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Weighted estimates for vector-valued multilinear square function

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Abstract

Let T be the multilinear square operator, respectively with certain smooth kernels and non-smooth kernels defined in (Xue and Yan in J. Math. Anal. Appl. 422:1342-1362, 2015) and (Hormozi *et al.* in arXiv preprint, 2015), and let T^* be its corresponding maximal operator. In this paper, we prove the vector-valued weighted norm boundedness for T and T^* and also establish multiple weighted inequalities for their corresponding iterated commutator generated by the vector-valued multilinear operator and BMO function.

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1 Introduction

The importance of the multilinear Littlewood-Paley g-function and related multilinear Littlewood-Paley type estimates was shown in PDE and other fields, one can see the works by Coifman *et al.* [3, 4], David and Journe [5], and also by Fabes *et al.* [6–8]. Moreover, a class of multilinear square functions was considered in [8], which was used for Kato's problem.

Recently, Xue *et al.* [9] introduced the multilinear-Paley g-function with a convolution-type kernel in the following way:

$$g(\vec{f})(x) = \left(\int_0^\infty \left| \frac{1}{t^{mn}} \int_{(\mathbb{R}^n)^m} \psi\left(\frac{y_1}{t}, \ldots, \frac{y_m}{t} \right) \prod_{j=1}^m f_j(x - y_j)\, d\vec{y} \right|^2 \frac{dt}{t} \right)^{1/2},$$

and obtained the strong $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ to $L^p(v_{\vec{\omega}})$ boundedness and the weak type results. Later, Xue and Yan [1] studied a class of multilinear square functions associated with the following more general non-convolution-type kernels.

Definition 1 (Integral smooth condition of C-Z type I) (see [1]) For any $v \in (0, \infty)$, let $K_v(x, y_1, \ldots, y_m)$ be a locally integrable function defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$ and denote $\vec{y} = (y_1, \ldots, y_m)$. We say that K_v satisfies the integral condition of C-Z type I, if for some positive constants γ, A, and $B > 1$, the following inequalities

hold:

$$\left(\int_0^\infty |K_\nu(x,\vec{y})|^2 \right)^{1/2} \frac{d\nu}{\nu} \leq \frac{A}{\left(\sum_{j=1}^m |x-y_j|\right)^{mn}}, \tag{1.1}$$

$$\left(\int_0^\infty |K_\nu(z,\vec{y}) - K_\nu(x,\vec{y})|^2 \frac{d\nu}{\nu} \right)^{1/2} \leq \frac{A|z-x|^\gamma}{\left(\sum_{j=1}^m |x-y_j|\right)^{mn+\gamma}}, \tag{1.2}$$

whenever $|z-x| \leq \frac{1}{B} \max_{j=1}^m |x-y_j|$; and

$$\left(\int_0^\infty |K_\nu(x,\vec{y}) - K_\nu(x,y_1,\ldots,y_i',\ldots,y_m)|^2 \frac{d\nu}{\nu} \right)^{1/2} \leq \frac{A|y_i-y_i'|^\gamma}{\left(\sum_{j=1}^m |x-y_j|\right)^{mn+\gamma}} \tag{1.3}$$

for any $i \in \{1,\ldots,m\}$, whenever $|y_i-y_i'| \leq \frac{1}{B}|x-y_i|$.

We define the multilinear square function T by

$$T(\vec{f})(x) = \left(\int_0^\infty \left| \int_{(\mathbb{R}^n)^m} K_\nu(x,y_1,\ldots,y_m) \prod_{j=1}^m f_j(y_j)\, d\vec{y} \right|^2 \frac{d\nu}{\nu} \right)^{1/2} \tag{1.4}$$

for any $\vec{f} = (f_1,\ldots,f_m) \in \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n)$ and for all $x \notin \bigcap_{j=1}^m \mathrm{supp} f_j$.

In order to state their results, we first give the definition of multiple weights $A_{\vec{p}}$.

Definition 2 (Multiple weights) (see [10]) Let $1 \leq p_1,\ldots,p_m < \infty$, and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. For any $\vec{\omega} = (\omega_1,\ldots,\omega_m)$, denote $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$. If

$$\sup_B \left(\frac{1}{|B|} \int_B \nu_{\vec{\omega}} \right)^{1/p} \prod_{i=1}^m \left(\frac{1}{|B|} \int_B \omega_i^{1-p_i'} \right)^{1/p_i'} < \infty \tag{1.5}$$

holds, we say that $\vec{\omega}$ satisfies the $A_{\vec{p}}$ condition. Specially, when $p_i = 1$, $(\frac{1}{|B|} \int_B \omega_i^{1-p_i'})^{1/p_i'}$ is understood as $(\inf_B \omega_i)^{-1}$.

We will need the easy fact: if each $\omega_j \in A_{p_j}$, then $\prod_{j=1}^m A_{p_j} \subset A_{\vec{p}}$.

In [10], the multilinear maximal operator $\mathcal{M}$ was defined by

$$\mathcal{M}(\vec{f})(x) = \sup_{x \in Q} \prod_{i=1}^m \frac{1}{|Q|} \int_Q |f_i(y_i)|\, dy_i, \tag{1.6}$$

where the supremum is taken over all cubes Q containing x. The easy fact is that $\mathcal{M}(\vec{f})(x) \leq \prod_{i=1}^m Mf_i(x)$, where M is the Hardy-Littlewood maximal operator.

Theorem A (see [1]) *Let T be the multilinear square operator defined in* (1.4) *with the kernel satisfying the integral smooth condition of C-Z type* I. *Let $1 < p_1, p_2, \ldots, p_m < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. If $\vec{\omega}$ satisfies the $A_{\vec{p}}$ condition, there exists a constant C such that*

$$\|T(\vec{f})\|_{L^p(\nu_{\vec{\omega}})} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i)}. \tag{1.7}$$

Theorem B (see [1]) *Let T be the operator defined in* (1.4) *with the kernel satisfying the integral smooth condition of C-Z type* I. *Let $0 < \delta < 1/m$, the following inequality holds*:

$$M_\delta^\sharp T(\vec{f})(x) \leq C\mathcal{M}(\vec{f})(x) \tag{1.8}$$

for any bounded and compact supported functions f_i, $i = 1, \ldots, m$.

