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Approximation of a kind of new type Bézier operators

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Abstract
In this paper, a kind of new type Bézier operators is introduced. The Korovkin type approximation theorem of these operators is investigated. The rates of convergence of these operators are studied by means of modulus of continuity. Then, by using the Ditzian-Totik modulus of smoothness, a direct theorem concerned with an approximation for these operators is also obtained.

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Keywords: Bézier type operators; Korovich type approximation theorem; rate of convergence; direct theorem; modulus of smoothness

1 Introduction

In view of the Bézier basis function, which was introduced by Bézier [1], in 1983, Chang [2] defined the generalized Bernstein-Bézier polynomials for any $\alpha > 0$, and a function f defined on $[0,1]$ as follows:

$$B_{n,\alpha}(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\left[J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)\right], \tag{1}$$

where $J_{n,n+1}(x) = 0$, and $J_{n,k}(x) = \sum_{i=k}^{n} P_{n,i}(x)$, $k = 0,1,\ldots,n$, $P_{n,i}(x) = \binom{n}{i}x^i(1-x)^{n-i}$. $J_{n,k}(x)$ is the Bézier basis function of degree n.

Obviously, when $\alpha = 1$, $B_{n,\alpha}(f;x)$ become the well-known Bernstein polynomials $B_n(f;x)$, and for any $x \in [0,1]$, we have $1 = J_{n,0}(x) > J_{n,1}(x) > \cdots > J_{n,n}(x) = x^n$, $J_{n,k}(x) - J_{n,k+1}(x) = P_{n,k}(x)$.

During the last ten years, the Bézier basis function was extensively used for constructing various generalizations of many classical approximation processes. Some Bézier type operators, which are based on the Bézier basis function, have been introduced and studied (*e.g.*, see [3–9]).

In 2012, Ren [10] introduced Bernstein type operators as follows:

$$L_n(f;x) = f(0)P_{n,0}(x) + \sum_{k=1}^{n-1} P_{n,k}(x)B_{n,k}(f) + f(1)P_{n,n}(x), \tag{2}$$

where $f \in C[0,1]$, $x \in [0,1]$, $P_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$, $k = 0,1,\ldots,n$, and $B_{n,k}(f) = \frac{1}{B(nk,n(n-k))}\int_0^1 t^{nk-1}(1-t)^{n(n-k)-1}f(t)\,dt$, $k = 1,\ldots,n-1$, $B(\cdot,\cdot)$ is the beta function.

The moments of the operators $L_n(f;x)$ were obtained as follows (see [10]).

Remark 1 For $L_n(t^j;x)$, $j = 0, 1, 2$, we have

(i) $L_n(1;x) = 1$;

(ii) $L_n(t;x) = x$;

(iii) $L_n(t^2;x) = \frac{n(n-1)}{n^2+1}x^2 + \frac{n+1}{n^2+1}x$.

In the present paper, we will study the Bézier variant of the Bernstein type operators $L_n(f;x)$, which have been given by (2). We introduce a new type of Bézier operators as follows:

$$L_{n,\alpha}(f;x) = f(0)Q_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x)B_{n,k}(f) + f(1)Q_{n,n}^{(\alpha)}(x), \tag{3}$$

where $f \in C[0,1]$, $x \in [0,1]$, $\alpha > 0$, $Q_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$, $J_{n,n+1}(x) = 0$, $J_{n,k}(x) = \sum_{i=k}^{n} P_{n,i}(x)$, $k = 0, 1, \dots, n$, $P_{n,i}(x) = \binom{n}{i}x^i(1-x)^{n-i}$, and $B_{n,k}(f) = \frac{1}{B(nk,n(n-k))}\int_0^1 t^{nk-1}(1-t)^{n(n-k)-1}f(t)\,dt$, $k = 1, \dots, n-1$, $B(\cdot,\cdot)$ is the beta function.

It is clear that $L_{n,\alpha}(f;x)$ are linear and positive on $C[0,1]$. When $\alpha = 1$, $L_{n,\alpha}(f;x)$ become the operators $L_n(f;x)$.

