\forall t \in I_k$, the estimation is obtained. We have

$$\begin{aligned} V(t) &\le \left[a_{k-1}(t)\exp\left(\int_{t_{k-1}}^{t} Q_2(s)\,ds\right)\right]^{\frac{q}{p}} \\ &\quad\times\left[1 + L\left(1-\frac{q}{p}\right)a_{k-1}^{\frac{p}{p-q}}(t)\int_{t_{k-1}}^{t}\exp\left(-\int_{t_{k-1}}^{s}\left(1-\frac{q}{p}\right)Q_2(\tau)\,d\tau\right)ds\right]^{\frac{q}{p-q}}. \end{aligned} \tag{28}$$

This completes the proof. □

We consider the inequality (3) now.

Theorem 2.3 *Suppose $\phi_i(t)$, $w_j(t)$, $a_0(t)$, p, q are defined as those in Theorem 2.2. $g_i(t,s)$, $b_j(t,s)$, $c_j(t,s)$ are nondecreasing functions with their two variables. $q_1(t)$, $q_2(t)$ are continuous and nondecreasing functions on $[t_0,\infty)$ and positive on $[t_0,\infty)$ and $u(t)$ is a piecewise continuous nonnegative function on $[t_0,\infty)$ with only the first discontinuous points t_i, $i=1,2,\ldots$, and satisfying* (3). *Then, for all $t\in I_k$,*

$$u(t)\le R_k^{q^{-1}}(t), \tag{29}$$

where

$$\begin{aligned} R_k(t) &= \left[a_{k-1}(t)q(t)\exp\left(\int_{t_{k-1}}^{t}\big(\tilde{Q}_1(s)+\tilde{B}_1(s)\big)\,ds\right)\right]^{\frac{q}{p}} \\ &\quad\times\left[1+\left(1-\frac{q}{p}\right)\int_{t_{k-1}}^{t}\exp\left(\int_{t_{k-1}}^{s}\left(\frac{q}{p}-1\right)\big(\tilde{Q}_1(\tau)+\tilde{B}_1(\tau)\big)\,d\tau\right)ds\right]^{\frac{q}{p-q}}, \\ a_{k-1}(t) &= a_0(t)q(t)\left\{1+\sum_{i=1}^{k}\int_{t_{k-1}}^{t_k}g_i(t,s)\left[\frac{R_i(\phi_i(s))}{a_0(s)}\right]ds\right. \\ &\quad\left.+\sum_{j=1}^{L}\int_{t_{k-1}}^{t_k}\left[\frac{b_j(t,s)}{a_0(s)}\right]\left(\int_{t_{k-1}}^{s}c_j(s,\theta)R_j\big(w_j(\theta)\big)\,d\theta\right)ds+\sum_{i=1}^{k}\beta_i\frac{R_i(t_i-0)}{a(t_i-0)}\right\}, \\ \tilde{Q}_1(t) &= \sum_{i=1}^{N}g_i(t,s)\frac{[q(\phi_i(t))a_0(\phi_i(t))]^{\frac{q}{p}}}{a_0(t)}, \\ \tilde{B}_1(t) &= \sum_{j=1}^{L}b_j(t,s)c_j(t,s)\frac{[q(w_j(t))a_0(w_j(t))]^{\frac{q}{p}}}{a_0(t)}. \end{aligned} \tag{30}$$

Proof Let $v=u^q$, so the inequality (3) is equivalent to

$$\begin{aligned} v^{\frac{p}{q}}(t) &\le a_0(t)+q_1(t)\sum_{i=1}^{N}\int_{t_0}^{t}g_i(t,s)v\big(\phi_i(s)\big)\,ds \\ &\quad+\sum_{j=1}^{L}\int_{t_0}^{t}b_j(t,s)\int_{t_0}^{s}c_j(s,\theta)v\big(w_j(\theta)\big)\,d\theta\,ds+q_2(t)\sum_{t_0<t_i<t}\beta_i v(t_i-0). \end{aligned} \tag{31}$$