In order to extend it to a more general case, we recall a class of integral operators $\{A_t\}_{t>0}$ defined in [11], where the operators A_t associated with the kernels $a_t(x, y)$ are defined by

$$A_t f(x) = \int_{\mathbb{R}^n} a_t(x, y) f(y)\, dy$$

for every function $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, and $a_t(x, y)$ satisfies the following size condition:

$$\left| a_t(x, y) \right| \leq h_t(x, y) := t^{-n/s} h\left(\frac{|x - y|}{t^{1/s}} \right) \quad \text{for a fixed constant } s > 0, \tag{1.9}$$

where h is a positive, bounded, decreasing function satisfying

$$\lim_{r \to 0} r^{n+\eta} h(r^s) = 0 \tag{1.10}$$

for some $\eta > 0$. The above conditions indicate that for some $C > 0$ and all $0 < \eta \leq \eta'$, the kernels $a_t(x, y)$ satisfy

$$\left| a_t(x, y) \right| \leq C t^{-n/s} \left(1 + t^{-1/s} |x - y| \right)^{-n-\eta'}.$$

Assumption (H1) Assume that for each $i = 1, \ldots, m$, there exist operators $\{A_t^{(i)}\}_{t>0}$ with kernels $a_t^{(i)}(x, y)$ satisfying conditions (1.9) and (1.10) with constants s and η and that for every $i = 1, \ldots, m$, there exist kernels $K_{t,v}^{(i)}$ such that

$$\left\langle T\left(f_1, \ldots, A_t^{(i)} f_i, \ldots, f_m \right), g \right\rangle$$

$$= \int_{\mathbb{R}^n} \left(\int_0^\infty \left| \int_{(\mathbb{R}^n)^m} K_{t,v}^{(i)}(x, y_1, \ldots, y_m) \prod_{i=1}^m f_i(y_i)\, d\vec{y} \right|^2 \frac{dv}{v} \right)^{1/2} g(x)\, dx \tag{1.11}$$

for all Schwartz functions $f_1, \ldots, f_m, g$ with $\bigcap_{k=1}^m \operatorname{supp} f_k \cap \operatorname{supp} g = \emptyset$.

There exists a function $\phi \in C(\mathbb{R})$ with $\operatorname{supp} \phi \in [-1, 1]$ and a constant $\epsilon > 0$ so that, for every $i = 1, \ldots, m$, we have

$$\left(\int_0^\infty \left| K_v(x, \vec{y}) - K_{t,v}^{(i)}(x, \vec{y}) \right|^2 \frac{dv}{v} \right)^{1/2}$$

$$\leq \frac{A}{(\sum_{j=1}^m |x - y_j|)^{mn}} \sum_{k=1, k \neq i}^m \phi\left(\frac{|y_i - y_k|}{t^{1/s}} \right) + \frac{A t^{\epsilon/s}}{(\sum_{j=1}^m |x - y_j|)^{mn+\epsilon}}, \tag{1.12}$$

whenever $t^{1/s} \leq |x - y_i|/2$.

Assumption (H2) Assume that there exist operators $\{A_t\}_{t>0}$ with kernels $a_t(x,y)$ that satisfy conditions (1.9) and (1.10) with constants s and η, and there exist kernels $K_{t,v}^{(0)}(x,\vec{y})$ such that

$$K_{t,v}^{(0)}(x,\vec{y}) = \int_{\mathbb{R}^n} K_v(z,\vec{y}) a_t(x,z)\, dz \tag{1.13}$$

makes sense for all $(x,\vec{y}) \in (\mathbb{R}^n)^{m+1}$ and $t > 0$. Assume also that there exists a function $\phi \in C(\mathbb{R})$ and $\operatorname{supp}\phi \subset [-1,1]$ and a constant $\epsilon > 0$ such that

$$\left(\int_0^\infty \left| K_{t,v}^{(0)}(x,\vec{y}) \right|^2 \frac{dv}{v} \right)^{1/2} \le \frac{A}{\left(\sum_{k=1}^m |x - y_k| \right)^{mn}}, \tag{1.14}$$

whenever $2t^{1/s} \le \min_{1 \le j \le m} |x - y_j|$ and

$$\left(\int_0^\infty \left| K_v(x,\vec{y}) - K_{t,v}^{(0)}(x,\vec{y}) \right|^2 \frac{dv}{v} \right)^{1/2}$$

$$\le \frac{A}{\left(\sum_{j=1}^m |x - y_j| \right)^{mn}} \sum_{k=1, k \ne i}^m \phi\left(\frac{|x - y_k|}{t^{1/s}} \right) + \frac{A t^{\epsilon/s}}{\left(\sum_{j=1}^m |x - y_j| \right)^{mn+\epsilon}} \tag{1.15}$$

for some $A > 0$, whenever $2t^{1/s} \le \max_{1 \le j \le m} |x - y_j|$.

Assumption (H3) Assume that there exist operators $\{A_t\}_{t>0}$ with kernels $a_t(x,y)$ that satisfy condition (1.9) and (1.10) with constant s and η. Also assume that there exist kernels $K_{t,v}^{(0)}$ satisfying (1.13) and positive constants A and ϵ such that

$$\left(\int_0^\infty \left| K_{t,v}^{(0)}(x,\vec{y}) - K_{t,v}^{(0)}(x',\vec{y}) \right|^2 \frac{dv}{v} \right)^{1/2} \le \frac{A t^{\epsilon/s}}{\left(\sum_{j=1}^m |x - y_j| \right)^{mn+\epsilon}}, \tag{1.16}$$

whenever $2t^{1/s} \le \min_{1 \le j \le m} |x - y_j|$ and $2|x - x'| \le t^{1/s}$.

We say that the kernels K_v generalized the square function kernels if they satisfy (1.1), (1.11), and (1.12) with parameters m, A, s, η, ϵ, and we denote their collection by $m - \mathrm{GSFK}(A, s, \eta, \epsilon)$. We say that T is of class $m - \mathrm{GSFO}(A, s, \eta, \epsilon)$ if T has an associated kernel K_v in $m - \mathrm{GSFK}(A, s, \eta, \epsilon)$.

Theorem C (see [2]) *Let T be a multilinear operator in $m - \mathrm{GSFO}(A, s, \eta, \epsilon)$ with a kernel satisfying Assumptions* (H2) *and* (H3)*. For $1 < p_1, \ldots, p_m < \infty$, $p \ge 1$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\omega \in A_p$, the following inequality holds*:

$$\left\| T(\vec{f}) \right\|_{L^p(\omega)} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega)}. \tag{1.17}$$

Theorem D (see [2]) *Let $0 < \delta < 1/m$ and T be a multilinear operator in $m - \mathrm{GSFO}(A, s, \eta, \epsilon)$ with a kernel satisfying Assumptions* (H2) *and* (H3)*. Then there exists a constant C such that*

$$M_\delta^\sharp T(\vec{f})(x) \le C \prod_{j=1}^m M f_j(x) \tag{1.18}$$

holds for any bounded and compact supported function f_i, $i = 1, 2, \ldots, m$.

