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RESEARCH

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Two-log-convexity of the Catalan-Larcombe-French sequence

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Abstract

The Catalan-Larcombe-French sequence $\{P_n\}_{n\geq 0}$ arises in a series expansion of the complete elliptic integral of the first kind. It has been proved that the sequence is log-balanced. In the paper, by exploring a criterion due to Chen and Xia for testing 2-log-convexity of a sequence satisfying three-term recurrence relation, we prove that the new sequence $\{P_n^2 - P_{n-1}P_{n+1}\}_{n\geq 1}$ are strictly log-convex and hence the Catalan-Larcombe-French sequence is strictly 2-log-convex.

MSC: 05A20; 11B37; 11B83

Keywords: log-balanced sequence; log-convex sequence; log-concave sequence; the Catalan-Larcombe-French sequence; three-term recurrence

1 Introduction

This paper is concerned with the log-behavior of the Catalan-Larcombe-French sequence. To begin with, let us recall that a sequence $\{z_n\}_{n\geq 0}$ is said to be log-concave if

$$z_n^2 \geq z_{n+1}z_{n-1}, \quad \text{for } n \geq 1, \tag{1.1}$$

and it is log-convex if

$$z_n^2 \leq z_{n+1}z_{n-1}, \quad \text{for } n \geq 1. \tag{1.2}$$

Meanwhile, the sequence $\{z_n\}_{n\geq 0}$ is called strictly log-concave (resp. log-convex) if the inequality in (1.1) (resp. (1.2)) is strict for all $n \geq 1$. We call $\{z_n\}_{n\geq 0}$ log-balanced if the sequence itself is log-convex while $\{\frac{z_n}{n!}\}_{n\geq 0}$ is log-concave.

Given a sequence $A = \{z_n\}_{n\geq 0}$, define the operator $\mathcal{L}$ by

$$\mathcal{L}(A) = \{s_n\}_{n\geq 0},$$

where $s_n = z_{n-1}z_{n+1} - z_n^2$ for $n \geq 1$. We say that $\{z_n\}_{n\geq 0}$ is k-log-convex (*resp. k-log-concave*) if $\mathcal{L}^j(A)$ is log-convex (*resp. log-concave*) for all $j = 0, 1, \ldots, k-1$, and that $A = \{z_n\}_{n\geq 0}$ is ∞-log-convex (*resp. ∞-log-concave*) if $\mathcal{L}^k(A)$ is log-convex (*resp. log-concave*) for any $k \geq 0$. Similarly, we can define strict k-log-concavity or strict k-log-convexity of a sequence.

It is worthy to mention that besides that they are fertile sources of inequalities, log-convexity and log-concavity have many applications in some different mathematical disciplines, such as geometry, probability theory, combinatorics, and so on. See the surveys due to Brenti [1] and Stanley [2] for more details. Additionally, it is clear that the log-balancedness implies the log-convexity and a sequence $\{z_n\}_{n\geq 0}$ is log-convex (resp. log-concave) if and only if its quotient sequence $\{\frac{z_n}{z_{n-1}}\}_{n\geq 1}$ is nondecreasing (resp. nonincreasing). It is also known that the quotient sequence of a log-balanced sequence does not grow too fast. Therefore, log-behavior are important properties of combinatorial sequences and they are instrumental in obtaining the growth rate of a sequence. Hence the log-behaviors of a sequence deserves to be investigated.

In this paper, we investigate the 2-log-behavior of the Catalan-Larcombe-French sequence, denoted by $\{P_n\}_{n\geq 0}$, which arises in connection with series expansions of the complete elliptic integrals of the first kind [3, 4]. To be precise, for $0 < |c| < 1$,

$$\int_0^{\pi/2} \frac{1}{\sqrt{1-c^2\sin^2\theta}}\,d\theta = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{1-\sqrt{1-c^2}}{16}\right)^n P_n.$$

Furthermore, the numbers P_n can be written as the following sum:

$$P_n = 2^n \sum_{k=0}^{n} (-4)^i \binom{n-k}{k}\binom{2n-2k}{n-k}^2,$$

see [5], A05317. Besides, the number P_n satisfies three-term recurrence relations [4] as follows:

$$(n+1)^2 P_{n+1} = 8\bigl(3n^2+3n+1\bigr)P_n - 128n^2 P_{n-1}, \quad \text{for } n \geq 1, \tag{1.3}$$

with the initial values $P_0 = 1$ and $P_1 = 8$.