Note that $w(t)=v^{\frac{p}{q}}(t)$, then from (31), we get

$$\begin{aligned} \frac{w(t)}{a_0(t)} &\le 1+q_1(t)\sum_{i=1}^{N}\int_{t_0}^{t}g_i(t,s)\left[\frac{v(\phi_i(s))}{a_0(s)}\right]ds \\ &\quad+\sum_{j=1}^{L}\int_{t_0}^{t}\left[\frac{b_j(t,s)}{a_0(s)}\right]\int_{t_0}^{s}c_j(s,\theta)v\big(w_j(\theta)\big)\,d\theta\,ds \\ &\quad+q_2(t)\sum_{t_0<t_i<t}\beta_i\frac{v(t_i-0)}{a_0(t_i-0)}. \end{aligned} \tag{32}$$

Moreover, with the assumption that $q(t) = \max\{q_1(t), q_2(t)\} + 1$, we see

$$\begin{aligned}\frac{w(t)}{a_0(t)} \le q(t)\Bigg\{&1 + \sum_{i=1}^{N}\int_{t_0}^{t} g_i(t,s)\left[\frac{v(\phi_i(s))}{a_0(s)}\right] ds \\ &+ \sum_{j=1}^{L}\int_{t_0}^{t}\left[\frac{b_j(t,s)}{a_0(s)}\right]\int_{t_0}^{s} c_j(s,\theta)v\big(w_j(\theta)\big)\,d\theta\,ds + \sum_{t_0<t_i<t}\beta_i\frac{v(t_i-0)}{a_0(t_i-0)}\Bigg\}.\end{aligned} \tag{33}$$

Let

$$\begin{aligned}\Gamma_1(t) = {}&1 + \sum_{i=1}^{N}\int_{t_0}^{t} g_i(t,s)\left[\frac{v(\phi_i(s))}{a_0(s)}\right] ds \\ &+ \sum_{j=1}^{L}\int_{t_0}^{t}\left[\frac{b_j(t,s)}{a_0(s)}\right]\int_{t_0}^{s} c_j(s,\theta)v\big(w_j(\theta)\big)\,d\theta\,ds + \sum_{t_0<t_i<t}\beta_i\frac{v(t_i-0)}{a_0(t_i-0)}.\end{aligned} \tag{34}$$

Then $\Gamma_1(t)$ is a positive and nondecreasing function on I with $\Gamma_1(t_0) = 1$ and

$$\frac{w(t)}{a_0(t)} \le q(t)\Gamma_1(t), w(t) \le q(t)\Gamma_1(t)a_0(t), \tag{35}$$

so $v(t) \le (q(t)\Gamma_1(t)a_0(t))^{\frac{q}{p}}$. Applying (31) to (35), we obtain

$$\begin{aligned}\Gamma_1(t) \le {}&1 + \sum_{i=1}^{N}\int_{t_0}^{t} g_i(t,s)\frac{[q(\phi_i(s))a_0(\phi_i(s))]^{\frac{q}{p}}}{a_0(s)}\Gamma_1^{\frac{q}{p}}(s)\,ds \\ &+ \sum_{j=1}^{L}\int_{t_0}^{t}\left[\frac{b_j(t,s)}{a_0(s)}\right]\left(\int_{t_0}^{s} c_j(s,\theta)\big[q\big(w_j(\theta)\big)a_0\big(w_j(\theta)\big)\big]^{\frac{q}{p}}\Gamma_1^{\frac{q}{p}}(\theta)\,d\theta\right) ds \\ &+ \sum_{t_0<t_i<t}\beta_i\frac{[q(t_i-0)a_0(t_i-0)\Gamma_1(t_i-0)]^{\frac{q}{p}}}{a_0(t_i-0)}.\end{aligned} \tag{36}$$