Moreover, the corresponding multilinear maximal square function T^* is defined by

$$T^*(\vec{f})(x) = \sup_{\delta > 0} \left(\int_0^\infty \left| \int_{\sum_{i=1}^m |x-y_i|^2 > \delta^2} K_\nu(x, \vec{y}) \prod_{k=1}^m f_j(y_j)\, d\vec{y} \right|^2 \frac{d\nu}{\nu} \right)^{1/2}. \tag{1.19}$$

Theorem E (see [2]) *Let T be a multilinear operator in m – GSFO(A, s, η, ϵ) with a kernel satisfying Assumptions* (H2) *and* (H3). *For $1 < p_1, \ldots, p_m < \infty$, $p \geq 1$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\omega \in A_p$, the following inequality holds:*

$$\left\| T^*(\vec{f}) \right\|_{L^p(\omega)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega)}. \tag{1.20}$$

Theorem F (see [2]) *Let T be a multilinear operator in m – GSFO(A, s, η, ϵ) with a kernel satisfying Assumptions* (H2) *and* (H3). *For any $\eta > 0$, there is a constant $C < \infty$ depending on η such that the following inequality holds:*

$$T^*(\vec{f})(x) \leq C \left(M_\eta T(\vec{f})(x) + \prod_{j=1}^m M f_j(x) \right), \quad \forall x \in \mathbb{R}^n \tag{1.21}$$

for all $\vec{f}$ in any product of $L^{q_j}(\mathbb{R}^n)$ spaces, with $1 \leq q_j < \infty$.

2 Main results

In this section, we first list some results about vector-valued multilinear operator T_q and the corresponding vector-valued maximal multilinear operator T_q^* which are defined, respectively, by

$$T_q(\vec{f})(x) = \left\| T(\vec{f})(x) \right\|_{\ell^q} = \left(\sum_{k=1}^\infty \left| T(f_{1k}, \ldots, f_{mk})(x) \right|^q \right)^{1/q}, \tag{2.1}$$

$$T_q^*(\vec{f})(x) = \left\| T^*(\vec{f})(x) \right\|_{\ell^q} = \left(\sum_{k=1}^\infty \left| T^*(f_{1k}, \ldots, f_{mk})(x) \right|^q \right)^{1/q}, \tag{2.2}$$

where $\vec{f} = (f_1, \ldots, f_m)$ with $f_i = \{f_{ik}\}_{k=1}^\infty$.

Theorem 3 *Assume that T is a multilinear square operator defined in* (1.4) *with the kernel satisfying the integral condition of C-Z type* I. *Let $1 < p_1, p_2, \ldots, p_m < \infty$, $1 < q_1, q_2, \ldots, q_m < \infty$, and $1/m < p, q < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$. If $(\omega_1^{p_1}, \ldots, \omega_m^{p_m}) \in (A_{p_1}, \ldots, A_{p_m})$, the following inequality holds:*

$$\left\| T_q(\vec{f}) \right\|_{L^p(\prod_{j=1}^m \omega_j^p)} \leq C \prod_{j=1}^m \left\| \|f_j\|_{\ell^{q_j}} \right\|_{L^{p_j}(\omega_j^{p_j})}. \tag{2.3}$$

Theorem 4 *Assume that T is a multilinear square operator defined in* (1.4) *with the kernel satisfying the integral condition of C-Z type* I. *Let $1 \leq p_1, p_2, \ldots, p_m < \infty$, $1 < q_1, q_2, \ldots, q_m < \infty$, and $0 < p, q < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$.*

(i) *If $1 \leq p_1, p_2, \ldots, p_m < \infty$ and $\omega \in A_{p_1} \cap \cdots \cap A_{p_m}$, there exists a constant $C > 0$ such that*

$$\left\| T_q(\vec{f}) \right\|_{L^p(\omega)} \leq C \prod_{j=1}^{m} \left\| \|f_j\|_{\ell^{q_j}} \right\|_{L^{p_j}(\omega)}. \tag{2.4}$$

(ii) *If at least one $p_j = 1$ and $\omega \in A_1$, there exists a constant $C > 0$ such that*

$$\left\| T_q(\vec{f}) \right\|_{L^{p,\infty}(\omega)} \leq C \prod_{j=1}^{m} \left\| \|f_j\|_{\ell^{q_j}} \right\|_{L^{p_j}(\omega)}. \tag{2.5}$$

Next, we show the results for the multilinear square operator T with non-smooth kernels and its corresponding maximal operator T^*. Meanwhile, we also establish multiple weighted inequalities for their corresponding iterated commutator generated by the vector-valued multilinear operator and BMO function. We will state our results as follows.

Theorem 5 *Let T be a multilinear operator in $m - \mathrm{GSFO}(A, s, \eta, \epsilon)$ with a kernel satisfying Assumptions* (H2) *and* (H3). *Let $1 < p_1, p_2, \ldots, p_m < \infty$, $1 < q_1, q_2, \ldots, q_m < \infty$ and $1/m < p, q < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$. If $(\omega_1^{p_1}, \ldots, \omega_m^{p_m}) \in (A_{p_1}, \ldots, A_{p_m})$, there exists a constant $C > 0$ such that*

$$\left\| T_q(\vec{f}) \right\|_{L^p(\prod_{j=1}^{m} \omega_j^p)} \leq C \prod_{j=1}^{m} \left\| \|f_j\|_{\ell^{q_j}} \right\|_{L^{p_j}(\omega_j^{p_j})}. \tag{2.6}$$

A similar estimate also holds true for the corresponding maximal operator T^*.

Theorem 6 *Let T be a multilinear operator in $m - \mathrm{GSFO}(A, s, \eta, \epsilon)$ with a kernel satisfying Assumptions* (H2) *and* (H3). *Let $1 \leq p_1, p_2, \ldots, p_m < \infty$, $1 < q_1, q_2, \ldots, q_m < \infty$ and $0 < p, q < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$.*
  (i) *If $1 \leq p_1, p_2, \ldots, p_m < \infty$ and $\omega \in A_{p_1} \cap \cdots \cap A_{p_m}$, the following inequality holds:*

$$\left\| T_q(\vec{f}) \right\|_{L^p(\omega)} \leq C \prod_{j=1}^{m} \left\| \|f_j\|_{\ell^{q_j}} \right\|_{L^{p_j}(\omega)}. \tag{2.7}$$

(ii) *If at least one $p_j = 1$ and $\omega \in A_1$, the following inequality holds:*

$$\left\| T_q(\vec{f}) \right\|_{L^{p,\infty}(\omega)} \leq C \prod_{j=1}^{m} \left\| \|f_j\|_{\ell^{q_j}} \right\|_{L^{p_j}(\omega)}. \tag{2.8}$$

Similar estimates also hold true for the corresponding maximal operators T^*.