Recently, Zhao [4] studied the log-behavior of the Catalan-Larcombe-French sequence and proved that the sequence $\{P_n\}_{n\geq 0}$ is log-balanced. What is more, the Catalan-Larcombe-French sequence has many interesting properties and the reader can refer [3, 4, 6]. In the sequel, we study the 2-log-behavior of the sequences and obtain the following result.

Theorem 1.1 *The Catalan-Larcombe-French sequence* $\{P_n\}_{n\geq 0}$ *is strictly* 2-*log-convex, that is,*

$$\mathcal{P}_n^2 < \mathcal{P}_{n-1}\mathcal{P}_{n+1}, \tag{1.4}$$

where $\mathcal{P}_n = P_n^2 - P_{n-1}P_{n+1}$.

We will give our proof of Theorem 1.1 in the third section by utilizing a testing criterion, which is proposed by Chen and Xia [7].

To make this paper self-contained, let us recall their criterion.

Theorem 1.2 (Chen and Xia [7]) *Suppose $\{z_n\}_{n\geq 0}$ is a positive log-convex sequence that satisfies the following three-term recurrence relation*:

$$z_n = a(n)z_{n-1} + b(n)z_{n-2}, \quad \text{for } n \geq 2. \tag{1.5}$$

Let

$$\begin{aligned}
c_0(n) &= -b^2(n+1)\bigl[a^2(n+2) + b(n+1) - a(n+2)a(n+3) - b(n+3)\bigr]; \\
c_1(n) &= b(n+1)\bigl[2a(n+2)b(n+1) + 2a(n+3)a(n+2)a(n+1) \\
&\quad + a(n+3)b(n+2) + 2a(n+1)b(n+3) - 2a^2(n+2)a(n+1) \\
&\quad - 2a(n+2)b(n+2) - 3a(n+1)b(n+1)\bigr]; \\
c_2(n) &= 4a(n+1)a(n+2)b(n+1) + 2b(n+1)b(n+2) + a^2(n+1)a(n+2)a(n+3) \\
&\quad + a(n+1)a(n+3)b(n+2) + a^2(n+1)b(n+3) - 3a^2(n+1)b(n+1) \\
&\quad - a(n+3)a(n+2)b(n+1) - a^2(n+2)a^2(n+1) - b(n+3)b(n+1) \\
&\quad - 2a(n+2)a(n+1)b(n+2) - b^2(n+2); \\
c_3(n) &= 2a^2(n+1)a(n+2) + 2a(n+1)b(n+2) - a(n+1)b(n+3) - a^3(n+1) \\
&\quad - a(n+1)a(n+2)a(n+3) - a(n+3)b(n+2);
\end{aligned}$$

and

$$\Delta(n) = 4c_2^2(n) - 12c_1(n)c_3(n).$$

Assume that $c_3(n) < 0$ and $\Delta(n) \geq 0$ for all $n \geq N$, where N is a positive integer. If there exist f_n and g_n such that, for all $n \geq N$,

(I) $f_n \leq \frac{z_n}{z_{n-1}} \leq g_n$;

(II) $f_n \geq \frac{-2c_2(n)-\sqrt{\Delta(n)}}{6c_3(n)}$;

(III) $c_3(n)g_n^3 + c_2(n)g_n^2 + c_1(n)g_n + c_0(n) \geq 0$,

then we see that $\{z_n\}_{n\geq N}$ is 2-log-convex, that is, for $n \geq N$,

$$\bigl(z_{n-1}z_{n+1} - z_n^2\bigr)\bigl(z_{n+1}z_{n+3} - z_{n+2}^2\bigr) > \bigl(z_nz_{n+2} - z_{n+1}^2\bigr)^2.$$

With respect to the theory in this field, it should be mentioned that the log-behavior of a sequence which satisfies a three-term recurrence has been extensively studied; see Liu and Wang [8], Chen *et al.* [9, 10], Liggett [11], Došlić [12], *etc.*

2 Bounds for $\frac{P_n}{P_{n-1}}$

Before proving Theorem 1.1, we need the following two lemmas.