Let $I_i = [t_{i-1}, t_i)$, first, we consider the condition under which, for all t in $[t_0, t_1)$, we have

$$\begin{aligned}\Gamma_1(t) \le {}&1 + \sum_{i=1}^{N}\int_{t_0}^{t} g_i(t,s)\frac{[q(\phi_i(s))a_0(\phi_i(s))]^{\frac{q}{p}}}{a_0(s)}\Gamma_1^{\frac{q}{p}}(s)\,ds \\ &+ \sum_{j=1}^{L}\int_{t_0}^{t}\left[\frac{b_j(t,s)}{a_0(s)}\right]\left(\int_{t_0}^{s} c_j(s,\theta)\big[q\big(w_j(\theta)\big)a_0\big(w_j(\theta)\big)\big]^{\frac{q}{p}}\Gamma_1^{\frac{q}{p}}(\theta)\,d\theta\right) ds.\end{aligned} \tag{37}$$

For all $t \in [t_0, T)$, where $T \in I_1$, we get

$$\begin{aligned}\Gamma_1(t) \le {}&1 + \sum_{i=1}^{N}\int_{t_0}^{t} g_i(T,s)\frac{[q(\phi_i(s))a_0(\phi_i(s))]^{\frac{q}{p}}}{a_0(s)}\Gamma_1^{\frac{q}{p}}(s)\,ds \\ &+ \sum_{j=1}^{L}\int_{t_0}^{t}\left[\frac{b_j(T,s)}{a_0(s)}\right]\left(\int_{t_0}^{s} c_j(s,\theta)\big[q\big(w_j(\theta)\big)a_0\big(w_j(\theta)\big)\big]^{\frac{q}{p}}\Gamma_1^{\frac{q}{p}}(\theta)\,d\theta\right) ds.\end{aligned} \tag{38}$$

Let $\Gamma_2(t)$ denote the function on the right-hand side of (38), which is a positive and nondecreasing function on I_1 with $\Gamma_2(t_0) = 1$, and $\Gamma_1(t) \leq \Gamma_2(t)$. For all $t \in [t_0, T)$, differentiating $\Gamma_2(t)$,

$$\begin{aligned}\frac{d\Gamma_2(t)}{dt} &\leq \sum_{i=1}^{N} g_i(T,t)\frac{[q(\phi_i(t))a_0(\phi_i(t))]^{\frac{q}{p}}}{a_0(t)}{\Gamma_2}^{\frac{q}{p}}(t) \\ &\quad + \sum_{j=1}^{L}\left[\frac{b_j(T,t)}{a_0(t)}\right]\int_{t_0}^{t} c_j(s,\theta)\bigl[q\bigl(w_j(\theta)\bigr)a_0\bigl(w_j(\theta)\bigr)\bigr]^{\frac{q}{p}}{\Gamma_2}^{\frac{q}{p}}(\theta)\,d\theta. \end{aligned} \tag{39}$$

Let

$$\begin{aligned} Q(t) &= \sum_{i=1}^{N} g_i(T,t)\frac{[q(\phi_i(t))a_0(\phi_i(t))]^{\frac{q}{p}}}{a_0(t)}, \\ B(t) &= \sum_{j=1}^{L} b_j(T,t)c_j(T,t)\frac{[q(w_j(t))a_0(w_j(t))]^{\frac{q}{p}}}{a_0(t)}. \end{aligned}$$

Since c_j, q, a_0 are nondecreasing functions, we can estimate (40) further to obtain

$$\frac{d\Gamma_2(t)}{dt} \leq Q(t){\Gamma_2}^{\frac{q}{p}}(t) + B(t)\int_{t_0}^{t}{\Gamma_2}^{\frac{q}{p}}(\theta)\,d\theta. \tag{40}$$

Moreover, we can get

$$\begin{aligned}\frac{d\Gamma_2(t)}{dt} &\leq \bigl(Q(t)+B(t)\bigr)\left(\Gamma_2^{\frac{q}{p}}(t) + \int_{t_0}^{t}\Gamma_2^{\frac{q}{p}}(\theta)\,d\theta\right) \\ &\leq \bigl(Q(t)+B(t)\bigr)\left(\Gamma_2(t) + \int_{t_0}^{t}{\Gamma_2}^{\frac{q}{p}}(\theta)\,d\theta\right). \end{aligned} \tag{41}$$