The commutator associated with T is given by

$$T_{\Pi \vec{b}}(\vec{f})(x) = \left[b_1, \left[b_2, \ldots \left[b_{\ell-1}, [b_\ell, T]_\ell \right]_{\ell-1} \cdots \right]_2 \right]_1 (\vec{f})(x)$$

$$= \int_{(\mathbb{R}^n)^m} \prod_{j=1}^{\ell} (b_j(x) - b_j(y)) K(x, y_1, \ldots, y_m) \prod_{i=1}^{m} f_i(y_i) \, d\vec{y},$$

where $1 \leq \ell \leq m$.

For simplicity of notation, for the sequence $\{\vec{f}_k\}_{k=1}^{\infty} = \{f_{1k}, \ldots, f_{mk}\}_{k=1}^{\infty}$ of vector functions, the commutator associated with a vector-valued T_q can be defined by

$$T_{\Pi\vec{b},q}(\vec{f})(x) = \big\| T_{\Pi\vec{b}}(\vec{f})(x) \big\|_{\ell^q} = \left(\sum_{k=1}^{\infty} \big| T_{\Pi\vec{b}}(\vec{f}_k)(x) \big|^q \right)^{1/q}. \tag{2.9}$$

Theorem 7 *Assume that T is a multilinear operator in $m-\mathrm{GSFO}(A, s, \eta, \epsilon)$ with kernel satisfying Assumptions (H2) and (H3). Let $1/m < p < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 < p_1, \ldots, p_m < \infty$, $1/m < q < \infty$, and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$ with $1 < q_1, \ldots, q_m < \infty$. Suppose that $\vec{\omega} \in A_{\vec{p}}$ and $\vec{b} \in (BMO)^{\ell}$.*

(i) *There exists a constant $C > 0$ such that*

$$\big\| T_{\Pi\vec{b},q}(\vec{f}) \big\|_{L^p(v_\omega)} \leq \prod_{j=1}^{\ell} \|b_j\|_{BMO} \prod_{j=1}^{m} \big\| \|f\|_{q_j} \big\|_{L^{p_j}(M\omega_j)}. \tag{2.10}$$

(ii) *If $\omega_j \in A_{p_j}$, there exists a constant $C > 0$ such that*

$$\big\| T_{\Pi\vec{b},q}(\vec{f}) \big\|_{L^p(v_\omega)} \leq \prod_{j=1}^{\ell} \|b_j\|_{BMO} \prod_{j=1}^{m} \big\| \|f\|_{q_j} \big\|_{L^{p_j}(\omega_j)}. \tag{2.11}$$

3 The proof of Theorem 3

Since $\omega_j \in A_{p_j}$, by the previous statement, $\prod_{j=1}^{m} \omega_j^{p_j} \in A_{\vec{p}}$. Writing $v_{\vec{\omega}} = \prod_{j=1}^{m} (\omega_j^{p_j})^{p/p_j} = \prod_{j=1}^{m} \omega_j^p$, Theorem A implies that

$$\big\| T(f) \big\|_{L^p(\prod_{j=1}^{m} \omega_j^p)} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}}(\omega_j^{p_j}). \tag{3.1}$$

We will apply the following lemma to get the desirable result.

Lemma 8 (see [12]) *Let $\mathcal{T}$ be an m-linear operator, and let $1 < s_1, \ldots, s_m < \infty$ and $1/m < s < \infty$ be fixed indices such that $\frac{1}{s} = \frac{1}{s_1} + \cdots + \frac{1}{s_m}$. For $(\omega_1^{s_1}, \ldots, \omega_m^{s_m}) \in (A_{s_1}, \ldots, A_{s_m})$, the following estimate holds:*

$$\big\| \mathcal{T}(\vec{f}) \big\|_{L^s(\prod_{j=1}^{m} \omega_j^s)} \prod_{j=1}^{m} \|f_j\|_{L^{s_j}}(\omega_j^{s_j}). \tag{3.2}$$

Then, for all indices, $1 < p_1, \ldots, p_m < \infty$ and $1/m < p < \infty$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $1 < q_1, \ldots, q_m < \infty$, and $1/m < q < \infty$ such that $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and all $(\omega_1^{p_1}, \ldots, \omega_m^{p_m}) \in (A_{p_1}, \ldots, A_{p_m})$. Then the following inequality holds:

$$\big\| \mathcal{T}_q(\vec{f}) \big\|_{L^p(\prod_{j=1}^{m} \omega_j^p)} \leq C \prod_{j=1}^{m} \big\| \|f_j\|_{\ell^{q_j}} \big\|_{L^{p_j}(\omega_j^{p_j})}. \tag{3.3}$$

4 The proof of Theorem 5

We first state the following Fefferman-Stein inequality.

Lemma 9 (see [13]) *Let $0 < p, \delta < \infty$ and ω be any Mockenhaupt A_∞ weight. Then there exists a constant C independent of f such that the inequality*

$$\int_{\mathbb{R}^n} \left(M_\delta f(x) \right)^p \omega(x)\, dx \le C \int_{\mathbb{R}^n} \left(M_\delta^\sharp f(x) \right)^p \omega(x)\, dx, \tag{4.1}$$

holds for any function f for which the left-hand side is finite.

Lemma 10 (see [12]) *For $(\omega_1, \ldots, \omega_m) \in (A_{p_1}, \ldots, A_{p_m})$ with $1 \le p_1, \ldots, p_m < \infty$ and for $0 < \theta_1, \ldots, \theta_m < 1$ such that $\theta_1 + \cdots + \theta_m = 1$, we have $\omega_1^{\theta_1} \cdots \omega_1^{\theta_1} \in A_{\max\{p_1, \ldots, p_m\}}$.*

Note that $(\omega_1^{p_1}, \ldots, \omega_m^{p_m}) \in (A_{p_1}, \ldots, A_{p_m})$, and Lemma 10 indicates that $\prod_{j=1}^{m} \omega_j^p = \prod_{j=1}^{m} (\omega_j^{p_j})^{p/p_j} \in A_{\max\{p_1, \ldots, p_m\}} \subset A_\infty$.

