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Some inequalities related to two expansions of $(1+1/x)^x$

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Abstract

We prove the following theorem: Let

$$\left(1+\frac{1}{x}\right)^x = e\left(1-\sum_{k=1}^{\infty}\frac{b_k}{(1+x)^k}\right) = e\left(1-\sum_{k=1}^{\infty}\frac{d_k}{(\frac{11}{12}+x)^k}\right),$$

$$\sigma_m(x) = \sum_{k=1}^{m}\frac{b_k}{(1+x)^k} \quad\text{and}\quad S_m(x) = \sum_{k=1}^{m}\frac{d_k}{(\frac{11}{12}+x)^k}.$$

(1) If $m \geq 6$ is even, we have $S_m(x) > \sigma_m(x)$ for all $x > 0$.
(2) If $m \geq 7$ is odd, we have $S_m(x) > \sigma_m(x)$ for all $x > 1$.

This provides an intuitive explanation for the main result in Mortici and Hu (On an infinite series for $(1+1/x)^x$, 2014, arXiv:1406.7814 [math.CA]).

MSC: 42B25; 42B35

Keywords: Carleman inequality; integral representation; series; number e

1 Introduction

The Carleman inequality [2]

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}a_n,$$

whenever $a_n \geq 0$, $n = 1, 2, 3, \ldots$, with $0 < \sum_{n=1}^{\infty} a_n < \infty$, has attracted the attention of many authors in the recent past [1, 3–10].

In [7], Yang proved

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}\left(1-\sum_{k=1}^{6}\frac{b_k}{(n+1)^k}\right)a_n$$

with $b_1 = 1/2$, $b_2 = 1/24$, $b_3 = 1/48$, $b_4 = 73/5{,}760$, $b_5 = 11/1{,}280$, $b_6 = 1{,}945/580{,}608$, and Yang conjectured that if

$$\left(1+\frac{1}{x}\right)^x = e\left(1-\sum_{k=1}^{\infty}\frac{b_k}{(x+1)^k}\right), \qquad x > 0, \tag{1.1}$$

then $b_k > 0$, $k = 1, 2, 3, \ldots$.

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Later, this conjecture was proved by Yang [8], Gylletberg and Yan [11], and Chen [9], respectively. As an application, Yang proved for any positive integer m

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}\left(1-\sum_{k=1}^{m}\frac{b_k}{(n+1)^k}\right)a_n, \tag{1.2}$$

whenever $a_n \geq 0$, $n = 1, 2, 3, \ldots$, and $0 < \sum_{n=1}^{\infty} a_n < \infty$, with $b_1 = \frac{1}{2}$ and

$$b_{n+1} = \frac{1}{n+1}\left(\frac{1}{n+2}-\sum_{k=1}^{n}\frac{b_k}{n+2-k}\right).$$

In the final part of his paper, Yang [8] remarked that in order to obtain better results, the right-hand side of (1.1) could be replaced by $e[1-\sum_{n=1}^{\infty}(d_n/(x+\varepsilon)^n)]$, where $\varepsilon \in (0,1]$ and $d_n = d_n(\varepsilon)$, but information about the values of ε are not provided.

Recently, Mortici and Hu [1] proved that $\varepsilon = 11/12$ provides the faster series

$$\sum_{n=1}^{\infty}\frac{d_n}{(x+\varepsilon)^n}$$

and therefore the following inequality is better than (1.2):

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}\left(1-\sum_{k=1}^{m}\frac{d_k}{(n+\frac{11}{12})^k}\right)a_n. \tag{1.3}$$

The proof of this conclusion is based on the following theorem [12], which is a powerful tool for measuring the speed of convergence.

Theorem *If $(\omega_n)_{n\geq 1}$ is convergent to zero and*

$$\lim_{n\to\infty} n^k(\omega_n - \omega_{n+1}) = l \in \mathbb{R},$$

with $k > 1$, then there exists the limit

$$\lim_{n\to\infty} n^{k-1}\omega_n = \frac{l}{k-1}.$$

The purpose of this paper is to establish some inequalities which explain Mortici's conclusion in a quantitative way. But our proof is not based on the theorem.

Our main result is the following theorem.

Theorem 1 *Let*

$$\left(1+\frac{1}{x}\right)^x = e\left(1-\sum_{k=1}^{\infty}\frac{b_k}{(1+x)^k}\right) = e\left(1-\sum_{k=1}^{\infty}\frac{d_k}{(\frac{11}{12}+x)^k}\right),$$

$$\sigma_m(x) = \sum_{k=1}^{m}\frac{b_k}{(1+x)^k} \quad \textit{and} \quad S_m(x) = \sum_{k=1}^{m}\frac{d_k}{(\frac{11}{12}+x)^k}.$$

(1) *If* $m \geq 6$ *is even, then* $S_m(x) > \sigma_m(x)$ *for all* $x > 0$.
(2) *If* $m \geq 7$ *is odd, then* $S_m(x) > \sigma_m(x)$ *for all* $x > 1$.