The goal of this paper is to study the approximation properties of these operators with the help of the Korovkin type approximation theorem. We also estimate the rates of convergence of these operators by using a modulus of continuity. Then we obtain the direct theorem concerned with an approximation for these operators by means of the Ditzian-Totik modulus of smoothness.

In the paper, for $f \in C[0,1]$, we denote $\|f\| = \max\{|f(x)| : x \in [0,1]\}$. $\omega(f,\delta)$ ($\delta > 0$) denotes the usual modulus of continuity of $f \in C[0,1]$.

2 Some lemmas

Now, we give some lemmas, which are necessary to prove our results.

Lemma 1 (see [2]) *Let $\alpha > 0$. We have*

(i) $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} J_{n,k}^{\alpha}(x) = x$ *uniformly on* $[0,1]$;

(ii) $\lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n} kJ_{n,k}^{\alpha}(x) = \frac{x^2}{2}$ *uniformly on* $[0,1]$.

Lemma 2 *Let $\alpha > 0$. We have*

(i) $L_{n,\alpha}(1;x) = 1$;

(ii) $\lim_{n\to\infty} L_{n,\alpha}(t;x) = x$ *uniformly on* $[0,1]$;

(iii) $\lim_{n\to\infty} L_{n,\alpha}(t^2;x) = x^2$ *uniformly on* $[0,1]$.

Proof By simple calculation, we obtain $B_{n,k}(1) = 1$, $B_{n,k}(t) = \frac{k}{n}$, $B_{n,k}(t^2) = \frac{1}{n^2+1}(k^2 + \frac{k}{n})$.

(i) Since $\sum_{k=0}^{n} Q_{n,k}^{(\alpha)}(x) = 1$, by (3) we can get $L_{n,\alpha}(1;x) = 1$.

(ii) By (3), we have

$$\begin{aligned} L_{n,\alpha}(t;x) &= \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x)\frac{k}{n} + Q_{n,n}^{(\alpha)}(x) \\ &= \left[J_{n,1}^{\alpha}(x) - J_{n,2}^{\alpha}(x)\right]\frac{1}{n} + \cdots + \left[J_{n,n-1}^{\alpha}(x) - J_{n,n}^{\alpha}(x)\right]\frac{n-1}{n} + J_{n,n}^{\alpha}(x)\frac{n}{n} \\ &= \frac{1}{n}\sum_{k=1}^{n} J_{n,k}^{\alpha}(x), \end{aligned}$$

thus, by Lemma 1(i), we have $\lim_{n\to\infty} L_{n,\alpha}(t;x) = x$ uniformly on $[0,1]$.

(iii) By (3), we have

$$\begin{aligned} L_{n,\alpha}\left(t^2;x\right) &= \frac{1}{n^2+1}\sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x)\left(k^2 + \frac{k}{n}\right) + Q_{n,n}^{(\alpha)}(x) \\ &= \frac{1}{n^2+1}\sum_{k=1}^{n}\left(2k - 1 + \frac{1}{n}\right)J_{n,k}^{\alpha}(x) \\ &= \frac{1}{n^2+1}\left[2n^2 \cdot \frac{1}{n^2}\sum_{k=1}^{n} kJ_{n,k}^{\alpha}(x) - n \cdot \frac{1}{n}\sum_{k=1}^{n} J_{n,k}^{\alpha}(x) + \frac{1}{n}\sum_{k=1}^{n} J_{n,k}^{\alpha}(x)\right], \end{aligned}$$

thus, by Lemma 1, we have $\lim_{n\to\infty} L_{n,\alpha}(t^2;x) = x^2$ uniformly on $[0,1]$. □

Lemma 3 (see [11]) *For* $x \in [0,1]$, $k = 0,1,\dots,n$, *we have*

$$0 \le Q_{n,k}^{(\alpha)}(x) \le \begin{cases} \alpha P_{n,k}(x), & \alpha \ge 1; \\ P_{n,k}^{\alpha}(x), & 0 < \alpha < 1. \end{cases}$$

Lemma 4 (see [12]) *For* $0 < \alpha < 1$, $\beta > 0$, *we have*

$$\sum_{k=0}^{n} |k - nx|^{\beta} P_{n,k}^{\alpha}(x) \le (n+1)^{1-\alpha}(A_{\frac{\beta}{\alpha}})^{\alpha} n^{\frac{\beta}{2}},$$

where the constant A_s *only depends on* s.