Exploiting Lemma 3.3 in [2] and the standard argument, we obtain $\|M_\delta T(\vec{f})\|_{L^p(\prod_{j=1}^{m} \omega_j^p)} < \infty$. Together with Theorem D, we have

$$\left\| T(\vec{f}) \right\|_{L^p(\prod_{j=1}^{m} \omega_j^p)} \le \left\| M_\delta T(\vec{f}) \right\|_{L^p(\prod_{j=1}^{m} \omega_j^p)}$$

$$\le C \left\| M_\delta^\sharp T(\vec{f}) \right\|_{L^p(\prod_{j=1}^{m} \omega_j^p)}$$

$$\le C \left\| \prod_{j=1}^{m} M f_j \right\|_{L^p(\prod_{j=1}^{m} \omega_j^p)}$$

$$\le C \prod_{j=1}^{m} \| M f_j \|_{L^{p_j}(\omega_j^{p_j})}$$

$$\le C \prod_{j=1}^{m} \| f_j \|_{L^{p_j}(\omega_j^{p_j})}.$$

By Lemma 8, we finish the proof of Theorem 5.

The estimate for T^* will follow from Lemma 8, Theorem F, and the following argument:

$$\left\| T^*(\vec{f}) \right\|_{L^p(\prod_{j=1}^{m} \omega_j^p)} \le C \left(\left\| M_\eta T(\vec{f}) \right\|_{L^p(\prod_{j=1}^{m} \omega_j^p)} + \left\| \prod_{j=1}^{m} M f_j \right\|_{L^p(\prod_{j=1}^{m} \omega_j^p)} \right)$$

$$\le C \left(\left\| |M|T(\vec{f})|^\eta \right\|_{L^{\frac{p}{\eta}}(\prod_{j=1}^{m} \omega_j^p)}^{\frac{1}{\eta}} + \left\| \prod_{j=1}^{m} M f_j \right\|_{L^p(\prod_{j=1}^{m} \omega_j^p)} \right)$$

$$\le C \left(\left\| |T(\vec{f})|^\eta \right\|_{L^{\frac{p}{\eta}}(\prod_{j=1}^{m} \omega_j^p)}^{\frac{1}{\eta}} + \left\| \prod_{j=1}^{m} M f_j \right\|_{L^p(\prod_{j=1}^{m} \omega_j^p)} \right)$$

$$\le C \left(\left\| T(\vec{f}) \right\|_{L^p(\prod_{j=1}^{m} \omega_j^p)} + \left\| \prod_{j=1}^{m} M f_j \right\|_{L^p(\prod_{j=1}^{m} \omega_j^p)} \right)$$

$$\le C \left\| \prod_{j=1}^{m} M f_j \right\|_{L^p(\prod_{j=1}^{m} \omega_j^p)}. \tag{4.2}$$

5 The proofs of Theorem 4 and Theorem 6

In order to prove these theorems, first we introduce the following lemmas.

Let $\mathcal{F}$ denote a family of ordered pairs of non-negative, measurable functions (f,g), if we say that for some p, $0 < p < \infty$, and $\omega \in A_\infty$,

$$\int_{\mathbb{R}^n} f(x)^p \omega(x)\,dx \leq C \int_{\mathbb{R}^n} g(x)^p \omega(x)\,dx, \tag{5.1}$$

and we denote it by $(f,g) \in \mathcal{F}$.

Lemma 11 (see [14]) *Given a family $\mathcal{F}$, suppose that for some p_0, $0 < p_0 < \infty$, and for every weight $\omega \in A_\infty$, $(f,g) \in \mathcal{F}$. Then we have, for all $0 < p, q < \infty$ and $\omega \in A_\infty$,*

$$\left\| \left(\sum_k (f_k)^q \right)^{1/q} \right\|_{L^p(\omega)} \leq C \left\| \left(\sum_k (g_k)^q \right)^{1/q} \right\|_{L^p(\omega)}, \quad \{(f_k, g_k)\}_k \subset \mathcal{F}. \tag{5.2}$$

For all $0 < p, q < \infty$, $0 < s \leq \infty$, and $\omega \in A_\infty$,

$$\left\| \left(\sum_k (f_k)^q \right)^{1/q} \right\|_{L^{p,s}(\omega)} \leq C \left\| \left(\sum_k (g_k)^q \right)^{1/q} \right\|_{L^{p,s}(\omega)}, \quad \{(f_k, g_k)\}_k \subset \mathcal{F}. \tag{5.3}$$

Lemma 12 (see [15])

(i) *Let $1 < q < \infty$ and $1 \leq p < \infty$, there is a constant $C_{r,p}$ such that*

$$\left\| \left(\sum_k |Mf_k(x)|^q \right)^{1/q} \right\|_{L^{p,\infty}(\omega)} \leq C_{q,p} \left\| \left(\sum_k |f_k(x)|^q \right)^{1/q} \right\|_{L^p(\omega)} \tag{5.4}$$

if and only if $\omega \in A_p$.

(ii) *Let $1 < q < \infty$ and $1 < p < \infty$, there is a constant $C_{q,p}$ such that*

$$\left\| \left(\sum_k |Mf_k(x)|^q \right)^{1/q} \right\|_{L^p(\omega)} \leq C_{q,p} \left\| \left(\sum_k |f_k(x)|^q \right)^{1/q} \right\|_{L^p(\omega)} \tag{5.5}$$

if and only if $\omega \in A_p$.

By Theorem B, Theorem D, and Theorem F, together with the argument from Section 3 and (4.2), we have

$$\left\| \mathcal{T}(\vec{f}) \right\|_{L^p(\omega)} \leq C \left\| \prod_{j=1}^m Mf_j \right\|_{L^p(\omega)}. \tag{5.6}$$

Here $\mathcal{T}$ can be replaced by T and T^* which are from Theorem 4 and Theorem 6.

We apply Lemma 11 to $(T(\vec{f}), \prod_{j=1}^m Mf_j) \in \mathcal{F}$, and by Lemma 12 we get the desirable results.

6 The proof of Theorem 7

In order to prove Theorem 7, first we will list some notations and lemmas:

$$\mathcal{M}(\|\vec{f}\|_{\ell^q})(x) := \sup_{x \in Q} \prod_{j=1}^m \frac{1}{|Q|} \int_Q \|f_j(y_j)\|_{\ell^{q_j}}\,dy_j, \tag{6.1}$$

$$\mathcal{M}_{L(\log L)}\big(\|\vec{f}\|_{\ell^q}\big)(x) := \sup_{x \in Q} \prod_{j=1}^{m} \big\| \|f_j\|_{\ell^{q_j}} \big\|_{L(\log L),Q}, \tag{6.2}$$

$$\mathcal{M}_{\rho}\big(\|\vec{f}\|_{\ell^q}\big)(x)$$

$$:= \sup_{x \in Q} \sum_{\nu=0}^{\infty} 2^{-\nu n \ell} \prod_{j \in \rho} \frac{1}{|Q|} \int_Q \|f_j(y_j)\|_{\ell^{q_j}} dy_j \prod_{j \in \rho'} \frac{1}{|2^\nu Q|} \int_{2^\nu Q} \|f_j(y_j)\|_{\ell^{q_j}} dy_j, \tag{6.3}$$

where $\rho = \{j_1, \ldots, j_\ell\} \subset \{1, \ldots, m\}$, $1 \le \ell < m$ and $\rho' = \{1, \ldots, m\} \backslash \rho$.