2 Lemmas

In order to prove our main results we need the following lemmas, and throughout this paper we set

$$g(s) = \frac{1}{\pi} s^s (1-s)^{1-s} \sin(\pi s),$$
$$h(s,x) = \frac{1}{1-s+x}\left(\frac{s}{1+x}\right)^{m-1} - \frac{1}{s+x}\left(\frac{12s-1}{11+12x}\right)^{m-1}.$$

Here $0 \leq s \leq 1$, $x > 0$, and $m \geq 1$ is an integer.

Lemma 1 *For* $x > 0$, *let*

$$\left(1+\frac{1}{x}\right)^x = e\left(1 - \sum_{k=1}^{\infty} \frac{b_k}{(1+x)^k}\right).$$

Then

$$b_k > 0, \quad k = 1,2,\ldots, \tag{2.1}$$
$$b_1 = \frac{1}{2}, \tag{2.2}$$
$$b_{n+1} = \frac{1}{n+1}\left(\frac{1}{n+2} - \sum_{j=1}^{n} \frac{b_j}{n+2-j}\right), \quad n = 1,2,\ldots,$$
$$eb_k = \int_0^1 g(s) s^{k-2}\, ds, \quad k = 1,2,\ldots. \tag{2.3}$$

Proof For (2.1) and (2.2), see [7]. For (2.3), see [13]. □

Remark 1 By (2.3) of Lemma 1, we have

$$\int_0^1 g(s)(s)^{n-2}\, ds = \int_0^1 g(s)(1-s)^{n-2}\, ds = eb_n \quad (n = 2,3,\ldots).$$

Example

$$\int_0^1 g(s)\, ds = eb_2 = \frac{e}{24}, \qquad \int_0^1 g(s) s\, ds = eb_3 = \frac{e}{48},$$
$$\begin{aligned}\int_0^1 \frac{1}{s} g(s)\, ds &= \int_0^1 \frac{1}{1-s} g(s)\, ds \\ &= \int_0^1 \left(1 + s + s^2 + \cdots\right) g(s)\, ds \\ &= e\sum_{n=2}^{\infty} b_n = e\sum_{n=1}^{\infty} b_n - eb_1 \\ &= e\left(1 - \frac{1}{e}\right) - \frac{e}{2} = \frac{e}{2} - 1.\end{aligned}$$

Lemma 2 *For $x > 0$, let*

$$\left(1+\frac{1}{x}\right)^x = e\left(1-\sum_{k=1}^{\infty}\frac{d_k}{(\frac{11}{12}+x)^k}\right).$$

Then

$$d_1 = \frac{1}{2}, \tag{2.4}$$

$$d_{n+1} = \frac{(-1)^{n+1}}{12^{n-1}e}\left(1+\int_0^1 g(s)\frac{(12s-1)^{n-1}}{s}\,ds\right), \quad n = 1, 2, \ldots,$$

$$d_n > 0, \quad n = 4, 5, \ldots, \tag{2.5}$$

$$b_n > d_n, \quad n = 2, 3, \ldots. \tag{2.6}$$

Proof For (2.4), see [1].

Now, we prove (2.5). If n is an odd, then d_{n+1} is obviously positive.

If n is an even, then we have for all $n \geq 4$

$$\int_0^{\frac{1}{12}} \frac{g(s)}{s}(1-12s)^{n-1}\,ds \leq \int_0^{\frac{1}{12}} \frac{g(s)}{s}(1-12s)^3\,ds, \tag{2.7}$$

$$\int_{\frac{1}{12}}^{\frac{1}{6}} \frac{g(s)}{s}(1-12s)^{n-1}\,ds < 0, \tag{2.8}$$

$$\int_{\frac{1}{6}}^{1} \frac{g(s)}{s}(1-12s)^{n-1}\,ds \leq \int_{\frac{1}{6}}^{1} \frac{g(s)}{s}(1-12s)^3\,ds. \tag{2.9}$$

From (2.7), (2.8), (2.9), and Remark 1, we get

$$\begin{aligned}\int_0^1 \frac{g(s)}{s}(1-12s)^{n-1}\,ds &\leq \int_0^1\left(\frac{1}{s}-36+432s-1{,}728s^2\right)g(s)\,ds \\ &\quad + \int_{\frac{1}{12}}^{\frac{1}{6}}\left(1{,}728s^2-432s+30\right)ds \\ &\leq \left(\frac{e}{2}-1\right)-\frac{3e}{2}+9e-21.9e+6\int_0^1 g(s)\,ds \\ &= -1-13.9e+\frac{e}{4} < -1.\end{aligned}$$

Thus from this and (2.4), we have $d_{n+1} > 0$. This proves (2.5). The proof of (2.6) is similar to (2.5). □

Remark 2 By Lemma 2, it is not obvious that $S_m(x) > \sigma_m(x)$.