Let $\Gamma_3(t) = \Gamma_2(t) + \int_{t_0}^{t}\Gamma_2^{\frac{q}{p}}(\theta)\,d\theta$. We see that $\Gamma_3(t)$ satisfies $\Gamma_2(t) \leq \Gamma_3(t)$, and differentiating $\Gamma_3(t)$, we can obtain

$$\begin{aligned}\frac{d\Gamma_3(t)}{dt} &= \frac{d\Gamma_2(t)}{dt} + {\Gamma_2}^{\frac{q}{p}}(t) \\ &\leq \bigl(Q(t)+B(t)\bigr)\Gamma_3(t) + {\Gamma_3}^{\frac{q}{p}}(t). \end{aligned} \tag{42}$$

Consider

$$\begin{cases} \frac{d\Gamma_4(t)}{dt} = (Q(t)+B(t))\Gamma_4(t) + \Gamma_4^{\frac{q}{p}}(t), \\ \Gamma_4(t_0) = 1. \end{cases} \tag{43}$$

Since (43) is a Bernoulli equation, we compute it to obtain

$$\begin{aligned}\Gamma_4(t) &= \exp\left(\int_{t_0}^{t}\bigl(Q(s)+B(s)\bigr)\,ds\right) \\ &\quad \times \left[1 + \left(1-\frac{q}{p}\right)\int_{t_0}^{t}\exp\left(\int_{t_0}^{s} -\left(1-\frac{q}{p}\right)\bigl(Q(\tau)+B(\tau)\bigr)\,d\tau\right)ds\right]^{\frac{p}{p-q}}. \end{aligned} \tag{44}$$

Then by comparison of the differential inequality, we have $\Gamma_3(t) \le \Gamma_4(t)$. Therefore,

$$\nu(t) \le \left(a_0(t)q(t)\exp\left(\int_{t_0}^{t}\big(Q(s)+B(s)\big)\,ds\right)\right)^{\frac{q}{p}} \times \left[1+\left(1-\frac{q}{p}\right)\int_{t_0}^{t}\exp\left(\int_{t_0}^{s}-\left(1-\frac{q}{p}\right)\big(Q(\tau)+B(\tau)\big)\,d\tau\right)ds\right]^{\frac{q}{p-q}}. \tag{45}$$

By taking $t = T$ in the above inequality, and noticing the definitions of $\tilde{Q}_1(t)$ and $\tilde{B}_1(t)$, we get

$$\nu(t) \le \left[a_0(t)q(t)\exp\left(\int_{t_0}^{t}\big(\tilde{Q}_1(s)+\tilde{B}_1(s)\big)\,ds\right)\right]^{\frac{q}{p}} \times \left[1+\int_{t_0}^{t}\left(1-\frac{q}{p}\right)\exp\left(\int_{t_0}^{t}-\left(1-\frac{q}{p}\right)\big(\tilde{Q}_1(s)+\tilde{B}_1(s)\big)\,ds\right)\right]^{\frac{q}{p-q}}. \tag{46}$$

Let $R_1(t)$ denote the function on the right-hand side of (46). When $t \in I_2$, we obtain

$$\begin{aligned}\nu^{\frac{p}{q}}(t) \le{}& a_0(t)q(t)\left\{1+\sum_{i=1}^{N}\int_{t_0}^{t_1} g_i(t,s)\left[\frac{R_1(\phi_i(s))}{a_0(s)}\right]ds\right.\\ &\left.+\sum_{j=1}^{L}\int_{t_0}^{t_1}\left[\frac{b_j(t,s)}{a_0(s)}\right]\left(\int_{t_0}^{s} c_j(s,\theta)R_1\big(w_j(\theta)\big)\,d\theta\right)ds+\sum_{i=1}^{k}\beta_i\frac{R_i(t_i-0)}{a(t_i-0)}\right\}\\ &+a_0(t)q(t)\left\{\sum_{i=1}^{N}\int_{t_1}^{t} g_i(t,s)\left[\frac{\nu(\phi_i(s))}{a_0(s)}\right]ds\right.\\ &\left.+\sum_{j=1}^{L}\int_{t_1}^{t}\left[\frac{b_j(t,s)}{a_0(s)}\right]\left(\int_{t_0}^{s} c_j(s,\theta)\nu\big(w_j(\theta)\big)\,d\theta\right)ds\right\}.\end{aligned} \tag{47}$$