Lemma 13 (see [10]) *Let* $1 < p_1, \ldots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\vec{P} = (p_1, \ldots, p_m)$, $\vec{\omega} \in A_{\vec{P}}$, *and* $\rho = \{j_1, \ldots, j_\ell\} \subset \{1, \ldots, m\}$, $1 \le \ell < m$. *Then* $\mathcal{M}$, $\mathcal{M}_{L(\log L)}$, $\mathcal{M}_\rho$ *are bounded from* $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ *to* $L^p(\nu_\omega)$.

Lemma 14 *Let* T *be a multilinear operator in* $m - \mathrm{GSFO}(A, s, \eta, \epsilon)$ *with kernel satisfying Assumptions* (H2) *and* (H3). *Assume that* $1 \le \ell < m$, $\rho = \{j_1, \ldots, j_\ell\}$, *and* $1/m < q < \infty$, $1 \le q_1, \ldots, q_m < \infty$ *with* $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$. *Then there exists a constant* $C > 0$ *such that*

$$M_\delta^\sharp T_q(\vec{f})(x) \le C\big(\mathcal{M}\big(\|\vec{f}\|_{\ell^q}\big)(x) + \mathcal{M}_\rho\big(\|\vec{f}\|_{\ell^q}\big)(x)\big). \tag{6.4}$$

Proof For a point x and a cube $Q \ni x$, to obtain (6.4), it suffices to prove for $0 < \delta < 1/m$,

$$\left(\frac{1}{|Q|} \int_Q \|T(\vec{f})(z) - c\|_{\ell^q}^\delta \, dz \right)^{1/\delta} \le C\big(\mathcal{M}\big(\|\vec{f}\|_{\ell^q}\big)(x) + \mathcal{M}_\rho\big(\|\vec{f}\|_{\ell^q}\big)(x)\big) \tag{6.5}$$

for some constant c to be determined later.

Write $\vec{f}_k = \vec{f}_k^0 + \vec{f}_k^\infty$, where $\{\vec{f}_k^0\}_{k=1}^\infty = \{\vec{f}_k^0 \chi_{Q^*}\}_{k=1}^\infty = \{f_{1k}\chi_{Q^*}, \ldots, f_{mk}\chi_{Q^*}\}$ and $Q^* = (8\sqrt{n}+4)Q$. Let $c = \sum_{\alpha_1,\ldots,\alpha_m} T(\vec{f}^\alpha)(x)$ and in the sum each $\alpha_j = 0$ or ∞ and in each term there is at least one $\alpha_j = \infty$. Then

$$\left(\frac{1}{|Q|} \int_Q \|T(\vec{f})(z) - c\|_{\ell^q}^\delta \, dz \right)^{1/\delta}$$

$$\le C\left(\frac{1}{|Q|} \int_Q |T_q(\vec{f}^0)(z)|^\delta \, dz \right)^{1/\delta}$$

$$+ C \sum_{\alpha_1,\ldots,\alpha_m} \left(\frac{1}{|Q|} \int_Q \|T(\vec{f}^\alpha)(z) - T(\vec{f}^\alpha)(x)\|_{\ell^q}^\delta \, dz \right)^{1/\delta}$$

$$:= I + \sum_{\alpha_1,\ldots,\alpha_m} II_{\alpha_1,\ldots,\alpha_m},$$

where in each term of the last sum there is at least one $\alpha_j = \infty$.

Kolmogorov's inequality and Theorem 6 implies that

$$I \le C\|T_q(\vec{f}^0)\|_{L^{1/m,\infty}(Q, \frac{dz}{|Q|})}$$

$$\le C \prod_{j=1}^{m} \frac{1}{|Q|} \int_Q \|f_j(z)\|_{\ell^{q_j}} \, dz$$

$$\le C\mathcal{M}\big(\|\vec{f}\|_{\ell^q}\big). \tag{6.6}$$

We proceed to the estimate for $II_{\alpha_1,\dots,\alpha_m}$. Here we choose $t = [2\sqrt{n}\ell(Q)]^s$. If $\alpha_1 = \cdots \alpha_m = \infty$, we have

$$II_{\infty,\dots,\infty} \leq \frac{C}{|Q|} \int_Q \left\| T(\vec{f}^\alpha)(z) - T(\vec{f}^\alpha)(x) \right\|_{\ell^q} dz$$

$$\leq \frac{C}{|Q|} \int_Q \left(\sum_{k=1}^\infty \left| T(\vec{f}_k^\infty)(z) - T(\vec{f}_k^\infty)(x) \right|^q \right)^{1/q} dz$$

$$\leq \frac{C}{|Q|} \int_Q \left(\sum_{k=1}^\infty \left| \int_0^\infty \left| \int_{(\mathbb{R}^n \setminus Q^*)^m} \left(K_\nu(z,\vec{y}) - K_\nu(x,\vec{y}) \right) \prod_{j=1}^m f_{jk}(y_j)\, d\vec{y} \right|^2 \frac{d\nu}{\nu} \right|^{q/2} \right)^{1/q} dz,$$

applying Minkowski's inequality, we get

$$\frac{C}{|Q|} \int_Q \left(\sum_{k=1}^\infty \left| \int_0^\infty \left| \int_{(\mathbb{R}^n \setminus Q^*)^m} \left(K_\nu(z,\vec{y}) - K_\nu(x,\vec{y}) \right) \prod_{j=1}^m f_{jk}(y_j)\, d\vec{y} \right|^2 \frac{d\nu}{\nu} \right|^{q/2} \right)^{1/q} dz$$

$$\leq \frac{C}{|Q|} \int_Q \left(\sum_{k=1}^\infty \left| \int_{(\mathbb{R}^n \setminus Q^*)^m} \left(\int_0^\infty \left| K_\nu(z,\vec{y}) - K_\nu(x,\vec{y}) \right|^2 \frac{d\nu}{\nu} \right)^{1/2} \prod_{j=1}^m f_{jk}(y_j)\, d\vec{y} \right|^q \right)^{1/q} dz$$

$$\leq \frac{C}{|Q|} \int_Q \left(\sum_{k=1}^\infty \left| \int_{(\mathbb{R}^n \setminus Q^*)^m} \left(\int_0^\infty \left| K_\nu(z,\vec{y}) - K_{t,\nu}^{(0)}(z,\vec{y}) \right|^2 \frac{d\nu}{\nu} \right)^{1/2} \prod_{j=1}^m f_{jk}(y_j)\, d\vec{y} \right|^q \right)^{1/q} dz$$