Lemma 5 *For* $\alpha \ge 1$, *we have*

$$\text{(i)} \quad L_{n,\alpha}\left((t-x)^2;x\right) \le \frac{\alpha}{2} \cdot \frac{1}{n};$$

$$\text{(ii)} \quad L_{n,\alpha}\left(|t-x|;x\right) \le \sqrt{\frac{\alpha}{2}} \cdot \sqrt{\frac{1}{n}}.$$

Proof (i) By (3), Lemma 3 and Remark 1, we obtain

$$\begin{aligned} &L_{n,\alpha}\left((t-x)^2;x\right) \\ &\quad = x^2 Q_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x) B_{n,k}\left((t-x)^2\right) + (1-x)^2 Q_{n,n}^{(\alpha)}(x) \end{aligned}$$

$$
\begin{aligned}
&\le \alpha\left[x^2P_{n,0}(x)+\sum_{k=1}^{n-1}P_{n,k}(x)B_{n,k}\bigl((t-x)^2\bigr)+(1-x)^2P_{n,n}(x)\right]\\
&=\alpha L_n\bigl((t-x)^2;x\bigr)\\
&=\frac{(n+1)\alpha}{n^2+1}x(1-x), \qquad (4)
\end{aligned}
$$

since $\max_{0\le x\le 1}x(1-x)=\frac{1}{4}$, and for any $n\in N$, one can get $\frac{n(n+1)}{n^2+1}\le 2$, so we have

$$L_{n,\alpha}\bigl((t-x)^2;x\bigr)\le\frac{\alpha}{2}\cdot\frac{1}{n}.$$

(ii) In view of $L_{n,\alpha}(1;x)=1$, by the Cauchy-Schwarz inequality, we have

$$L_{n,\alpha}\bigl(|t-x|;x\bigr)\le\sqrt{L_{n,\alpha}(1;x)}\sqrt{L_{n,\alpha}\bigl((t-x)^2;x\bigr)},$$

thus, we get

$$L_{n,\alpha}\bigl(|t-x|;x\bigr)\le\sqrt{\frac{\alpha}{2}}\cdot\sqrt{\frac{1}{n}}. \qquad \square$$

Lemma 6 *For* $0<\alpha<1$, *we have*

(i) $L_{n,\alpha}\bigl((t-x)^2;x\bigr)\le M_\alpha n^{-\alpha}$;

(ii) $L_{n,\alpha}\bigl(|t-x|;x\bigr)\le\sqrt{M_\alpha}\cdot n^{-\frac{\alpha}{2}}$.

Here the constant M_α *only depends on* α.

Proof (i) By (3) and Lemma 3, we obtain

$$
\begin{aligned}
&L_{n,\alpha}\bigl((t-x)^2;x\bigr)\\
&\quad=x^2Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)B_{n,k}\bigl((t-x)^2\bigr)+(1-x)^2Q_{n,n}^{(\alpha)}(x)\\
&\quad\le x^2P_{n,0}^{\alpha}(x)+\sum_{k=1}^{n-1}P_{n,k}^{\alpha}(x)B_{n,k}\bigl((t-x)^2\bigr)+(1-x)^2P_{n,n}^{\alpha}(x)\\
&\quad=\sum_{k=0}^{n}P_{n,k}^{\alpha}(x)\left[\frac{1}{n^2+1}\left(k^2+\frac{k}{n}\right)-2x\frac{k}{n}+x^2\right]\\
&\quad=\frac{1}{n^2+1}\sum_{k=0}^{n}(k-nx)^2P_{n,k}^{\alpha}(x)+\frac{1}{n^2+1}\sum_{k=0}^{n}P_{n,k}^{\alpha}(x)\left(\frac{k}{n}-2x\frac{k}{n}+x^2\right)\\
&\quad:=I_1+I_2.
\end{aligned}
$$

By Lemma 4, we have $I_1\le\frac{n(n+1)}{n^2+1}(n+1)^{-\alpha}(A_{\frac{2}{\alpha}})^{\alpha}\le 2(A_{\frac{2}{\alpha}})^{\alpha}n^{-\alpha}$, where the constant $A_{\frac{2}{\alpha}}$ only depends on α.