Let

$$\begin{aligned}a_1(t) ={}& a_0(t)q(t)\left\{1+\sum_{i=1}^{N}\int_{t_0}^{t_1} g_i(t,s)\left[\frac{R_1(\phi_i(s))}{a_0(s)}\right]ds\right.\\ &+\sum_{j=1}^{L}\int_{t_0}^{t_1}\left[\frac{b_j(t,s)}{a_0(s)}\right]\left(\int_{t_0}^{s} c_j(s,\theta)R_1\big(w_j(\theta)\big)\,d\theta\right)ds\\ &\left.+\sum_{i=1}^{k}\beta_i\frac{R_i(t_i-0)}{a(t_i-0)}\right\}.\end{aligned} \tag{48}$$

Obviously, $a_1(t) \ge a_0(t)$. Since $q(t) \ge 1$, we can go further to obtain

$$\begin{aligned}\nu^{\frac{p}{q}}(t) \le{}& a_1(t)q(t)\left\{1+\sum_{i=1}^{N}\int_{t_1}^{t} g_i(t,s)\left[\frac{\nu(\phi_i(s))}{a_0(s)}\right]ds\right.\\ &\left.+\sum_{j=1}^{L}\int_{t_1}^{t}\left[\frac{b_j(t,s)}{a_0(s)}\right]\left(\int_{t_0}^{s} c_j(s,\theta)\nu\big(w_j(\theta)\big)\,d\theta\right)ds\right\}.\end{aligned} \tag{49}$$

Since (49) has the same style as (38), we can use the same solution to deal with it, finally the estimation of the unknown function in the inequality (3) is obtained. We have

$$\begin{aligned} v(t) &\leq \left(a_{k-1}(t)q(t)\exp\left(\int_{t_{k-1}}^{t}\left(\tilde{Q}_1(s)+\tilde{B}_1(s)\,ds\right)\right)\right)^{\frac{q}{p}} \\ &\quad\times\left[1+\left(1-\frac{q}{p}\right)\int_{t_{k-1}}^{t}\exp\left(-\left(1-\frac{q}{p}\right)\int_{t_{k-1}}^{s}\left(\tilde{Q}_1(\tau)+\tilde{B}_1(\tau)\right)d\tau\right)ds\right]^{\frac{q}{p-q}}. \end{aligned} \tag{50}$$

Let $R_k(t)$ denote the function on the right-hand side, and

$$\begin{aligned} a_{k-1}(t) &= a_0(t)q(t)\left\{1+\sum_{i=1}^{k}\int_{t_{k-1}}^{t_k} g_i(t,s)\left[\frac{R_i(\phi_i(s))}{a_0(s)}\right]ds \right. \\ &\quad\left. +\sum_{j=1}^{L}\int_{t_{k-1}}^{t_k}\left[\frac{b_j(t,s)}{a_0(s)}\right]\left(\int_{t_{k-1}}^{s} c_j(s,\theta)R_j\left(w_j(\theta)\right)d\theta\right)ds+\sum_{i=1}^{k}\beta_i\frac{R_i(t_i-0)}{a(t_i-0)}\right\}. \end{aligned} \tag{51}$$

So we obtain

$$u(t)\leq R_k^{q^{-1}}(t).$$

This proves Theorem 2.3. □

3 Applications

In this section we will apply our Theorem 2.1 and Theorem 2.2 to discuss the following differential-integral equation and retarded differential equation for discontinuous functions, respectively. We present the following propositions.