$$+ \frac{C}{|Q|} \int_Q \left(\sum_{k=1}^\infty \left| \int_{(\mathbb{R}^n \setminus Q^*)^m} \left(\int_0^\infty \left| K_{t,\nu}^{(0)}(z,\vec{y}) - K_{t,\nu}^{(0)}(x,\vec{y}) \right|^2 \frac{d\nu}{\nu} \right)^{1/2} \right. \right.$$

$$\left. \left. \times \prod_{j=1}^m f_{jk}(y_j)\, d\vec{y} \right|^q \right)^{1/q} dz$$

$$+ \frac{C}{|Q|} \int_Q \left(\sum_{k=1}^\infty \left| \int_{(\mathbb{R}^n \setminus Q^*)^m} \left(\int_0^\infty \left| K_{t,\nu}^{(0)}(x,\vec{y}) - K_\nu(x,\vec{y}) \right|^2 \frac{d\nu}{\nu} \right)^{1/2} \prod_{j=1}^m f_{jk}(y_j)\, d\vec{y} \right|^q \right)^{1/q} dz$$

$$:= II_{\infty,\dots,\infty}^1 + II_{\infty,\dots,\infty}^2 + II_{\infty,\dots,\infty}^3.$$

Because of $z \in Q$ and $y_j \in \mathbb{R}^n \setminus (8\sqrt{n} + 4)Q$, we obtain $|y_j - z| > (4\sqrt{n} + 1)\ell(Q) > 2t^{1/s}$ for all $j = 1,\dots,m$. Assumption (H2) gives

$$II_{\infty,\dots,\infty}^1 \leq \frac{C}{|Q|} \int_Q \int_{(\mathbb{R}^n \setminus Q^*)^m} \frac{At^{\epsilon/s}}{(\sum_{j=1}^m |z - y_j|)^{mn+\epsilon}} \prod_{j=1}^m \left\| f_j(y_j) \right\|_{\ell^{q_j}} d\vec{y}\, dz$$

$$\leq \sum_{k=1}^\infty \frac{1}{2^{k\epsilon}} \prod_{j=1}^m \frac{1}{2^{(k+1)n}|Q^*|} \int_{2^{k+1}Q^*} \left\| f_j(y_j) \right\|_{\ell^{q_j}} dy_j$$

$$\leq C \mathcal{M}\left(\|\vec{f}\|_{\ell^q} \right)(x).$$

Since $x, z \in Q$, $|z - x| \leq \sqrt{n}\ell(Q) \leq 1/2 t^{1/s}$. Noting that $|y_j - z| > (4\sqrt{n} + 1)\ell(Q) > 2t^{1/s}$ for all $j = 1,\dots,m$, applying Assumption (H3) and a similar argument to II^1, we have $II_{\infty,\dots,\infty}^2 \leq C \mathcal{M}(\|\vec{f}\|_{\ell^q})(x)$. Similarly, we also get $II_{\infty,\dots,\infty}^3 \leq C \mathcal{M}(\|\vec{f}\|_{\ell^q})(x)$.

Now let us consider the typical case of $II_{\alpha_1,\ldots,\alpha_m}$, that is, $\alpha_1 = \cdots = \alpha_h = \infty$ and $\alpha_{h+1} = \cdots = \alpha_m = 0$, $1 \le h < m$,

$$
\begin{aligned}
II_{\infty,\ldots,0} \le{}& \frac{C}{|Q|} \int_Q \Bigg(\sum_{k=1}^{\infty} \bigg| \int_{(\mathbb{R}^n)^m} \bigg(\int_0^{\infty} \big|K_\nu(z,\vec{y}) - K_{t,\nu}^{(0)}(z,\vec{y})\big|^2 \frac{d\nu}{\nu} \bigg)^{1/2} \\
& \times \prod_{j=1}^{h} f_{jk}^{\infty} \prod_{j=h+1}^{m} f_{jk}^0 \, d\vec{y} \bigg|^q \Bigg)^{1/q} dz \\
& + \frac{C}{|Q|} \int_Q \Bigg(\sum_{k=1}^{\infty} \bigg| \int_{(\mathbb{R}^n)^m} \bigg(\int_0^{\infty} \big|K_{t,\nu}^{(0)}(z,\vec{y}) - K_{t,\nu}^{(0)}(x,\vec{y})\big|^2 \frac{d\nu}{\nu} \bigg)^{1/2} \\
& \times \prod_{j=1}^{h} f_{jk}^{\infty} \prod_{j=h+1}^{m} f_{jk}^0 \, d\vec{y} \bigg|^q \Bigg)^{1/q} dz \\
& + \frac{C}{|Q|} \int_Q \Bigg(\sum_{k=1}^{\infty} \bigg| \int_{(\mathbb{R}^n)^m} \bigg(\int_0^{\infty} \big|K_{t,\nu}^{(0)}(x,\vec{y}) - K_\nu(x,\vec{y})\big|^2 \frac{d\nu}{\nu} \bigg)^{1/2} \\
& \times \prod_{j=1}^{h} f_{jk}^{\infty} \prod_{j=h+1}^{m} f_{jk}^0 \, d\vec{y} \bigg|^q \Bigg)^{1/q} dz \\
:={}& II_{\infty,\ldots,0}^1 + II_{\infty,\ldots,0}^2 + II_{\infty,\ldots,0}^3.
\end{aligned}
$$