Lemma 3 *Let $m \geq 3$ be an integer, we have*

$$(-1)^{m-1} + \int_0^1 \frac{g(s)}{1-s}(12s-1)^{m-1}\,ds > 0. \tag{2.10}$$

Proof The proof is similar to the proof of (2.5). □

Lemma 4 *Let $x > 0$, and $m \geq 6$ be an integer. Then we have for all $\frac{1}{12} \leq s \leq \frac{1}{2}$*

$$h(s,x) > 0. \tag{2.11}$$

Proof Noting that $\frac{s(\frac{11}{12}+x)}{(s-\frac{1}{12})(1+x)} > 1$ for all $x > 0$, the inequality (2.11) is equivalent to

$$\left(\frac{(s-\frac{1}{12})(1+x)}{s(\frac{11}{12}+x)}\right)^5 < \frac{s+x}{1-s+x}. \tag{2.12}$$

To prove (2.12), we define $h_1(s,x)$ as

$$h_1 = \ln(1-s+x) - \ln(s+x) + 5\ln\left(s-\frac{1}{12}\right) + 5\ln(1+x) - 5\ln s - 5\ln\left(\frac{11}{12}+x\right). \tag{2.13}$$

Easy computations reveal that

$$h_1\left(\frac{1}{2},0\right) = 5\ln\left(\frac{10}{11}\right) < 0, \tag{2.14}$$

$$\frac{\partial h_1(s,0)}{\partial s} > 0, \tag{2.15}$$

$$\frac{\partial h_1(s,x)}{\partial x} < 0. \tag{2.16}$$

Thus from (2.14), (2.15), and (2.16), we have

$$h_1(s,x) \leq h_1(s,0) \leq h_1\left(\frac{1}{2},0\right) < 0,$$

which implies

$$h(s,x) > 0. \qquad \square$$

Lemma 5 *Let $x > 0$, and $m \geq 2$ be an integer, then $h(s,x)$ is a monotonic increasing function of s on $[\frac{1}{2},1]$. If m is an odd, then $h(s,x)$ is a monotonic increasing function of s on $[0,\frac{1}{12}]$.*

Proof It suffices to show that $\frac{\partial h(s,x)}{\partial s} > 0$. Partial differentiation yields

$$\begin{aligned}\frac{\partial h(s,x)}{\partial s} \geq{}& \frac{m-1}{(1+x)(1-s+x)}\left(\frac{s}{1+x}\right)^{m-2} \\ &- \frac{m-1}{(11+12x)(s+x)}\left(\frac{12s-1}{11+12x}\right)^{m-2}.\end{aligned} \tag{2.17}$$

If $\frac{1}{2} \leq s \leq 1$, then for all $x > 0$ and $m \geq 2$, we have

$$\frac{m-1}{(1+x)(1-s+x)} > \frac{m-1}{(11+12x)(s+x)},$$

$$\frac{s}{1+x} > \frac{12s-1}{11+12x}.$$

Thus

$$\frac{\partial h(s,x)}{\partial s} > 0.$$

If m is an odd, then for $0 < s < \frac{1}{12}$, we have

$$(12s-1)^{m-2} < 0.$$

From this and (2.17), we get

$$\frac{\partial h(s,x)}{\partial s} > 0.$$

This completes the proof of Lemma 5. □

Lemma 6 *Let $m \geq 3$ be an integer, then we have for all $x \geq 1$*

$$h\left(\frac{1}{2},x\right) > \left|h(0,x)\right|. \tag{2.18}$$

Proof Because of $x > 1$, (2.18) is equivalent to

$$\left(5+\frac{1+2x}{2+2x}\right)^{m-1} - 5^{m-1} > 1 + \frac{1}{2x}. \tag{2.19}$$