Using the Hölder inequality, we have $\sum_{k=0}^{n} P_{n,k}^{\alpha}(x) \leq (n+1)^{1-\alpha}[\sum_{k=0}^{n} P_{n,k}(x)]^{\alpha}$, and $|\frac{k}{n} - 2x\frac{k}{n} + x^2| \leq 4$, so we have

$$I_2 \leq \frac{4}{n^2+1}(n+1)^{1-\alpha}\left[\sum_{k=0}^{n} P_{n,k}(x)\right]^{\alpha} = \frac{4}{n^2+1}(n+1)^{1-\alpha} \leq 4n^{-\alpha}.$$

Denote $M_\alpha = 2(A_{\frac{2}{\alpha}})^{\alpha} + 4$, then we can get

$$L_{n,\alpha}\bigl((t-x)^2;x\bigr) \leq M_\alpha n^{-\alpha}.$$

(ii) Since

$$L_{n,\alpha}\bigl(|t-x|;x\bigr) \leq \sqrt{L_{n,\alpha}(1;x)}\sqrt{L_{n,\alpha}\bigl((t-x)^2;x\bigr)},$$

thus, we get

$$L_{n,\alpha}\bigl(|t-x|;x\bigr) \leq \sqrt{M_\alpha} \cdot n^{-\frac{\alpha}{2}}. \qquad \square$$

Lemma 7 *For* $f \in C[0,1]$, $x \in [0,1]$ *and* $\alpha > 0$, *we have*

$$\bigl|L_{n,\alpha}(f;x)\bigr| \leq \|f\|.$$

Proof By (3) and Lemma 2(i), we have

$$\bigl|L_{n,\alpha}(f;x)\bigr| \leq \|f\| L_{n,\alpha}(1;x) = \|f\|. \qquad \square$$

3 Main results

First of all we give the following convergence theorem for the sequence $\{L_{n,\alpha}(f;x)\}$.

Theorem 1 *Let* $\alpha > 0$. *Then the sequence* $\{L_{n,\alpha}(f;x)\}$ *converges to* f *uniformly on* $[0,1]$ *for any* $f \in C[0,1]$.

Proof Since $L_{n,\alpha}(f;x)$ is bounded and positive on $C[0,1]$, and by Lemma 2, we have $\lim_{n\to\infty} \|L_{n,\alpha}(e_j;\cdot) - e_j\| = 0$ for $e_j(t) = t^j$, $j = 0,1,2$. So, according to the well-known Bohman-Korovkin theorem ([13], p.40, Theorem 1.9), we see that the sequence $\{L_{n,\alpha}(f;x)\}$ converges to f uniformly on $[0,1]$ for any $f \in C[0,1]$. $\square$

Next we estimate the rates of convergence of the sequence $\{L_{n,\alpha}\}$ by means of the modulus of continuity.

Theorem 2 *Let* $f \in C[0,1]$, $x \in [0,1]$. *Then*

(i) *when* $\alpha \geq 1$, *we have* $\|L_{n,\alpha}(f;\cdot) - f\| \leq (1 + \sqrt{\frac{\alpha}{2}})\omega(f, \frac{1}{\sqrt{n}})$;

(ii) *when* $0 < \alpha < 1$, *we have* $\|L_{n,\alpha}(f;\cdot) - f\| \leq (1 + \sqrt{M_\alpha})\omega(f, n^{-\frac{\alpha}{2}})$.

Here the constant M_α *only depends on* α.