Proposition 3.1 *Consider the following equation*:

$$\begin{cases} \frac{dx(t)}{dt}=H(t,x(\phi_1(t)),\ldots,x(\phi_N(t)),\int_0^t K(s,x(w_1(s)))\,ds,\ldots,\int_0^t K(s,x(w_L(s)))\,ds), \\ x(0)=x_0, \quad \forall t\in I=[0,\infty), \end{cases} \tag{52}$$

where the function K *is in* $C(R\times R,R_+)$ *and* $\phi_i(t)\leq t$, $w_j(t)\leq t$, *for* $t>0$, H *satisfies the following condition*:

$$\begin{aligned} &\left|H\left(t,u_1,u_2,\ldots,u_N,\int_0^t K(s,v_1)\,ds,\ldots,\int_0^t K(s,v_L)\,ds\right)\right| \\ &\quad\leq\sum_{i=1}^{N}g_i(t)u_i+\sum_{j=1}^{L}b_j(t)\int_0^t c_j(\theta)v_j\,d\theta, \end{aligned} \tag{53}$$

where $g_i(t)$, $b_j(t)$, $c_j(t)$, $w_j(t)$ *are defined as in Theorem* 2.1. *If*

$$\int_{t_0}^{t}Q_1(s)\,ds<\infty, \qquad \int_{t_0}^{t}Y(s)\exp\left(-\int_{t_0}^{s}Q_1(\tau)\,d\tau\right)ds<\infty.$$

Then all the solutions of equation (53) *exist on* I *and for all* t *in* $I=[0,\infty)$, *and they satisfy the following estimate*:

$$\left|x(t)\right|\leq x_0+\exp\left(\int_{t_0}^{t}Q_1(s)\,ds\right)\left(\int_{t_0}^{t}Y(s)\exp\left(-\int_{t_0}^{s}Q_1(\tau)\,d\tau\right)ds\right). \tag{54}$$

Proof Integrating both sides of equation (52) from 0 to t, we get

$$x(t) = x_0 + \int_0^t H\Bigl(s, x(\phi_1(s)), \ldots, x(\phi_N(s)), \int_0^s K(\tau, x(w_1(\tau)))\,d\tau, \ldots, \int_0^s K(\tau, x(w_L(\tau)))\,d\tau\Bigr)\,ds. \tag{55}$$

Using the conditions (52) and (53), we can obtain

$$|x(t)| \le x_0 + \sum_{i=1}^N \int_{t_0}^t g_i(s)x(\phi_i(s))\,ds + \sum_{j=1}^L \int_{t_0}^t b_j(s)\int_{t_0}^s c_j(\theta)x(w_j(\theta))\,d\theta\,ds. \tag{56}$$

Applying Theorem 2.1 to (56), we can obtain the estimate. □

Proposition 3.2 *Consider the impulsive differential system*

$$\begin{cases} \frac{dx^p(t)}{dt} = H(t, x(\phi_1(t)), \ldots, x(\phi_N(t)), \int_0^t K(s, x(w_1(s)))\,ds, \ldots, \int_0^t K(s, x(w_L(s)))\,ds), \\ \Delta(x)|_{t=t_0} = \beta_i x^q(t_i - 0), \\ x(0) = c, \quad \forall t \in I = [0, \infty), \end{cases} \tag{57}$$

where $\phi_i(t)$, $w_j(t)$ are defined as in Theorem 2.2 and the function K is in $C(R \times R, R_+)$. Furthermore, H satisfies

$$\Bigl|H\Bigl(t, x(\phi_1(t)), \ldots, x(\phi_N(t)), \int_0^t K(s, x(w_1(s)))\,ds, \ldots, \int_0^t K(s, x(w_L(s)))\,ds\Bigr)\Bigr| \le \sum_{i=1}^N g_i(t)x^q(\phi_i(t)) + \sum_{j=1}^L b_j(s)\int_0^s c_j(t)x^q(w_j(t))\,dt. \tag{58}$$

Then all the solutions of equation (57) *exist on I and satisfy $|x(t)| \le \tau_k(t)$ for all $t \in I_k$, where $\tau_k(t)$ is defined as in Theorem* 2.2.