Lemma 2.1 *Let*

$$f_n = \frac{232n}{15(n+2)},$$

and P_n be the sequence defined by the recurrence relation (1.3). *Then we have, for all $n \geq 1$,*

$$\frac{P_n}{P_{n-1}} > f_n. \tag{2.1}$$

Proof We proceed the proof by induction. First note that, for $n = 1$ and $n = 2$, we have $\frac{P_1}{P_0} = 8 > \frac{232}{45}$ and $\frac{P_2}{P_1} = 10 > \frac{464}{60}$. Assume that the inequality (2.1) is valid for $n \leq k$. We will show that

$$\frac{P_{k+1}}{P_k} > f_{k+1}.$$

By the recurrence (1.3), we have

$$\begin{aligned}\frac{P_{k+1}}{P_k} &= \frac{8(3k^2+3k+1)}{(k+1)^2} - \frac{128k^2}{(k+1)^2}\frac{P_{k-1}}{P_k} > \frac{8(3k^2+3k+1)}{(k+1)^2} - \frac{128k^2}{(k+1)^2}\frac{1}{f_k} \\ &= \frac{8(57k^2+27k+29)}{29(k+1)^2} \\ &> f_{k+1},\end{aligned}$$

in which the last inequality follows by

$$\begin{aligned}\frac{8(57k^2+27k+29)}{29(k+1)^2} - f_{k+1} &= \frac{8(14k^3+447k^2-873k+464)}{435(k+1)^2(k+3)} \\ &> 0,\end{aligned}$$

for all $k \geq 1$. This completes the proof. □

Lemma 2.2 *Let*

$$g_n = 16 - \frac{16}{n} - \frac{16}{n^3},$$

and P_n be the sequence defined by the recurrence relation (1.3). *Then we have, for all $n \geq 6$,*

$$\frac{P_n}{P_{n-1}} \leq g_n. \tag{2.2}$$

Proof First note that, for $n = 6$, we have $\frac{P_6}{P_5} = \frac{3{,}562}{269} < g_6 = \frac{358}{27}$. Assume that, for $k \geq 6$, the inequality (2.2) is valid for $n \leq k$. We will show that

$$\frac{P_{k+1}}{P_k} < g_{k+1}.$$

By the recurrence (1.3), we have

$$\begin{aligned}\frac{P_{k+1}}{P_k} &= \frac{8(3k^2+3k+1)}{(k+1)^2} - \frac{128k^2}{(k+1)^2}\frac{P_{k-1}}{P_k} < \frac{8(3k^2+3k+1)}{(k+1)^2} - \frac{128k^2}{(k+1)^2}\frac{1}{g_k} \\ &= \frac{8(2k^5-2k^3-4k^2-3k-1)}{(k+1)^2(k^3-k^2-1)}.\end{aligned} \tag{2.3}$$

Consider

$$\frac{8(2k^5-2k^3-4k^2-3k-1)}{(k+1)^2(k^3-k^2-1)}-g_{k+1}=-\frac{8(5k^2+2k+3)}{(k+1)^3(k^3-k^2-1)}<0, \tag{2.4}$$

for all $k\geq 2$. So we see that, for all $n\geq 6$, the inequality (2.2) holds by induction. □

With the above lemmas in hand, we are now in a position to prove our main result in the next section.

3 Proof of Theorem 1.1

In this section, by using the criterion of Theorem 1.2, we can show that the Catalan-Larcombe-French sequence is strictly 2-log-convex.

To begin with, the following lemma, which is obtained by Zhao [4], is indispensable for us.

Lemma 3.1 (Zhao [4]) *The Catalan-Larcombe-French sequence is log-balanced.*

By the definition of log-balanced sequence, we know that $\{P_n\}_{n\geq 0}$ is log-convex.

Proof of Theorem 1.1 By Lemma 3.1, it suffices for us to show that

$$\left(P_{n-1}P_{n+1}-P_n^2\right)\left(P_{n+1}P_{n+3}-P_{n+2}^2\right)-\left(P_nP_{n+2}-P_{n+1}^2\right)^2>0.$$

According to the recurrence relation (1.3), we see that

$$a(n)=\frac{8(3n^2-3n+1)}{n^2};$$
$$b(n)=-\frac{128(n-1)^2}{n^2}.$$

By taking $a(n)$, $b(n)$ in $c_0,\dots,c_3$, we can obtain

$$\begin{aligned}c_3(n)&=-\frac{512}{(n+1)^6(n+2)^2(n+3)^2}\\&\quad\times\left(3n^8+5n^7-27n^6-32n^5+112n^4+234n^3+177n^2+63n+9\right)\\&<0,\end{aligned}$$

for all $n\geq 1$. Besides, we have to verify that, for some positive integer N, the conditions (II) and (III) in Theorem 1.2 hold for all $n\geq N$. That is,