Proof (i) When $\alpha \geq 1$, by Lemma 2(i), we have

$$\begin{aligned}
&\left|L_{n,\alpha}(f;x)-f(x)\right| \\
&\leq \left|f(0)-f(x)\right|Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)B_{n,k}\left(\left|f(t)-f(x)\right|\right)+\left|f(1)-f(x)\right|Q_{n,n}^{(\alpha)}(x) \\
&\leq \omega\left(f,|0-x|\right)Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)B_{n,k}\left(\omega\left(f,|t-x|\right)\right)+\omega\left(f,|1-x|\right)Q_{n,n}^{(\alpha)}(x) \\
&\leq \left(1+\sqrt{n}|0-x|\right)\omega\left(f,\frac{1}{\sqrt{n}}\right)Q_{n,0}^{(\alpha)}(x) \\
&\quad+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)B_{n,k}\left(\left(1+\sqrt{n}|t-x|\right)\omega\left(f,\frac{1}{\sqrt{n}}\right)\right) \\
&\quad+\left(1+\sqrt{n}|1-x|\right)\omega\left(f,\frac{1}{\sqrt{n}}\right)Q_{n,n}^{(\alpha)}(x) \\
&\leq \omega\left(f,\frac{1}{\sqrt{n}}\right)+\sqrt{n}\omega\left(f,\frac{1}{\sqrt{n}}\right)L_{n,\alpha}\left(|t-x|;x\right),
\end{aligned}$$

so, by Lemma 5(ii), we obtain $|L_{n,\alpha}(f;x)-f(x)| \leq (1+\sqrt{\frac{\alpha}{2}})\omega(f,\frac{1}{\sqrt{n}})$. The desired result follows immediately.

(ii) When $0<\alpha<1$, by Lemma 2(i), we have

$$\begin{aligned}
&\left|L_{n,\alpha}(f;x)-f(x)\right| \\
&\leq \omega\left(f,|0-x|\right)Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)B_{n,k}\left(\omega\left(f,|t-x|\right)\right)+\omega\left(f,|1-x|\right)Q_{n,n}^{(\alpha)}(x) \\
&\leq \left(1+n^{\frac{\alpha}{2}}|0-x|\right)\omega\left(f,n^{-\frac{\alpha}{2}}\right)Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)B_{n,k}\left(1+n^{\frac{\alpha}{2}}|t-x|\right)\omega\left(f,n^{-\frac{\alpha}{2}}\right) \\
&\quad+\left(1+n^{\frac{\alpha}{2}}|1-x|\right)\omega\left(f,n^{-\frac{\alpha}{2}}\right)Q_{n,n}^{(\alpha)}(x) \\
&=\omega\left(f,n^{-\frac{\alpha}{2}}\right)+n^{\frac{\alpha}{2}}\omega\left(f,n^{-\frac{\alpha}{2}}\right)L_{n,\alpha}\left(|t-x|;x\right),
\end{aligned}$$

so, by Lemma 6(ii), we obtain $|L_{n,\alpha}(f;x)-f(x)| \leq (1+\sqrt{M_\alpha})\omega(f,n^{-\frac{\alpha}{2}})$. The desired result follows immediately. □

Theorem 3 *Let $f \in C^1[0,1]$, $x \in [0,1]$. Then*

(i) *when $\alpha \geq 1$, we have*

$$\left|L_{n,\alpha}(f;x)-f(x)\right| \leq \|f'\|\sqrt{\frac{\alpha}{2n}}+\omega\left(f',\frac{1}{\sqrt{n}}\right)\left(1+\sqrt{\frac{\alpha}{2}}\right)\sqrt{\frac{\alpha}{2n}};$$

(ii) *when $0<\alpha<1$, we have*

$$\left|L_{n,\alpha}(f;x)-f(x)\right| \leq \|f'\|\sqrt{M_\alpha n^{-\alpha}}+\omega\left(f',n^{-\frac{\alpha}{2}}\right)(1+\sqrt{M_\alpha})\sqrt{M_\alpha n^{-\alpha}},$$

where the constant M_α only depends on α.