The equality (2.19) follows immediately from

$$(m-1)5^{m-2}\frac{1+2x}{2+2x} > (m-1)5^{m-2}\frac{3}{4} > \frac{3}{2}.$$ □

3 Proof of Theorem 1

Proof By Lemma 1 and Lemma 2, we get for $m \geq 2$

$$\begin{aligned}
\sigma_m(x) &= \frac{e/2}{1+x} + \sum_{k=2}^{m}\int_0^1 \frac{g(s)}{s^2}\left(\frac{s}{1+x}\right)^k ds \\
&= \frac{e}{2(1+x)} + \int_0^1 \frac{g(s)}{s^2}\sum_{k=2}^{m}\left(\frac{s}{1+x}\right)^k ds \\
&= \frac{e}{2(1+x)} + \int_0^1 \frac{g(s)}{(1+x)(1+x-s)}\left(1-\left(\frac{s}{1+x}\right)^{m-1}\right) ds \\
&= \frac{e}{2(1+x)} + \int_0^1 \frac{g(s)}{(1+x)(x+s)}\,ds - \int_0^1 \frac{g(s)}{(1+x)(1-s+x)}\left(\frac{s}{1+x}\right)^{m-1} ds, \\
S_m(x) &= \frac{e}{2(1+x)} + \int_0^1 \frac{g(s)}{(x+s)(1+x)} + \frac{(-1)^{m-1}}{(11+12x)^m(1+x)} \\
&\quad - \int_0^1 \frac{g(s)}{(1+x)(x+s)}\left(\frac{12s-1}{11+12x}\right)^{m-1} ds \\
&\quad + \int_0^1 \frac{g(s)}{(11+12x)(1+x)(1-s)}\left(\frac{12s-1}{11+12x}\right)^{m-1} ds.
\end{aligned}$$

To prove our result, we consider

$$\begin{aligned} S_m(x) - \sigma_m(x) &= \frac{1}{(11+12x)^m(1+x)}\left((-1)^{m-1} + \int_0^1 \frac{g(s)}{1-s}(12s-1)^{m-1}\right) \\ &\quad + \frac{1}{1+x}\int_0^1 g(s)\left(\frac{1}{1-s+x}\left(\frac{s}{1+x}\right)^{m-1} - \frac{1}{x+s}\left(\frac{12s-1}{11+12x}\right)^{m-1}\right) ds \\ &= \frac{1}{(11+12x)^m(1+x)}\left((-1)^{m-1} + \int_0^1 \frac{g(s)}{1-s}(12s-1)^{m-1}\right) \\ &\quad + \frac{1}{1+x}\int_0^1 g(s)h(s,x)\, ds. \end{aligned}$$

By Lemma 3, it suffices to show that

$$\int_0^1 g(s)h(s,x)\, ds > 0.$$

Let first $m \geq 6$ be even. From Lemma 4 and Lemma 5, for all $x > 0$, we have

$$\int_0^1 g(s)h(s,x)\, ds = \int_0^{\frac{1}{12}} g(s)h(s,x)\, ds + \int_{\frac{1}{12}}^{\frac{1}{2}} g(s)h(s,x)\, ds + \int_{\frac{1}{2}}^1 g(s)h(s,x)\, ds > 0.$$

Here we used the fact that if m is an even and $0 \leq s \leq 1/12$, then $h(s,x) > 0$ for any $x > 0$.

Now let $m \geq 7$ be odd. From Lemma 4, Lemma 5, and Lemma 6 for all $x \geq 1$ we have

$$\begin{aligned} \int_0^1 g(s)h(s,x)\, ds &= \int_0^{\frac{1}{12}} g(s)h(s,x)\, ds + \int_{\frac{1}{12}}^{\frac{1}{2}} g(s)h(s,x)\, ds + \int_{\frac{1}{2}}^1 g(s)h(s,x)\, ds \\ &\geq \int_0^{\frac{1}{12}} g(s)h(0,x)\, ds + \int_{\frac{1}{12}}^{\frac{1}{2}} g(s)h(s,x)\, ds + \int_{\frac{1}{2}}^1 g(s)h\left(\frac{1}{2},x\right) ds \\ &\geq \left(h(0,x) + h\left(\frac{1}{2},x\right)\right)\int_0^{\frac{1}{12}} g(s)\, ds + \int_{\frac{1}{12}}^{\frac{1}{2}} g(s)h(s,x)\, ds > 0. \end{aligned}$$

This completes the proof of Theorem 1. □

Remark 3 By using computer simulation, we find $S_m(x) > \sigma_m(x)$ for all $x > 0$ and all $m \geq 1$, but we leave as an open problem the rigorous proof of this fact.

4 Conclusions

In this paper, we have established some inequalities which explain Mortici's result in a quantitative way. The authors believe that the present analysis will lead to a significant contribution toward the study of the Carleman inequality.

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions
The authors completed the paper together. They each read and approved the final manuscript.

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