Proof Integrating (57) we obtain

$$|x^p(t)| \le c^p + \int_0^t H\Bigl(s, x(\phi_1(s)), \ldots, x(\phi_N(s)), \int_0^s K(\tau, x(w_1(\tau)))\,d\tau, \ldots, \int_0^s K(\tau, x(w_L(\tau)))\,d\tau\Bigr)\,ds + \sum_{t_0<t_i<t} \beta_i u^q(t_i - 0), \quad \forall t \in I. \tag{59}$$

Furthermore, we get

$$|x^p(t)| \le c^p + \sum_{i=1}^N \int_0^t g_i(s)x^q(\phi_i(s))\,ds + \sum_{j=1}^L \int_0^t b_j(s)\int_0^s c_j(\theta)x^q(w_j(\theta))\,d\theta\,ds + \sum_{t_0<t_i<t} \beta_i u^q(t_i - 0). \tag{60}$$

Then we use Theorem 2.2 to obtain the estimation. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
ZZ came up with the main ideas and helped to draft the manuscript. XG proved the main theorems. JS revised the paper. All authors read and approved the final manuscript.

Author details
[1]School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, P.R. China. [2]Department of Mathematics, Jining University, Qufu 273155, P.R. China.

Acknowledgements
The authors sincerely thank the referees for their constructive suggestions and corrections. This project is supported by the NNSF of China (Grants 11171178 and 11271225), NSF of Shandong Province (ZR2015PA005).

Received: 30 June 2015 Accepted: 13 December 2015 Published online: 04 January 2016

References

1. Lipovan, O: A retarded Gronwall-like inequality and its applications. J. Math. Anal. Appl. **252**, 389-401 (2000). doi:10.1006/jmaa.2000.7085
2. Agarwal, RP, Deng, S, Zhang, W: Generalization of a retarded Gronwall-like inequality and its applications. Appl. Math. Comput. **165**, 599-612 (2005). doi:10.1016/j.amc.2004.04.067
3. Borysenko, DS: About one integral inequality for piece-wise continuous functions. In: Proceedings of 10th International Kravchuk Conference, Kyiv, p. 323 (2004)
4. Iovane, G: Some new integral inequalities of Bellman-Bihari type with delay for discontinuous functions. Nonlinear Anal. **66**(2), 498-508 (2007)
5. Wang, W-S, Li, Z: A new nonlinear retarded integral inequality and its application. J. Sichuan Normal Univ. Nat. Sci. **35**, 180-183 (2012) (in Chinese)
6. Yan, Y: Some new Gronwall-Bellman type impulsive integral inequality and its application. J. Sichuan Normal Univ. Nat. Sci. **36**, 603-608 (2013) (in Chinese)
7. Gronwall, TH: Note on the derivatives with respect to a parameter of the solutions of a system of differential equations. Ann. Math. **20**, 292-296 (1919). doi:10.2307/1967124
8. Bellman, R: The stability of solutions of linear differential equations. Duke Math. J. **10**, 643-647 (1943). doi:10.1215/S0012-7094-43-01059-2
9. Wang, W: Some new generalized retarded nonlinear integral inequalities with iterated integrals and its applications. J. Inequal. Appl. **2012**, 31 (2012)
10. Abdeldaim, A, Yakout, M: On some new integral inequalities of Gronwall-Bellman-Pachpatte type. Appl. Math. Comput. **217**, 7887-7889 (2011). doi:10.1016/j.amc.2011.02.093
11. Agarwal, RP, Kim, YH, Sen, SK: New retarded integral inequalities with application. J. Inequal. Appl. **2008**, Article ID 908784 (2008)
12. Liu, C, Li, Z: A new class of integral inequality for discontinuous function and its application. Pure Math. **3**, 4-8 (2013) (in Chinese)