Proof Let $f \in C^1[0,1]$. For any $t,x \in [0,1]$, $\delta > 0$, we have

$$\begin{aligned}\bigl|f(t)-f(x)-f'(x)(t-x)\bigr| &\le \left|\int_x^t \bigl|f'(u)-f'(x)\bigr|\,du\right| \\ &\le \omega\bigl(f',|t-x|\bigr)|t-x| \\ &\le \omega\bigl(f',\delta\bigr)\bigl(|t-x|+\delta^{-1}(t-x)^2\bigr),\end{aligned}$$

hence, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\bigl|L_{n,\alpha}\bigl(f(t)-f(x)-f'(x)(t-x);x\bigr)\bigr| \\ &\quad\le \omega\bigl(f',\delta\bigr)\bigl(L_{n,\alpha}\bigl(|t-x|;x\bigr)+\delta^{-1}L_{n,\alpha}\bigl((t-x)^2;x\bigr)\bigr) \\ &\quad\le \omega\bigl(f',\delta\bigr)\bigl(\sqrt{L_{n,\alpha}(1;x)} \\ &\qquad+\delta^{-1}\sqrt{L_{n,\alpha}\bigl((t-x)^2;x\bigr)}\bigr)\sqrt{L_{n,\alpha}\bigl((t-x)^2;x\bigr)}.\end{aligned}$$

So we get

$$\begin{aligned}&\bigl|L_{n,\alpha}(f;x)-f(x)\bigr| \\ &\quad\le \|f'\|L_{n,\alpha}\bigl(|t-x|;x\bigr) \\ &\qquad+\omega\bigl(f',\delta\bigr)\bigl(1+\delta^{-1}\sqrt{L_{n,\alpha}\bigl((t-x)^2;x\bigr)}\bigr)\sqrt{L_{n,\alpha}\bigl((t-x)^2;x\bigr)}.\end{aligned} \tag{5}$$

(i) When $\alpha \ge 1$, taking $\delta = \frac{1}{\sqrt{n}}$ in (5), by Lemma 5 and inequality (5), we obtain the desired result.
(ii) When $0 < \alpha < 1$, taking $\delta = n^{-\frac{\alpha}{2}}$ in (5), by Lemma 6 and inequality (5), we obtain the desired result. □

Finally we study the direct theorem concerned with an approximation for the sequence $\{L_{n,\alpha}\}$ by means of the Ditzian-Totik modulus of smoothness. For the next theorem we shall use some notations.

For $f \in C[0,1]$, $\varphi(x) = \sqrt{x(1-x)}$, $0 \le \lambda \le 1$, $x \in [0,1]$, let

$$\omega_{\varphi^\lambda}(f,t) = \sup_{0<h\le t}\ \sup_{x\pm\frac{h\varphi^\lambda(x)}{2}\in[0,1]}\left|f\left(x+\frac{h\varphi^\lambda(x)}{2}\right)-f\left(x-\frac{h\varphi^\lambda(x)}{2}\right)\right|$$

be the Ditzian-Totik modulus of first order, and let

$$K_{\varphi^\lambda}(f,t) = \inf_{g\in W_\lambda}\bigl\{\|f-g\|+t\bigl\|\varphi^\lambda g'\bigr\|\bigr\} \tag{6}$$

be the corresponding K-functional, where $W_\lambda = \{f | f \in AC_{\mathrm{loc}}[0,1], \|\varphi^\lambda f'\| < \infty, \|f'\| < \infty\}$.

It is well known that (see [14])

$$K_{\varphi^\lambda}(f,t) \le C\omega_{\varphi^\lambda}(f,t), \tag{7}$$

for some absolute constant $C > 0$.

Now we state our next main result.

Theorem 4 *Let* $f \in C[0,1]$, $\alpha \geq 1$, $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$, $0 \leq \lambda \leq 1$. *Then there exists an absolute constant* $C > 0$ *such that*

$$\left|L_{n,\alpha}(f;x) - f(x)\right| \leq C\omega_{\varphi^\lambda}\left(f, \frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\right).$$

Proof Let $g \in W_\lambda$, by Lemma 2(i) and Lemma 7, we have

$$\begin{aligned}
&\left|L_{n,\alpha}(f;x) - f(x)\right| \\
&\quad \leq \left|L_{n,\alpha}(f-g;x)\right| + \left|f(x) - g(x)\right| + \left|L_{n,\alpha}(g;x) - g(x)\right| \\
&\quad \leq 2\|f-g\| + \left|L_{n,\alpha}(g;x) - g(x)\right|.
\end{aligned} \tag{8}$$