$$f_n\geq\frac{-2c_2(n)-\sqrt{\Delta(n)}}{6c_3(n)}; \tag{3.1}$$

$$c_3(n)g_n^3+c_2(n)g_n^2+c_1(n)g_n+c_0(n)\geq 0. \tag{3.2}$$

Let

$$\delta(n)=-6c_3(n)f_n-2c_2(n)$$

and

$$f(g_n) = c_3(n)g_n^3 + c_2(n)g_n^2 + c_1(n)g_n + c_0(n).$$

To show (3.1), it is equivalent to show that, for some positive integers N, $\delta(n) \geq 0$ and $\delta^2(n) \geq \Delta(n)$. By calculating, we easily find that, for all $n \geq 1$,

$$\begin{aligned}\delta(n) &= \frac{8{,}192}{5(n+1)^6(n+2)^4(n+3)^2}\bigl(32n^{10} + 129n^9 + 472n^8 + 3{,}556n^7 + 12{,}157n^6 \\ &\quad + 17{,}632n^5 + 10{,}550n^4 + 1{,}293n^3 - 1{,}500n^2 - 798n - 135\bigr) \\ &\geq 0,\end{aligned}$$

and for all $n \geq 3$,

$$\begin{aligned}\delta^2(n) - \Delta(n) &= \frac{6{,}7108{,}864n}{25(n+3)^4(n+2)^7(n+1)^{12}}\bigl(699n^{18} + 2{,}158n^{17} + 6{,}983n^{16} \\ &\quad + 97{,}994n^{15} + 155{,}517n^{14} - 1{,}256{,}916n^{13} - 3{,}302{,}168n^{12} \\ &\quad + 5{,}191{,}280n^{11} + 25{,}505{,}142n^{10} + 14{,}486{,}584n^9 - 63{,}005{,}002n^8 \\ &\quad - 153{,}766{,}236n^7 - 178{,}037{,}517n^6 - 131{,}841{,}558n^5 - 68{,}012{,}397n^4 \\ &\quad - 24{,}910{,}146n^3 - 6{,}269{,}211n^2 - 975{,}888n - 70{,}470\bigr) \\ &\geq 0.\end{aligned}$$

Thus, take $N = 3$ and, for all $n \geq N$, we have $\delta(n) \geq 0$, $\delta^2(n) \geq \Delta(n)$, which follows from the inequality (3.1). We show the inequality (3.2) for some positive integer M. Note that, by Lemma 2.2 and some calculations, we have

$$\begin{aligned}f(g_n) &= c_3(n)g_n^3 + c_2(n)g_n^2 + c_1(n)g_n + c_0(n) \\ &= \frac{1{,}048{,}576}{n^9(n+1)^6(n+2)^4(n+3)^2}\bigl(54n^{15} + 378n^{14} + 916n^{13} + 644n^{12} - 1{,}529n^{11} \\ &\quad - 5{,}340n^{10} - 8{,}383n^9 - 7{,}416n^8 - 2{,}284n^7 + 4{,}156n^6 + 7{,}969n^5 + 7{,}688n^4 \\ &\quad + 4{,}953n^3 + 2{,}154n^2 + 576n + 72\bigr).\end{aligned}$$

Take $M = 6$, it is not difficult to verify that, for all $n \geq M$,

$$f(g_n) > 0.$$

Let $N_0 = \max\{N, M\} = 6$, then for all $n \geq 6$, all of the above inequalities hold. By Lemma 3.1 and Theorem 1.2, the Catalan-Larcombe-French sequence $\{P_n\}_{n\geq 6}$ is strictly 2-log-convex for all $n \geq 6$. What is more, one can easily test that these numbers $\{P_n\}_{0\leq n\leq 8}$ also satisfy the property of 2-log-convexity by simple calculations. Therefore, the whole sequence $\{P_n\}_{n\geq 0}$ is strictly 2-log-convex. This completes the proof. □

It deserves to be mentioned that by considerable calculations and plenty of verifications, the following conjectures should be true.

Conjecture 3.2 *The Catalan-Larcombe-French sequence is ∞-log-convex.*

Conjecture 3.3 *The quotient sequence $\{\frac{P_n}{P_{n-1}}\}_{n\geq 1}$ of the Catalan-Larcombe-French sequence is log-concave, equivalently, for all $n \geq 2$,*

$$P_{n-2}P_n^3 \geq P_{n+1}P_{n-1}^3.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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