Since $g(t) = \int_x^t g'(u)\,du + g(x)$, $L_{n,\alpha}(1;x) = 1$, so, we have

$$\begin{aligned}
\left|L_{n,\alpha}(g;x) - g(x)\right| &\leq \left|L_{n,\alpha}\left(\int_x^t \left|g'(u)\right|\,du;x\right)\right| \\
&\leq \left\|\varphi^\lambda g'\right\| L_{n,\alpha}\left(\left|\int_x^t \varphi^{-\lambda}(u)\,du\right|;x\right).
\end{aligned} \tag{9}$$

By the Hölder inequality, we get

$$\left|\int_x^t \varphi^{-\lambda}(u)\,du\right| \leq \left|\int_x^t \frac{1}{\sqrt{u(1-u)}}\,du\right|^\lambda |t-x|^{1-\lambda}, \tag{10}$$

also, in view of $1 \leq \sqrt{u} + \sqrt{1-u} < 2$, $0 \leq u \leq 1$, we have

$$\begin{aligned}
\left|\int_x^t \frac{1}{\sqrt{u(1-u)}}\,du\right| &\leq \left|\int_x^t \left(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1-u}}\right)du\right| \\
&\leq 2\left(|\sqrt{t} - \sqrt{x}| + |\sqrt{1-x} - \sqrt{1-t}|\right) \\
&\leq 2\left(\frac{|t-x|}{\sqrt{t} + \sqrt{x}} + \frac{|t-x|}{\sqrt{1-t} + \sqrt{1-x}}\right) \\
&\leq 2|t-x|\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}}\right) \\
&\leq 4|t-x|\varphi^{-1}(x),
\end{aligned} \tag{11}$$

thus, by (10) and (11), we obtain

$$\left|\int_x^t \varphi^{-\lambda}(u)\,du\right| \leq C\varphi^{-\lambda}(x)|t-x|, \tag{12}$$

also, by (9) and (12), we have

$$\begin{aligned}
\left|L_{n,\alpha}(g;x) - g(x)\right| &\leq C\left\|\varphi^\lambda g'\right\| L_{n,\alpha}\left(\varphi^{-\lambda}(x)|t-x|;x\right) \\
&= C\left\|\varphi^\lambda g'\right\| \varphi^{-\lambda}(x) L_{n,\alpha}\left(|t-x|;x\right).
\end{aligned} \tag{13}$$

In view of (4) and Lemma 2(i), by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} L_{n,\alpha}\big(|t-x|;x\big) &\le \sqrt{L_{n,\alpha}(1;x)}\sqrt{L_{n,\alpha}\big((t-x)^2;x\big)} \\ &\le \sqrt{\frac{(n+1)\alpha}{n^2+1}x(1-x)} \\ &\le C\frac{\varphi(x)}{\sqrt{n}}, \end{aligned} \tag{14}$$

so, by (13) and (14), we obtain

$$\big|L_{n,\alpha}(g;x)-g(x)\big| \le C\big\|\varphi^{\lambda}g'\big\|\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}, \tag{15}$$

thus, by (8) and (15), we have

$$\big|L_{n,\alpha}(f;x)-f(x)\big| \le 2\|f-g\| + C\big\|\varphi^{\lambda}g'\big\|\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}. \tag{16}$$

Then, in view of (16), (6), and (7), we obtain

$$\big|L_{n,\alpha}(f;x)-f(x)\big| \le CK_{\varphi^{\lambda}}\left(f,\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\right) \le C\omega_{\varphi^{\lambda}}\left(f,\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\right),$$

where C is a positive constant, in different places the value of C may be different. □

4 Conclusions

In this paper, a new kind of type Bézier operators is introduced. The Korovkin type approximation theorem of these operators is investigated. The rates of convergence of these operators are studied by means of the modulus of continuity. Then, by using the Ditzian-Totik modulus of smoothness, a direct theorem concerned with an approximation for these operators is obtained. Further, we can also study the inverse theorem and an equivalent theorem concerned with an approximation for these operators.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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