For $II_{\infty,\ldots,0}^1$, by Assumption (H2), we have

$$
\begin{aligned}
II_{\infty,\ldots,0}^1 \le{}& \frac{C}{|Q|} \int_Q \Bigg(\int_{(\mathbb{R}^n \setminus Q^*)^h} \frac{At^{\epsilon/s} \prod_{j=1}^{h} \|f_j\|_{\ell^{q_j}} \, dy_j}{(\sum_{j \in \{1,\ldots,h\}} |z - y_j|)^{mn+\epsilon}} \\
& + \int_{(\mathbb{R}^n \setminus Q^*)^h} \frac{A \prod_{j=1}^{h} \|f_j\|_{\ell^{q_j}} \, dy_j}{(\sum_{j \in \{1,\ldots,h\}} |z - y_j|)^{mn}} \Bigg) \prod_{j=h+1}^{m} \int_{Q^*} \|f_j\|_{\ell^{q_j}} \, dy_j \, dz \\
\le{}& \Bigg(\sum_{k=1}^{\infty} \frac{A|Q^*|^{\epsilon/n}}{(2^k |Q^*|^{1/n})^{mn+\epsilon}} \int_{(2^k Q^*)^h} \prod_{j=1}^{h} \|f_j\|_{\ell^{q_j}} \, dy_j \\
& + \sum_{k=1}^{\infty} \frac{A}{(2^k |Q^*|^{1/n})^{mn}} \int_{(2^k Q^*)^h} \prod_{j=1}^{h} \|f_j\|_{\ell^{q_j}} \, dy_j \Bigg) \prod_{j=h+1}^{m} \int_{Q^*} \|f_j\|_{\ell^{q_j}} \, dy_j \\
\le{}& C \sum_{k=0}^{\infty} \prod_{j=1}^{m} \frac{1}{2^{k\epsilon}} \frac{1}{(2^{k+1}|Q^*|^{1/n})^n} \int_{2^{k+1}Q^*} \|f_j\|_{\ell^{q_j}} \, dy_j \\
& + C \sum_{k=0}^{\infty} \frac{1}{2^{kn(m-h)}} \prod_{j=h+1}^{m} \frac{1}{|Q^*|} \int_{Q^*} \|f_j\|_{\ell^{q_j}} \, dy_j \\
& \times \prod_{j=1}^{h} \frac{1}{(2^{k+1}|Q^*|^{1/n})^n} \int_{2^{k+1}Q^*} \|f_j\|_{\ell^{q_j}} \, dy_j \\
\le{}& C\mathcal{M}\big(\|\vec{f}\|_{\ell^q}\big)(x) + \mathcal{M}_\rho\big(\|\vec{f}\|_{\ell^q}\big)(x).
\end{aligned}
$$

By a similar argument, we deduce that $II_{\infty,\ldots,0}^3 \le C\mathcal{M}(\|\vec{f}\|_{\ell^q})(x) + \mathcal{M}_\rho(\|\vec{f}\|_{\ell^q})(x)$ and $II_{\infty,\ldots,0}^2 \le C\mathcal{M}(\|\vec{f}\|_{\ell^q})(x)$. $\qquad\square$

Given any positive integer m, $\forall 1 \le j \le m$, let C_j^m denote the family of all finite subset $\sigma = \{\sigma(1), \ldots, \sigma(j)\}$ of j different elements. For any $\sigma \in C_j^m$ we associate the complementary sequence σ' given by $\sigma' = \{1, 2, \ldots, m\} \backslash \sigma$.

Lemma 15 *Let $0 < \delta < \epsilon < 1/m$, $1/m < q < \infty$, and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$ with $1 < q_1, \ldots, q_m < \infty$. Suppose that $\vec{b} \in (BMO)^\ell$. Then there exists a constant $C > 0$ depending only on δ and ϵ such that*

$$
M_\delta^\sharp (T_{\Pi \vec{b}, q} \vec{f})(x) \le C \prod_{j=1}^\ell \|b_j\|_{BMO} \big(\mathcal{M}_{L(\log L)} \|\vec{f}\|_{\ell^q}(x) + M_\epsilon (T_q(\vec{f}))(x) \big)
$$

$$
+ C \sum_{i=1}^{\ell-1} \sum_{\sigma \in C_i^\ell} C_{i,\ell} \prod_{j \in \sigma} \|b_j\|_{BMO} M_\epsilon (T_{\Pi \vec{b}_{\sigma'}, q} \vec{f})(x) \tag{6.7}
$$

for any smooth vector function $\{f_k\}_{k=1}^\infty$ for any $x \in \mathbb{R}^n$.

Proof For simplicity of notation, we replace $\prod_{j=1}^m f_j(y_j)$ by $F(\vec{y})$ and let $\lambda_j = \frac{1}{2|Q|} \int_{2Q} b_j(z)\, dz$, for $j = 1, \ldots, \ell$. Let $x \in \mathbb{R}^n$ and Q be a cube centered at x. We have

$$
T_{\Pi \vec{b}}(\vec{f})(x) = \left(\int_0^\infty \left| \int_{(\mathbb{R}^n)^m} \prod_{j=1}^\ell (b_j(x) - b_j(y_j)) K_\nu(x, \vec{y}) F(\vec{y})\, d\vec{y} \right|^2 \frac{d\nu}{\nu} \right)^{1/2}
$$

$$
= \left(\int_0^\infty \left| \int_{(\mathbb{R}^n)^m} \prod_{j=1}^\ell \big((b_j(x) - \lambda_j) - (b_j(y_j) - \lambda_j) \big) K_\nu(x, \vec{y}) F(\vec{y})\, d\vec{y} \right|^2 \frac{d\nu}{\nu} \right)^{1/2}
$$

$$
\le \sum_{i=0}^\ell \sum_{\sigma \in C_i^\ell} \prod_{j \in \sigma} |b_j(x) - \lambda_j|
$$

$$
\times \left(\int_0^\infty \left| \int_{(\mathbb{R}^n)^m} \prod_{j \in \sigma'} (b_j(y_j) - \lambda_j) K_\nu(x, \vec{y}) F(\vec{y})\, d\vec{y} \right|^2 \frac{d\nu}{\nu} \right)^{1/2}
$$

$$
= \prod_{j=1}^\ell |b_j(x) - \lambda_j| \, T(\vec{f})(x) + T\left(\prod_{j=1}^\ell (b_j(\cdot_j) - \lambda_j) \vec{f} \right)(x)
$$

$$
+ \sum_{i=1}^{\ell-1} \sum_{\sigma \in C_i^\ell} \prod_{j \in \sigma} |b_j(x) - \lambda_j|
$$

$$
\times \left(\int_0^\infty \left| \int_{(\mathbb{R}^n)^m} \prod_{j \in \sigma'} (b_j(y_j) - \lambda_j) K_\nu(x, \vec{y}) F(\vec{y})\, d\vec{y} \right|^2 \frac{d\nu}{\nu} \right)^{1/2}.
$$

Noting that $b_j(y_j) - \lambda_j = (b_j(y_j) - b_j(x)) + (b_j(x) - \lambda_j)$, we get

$$
T_{\Pi \vec{b}, q}(\vec{f})(z) = \prod_{j=1}^\ell |b_j(z) - \lambda_j| \, T_q(\vec{f})(z) + T_q\left(\prod_{j=1}^\ell (b_j(\cdot_j) - \lambda_j) \vec{f} \right)(z)
$$

$$
+ \sum_{i=1}^{\ell-1} \sum_{\sigma \in C_i^\ell} C_{i,\ell} \prod_{j \in \sigma} |b_j(z) - \lambda_j| \, T_{\Pi \vec{b}_{\sigma'}, q} \vec{f}(z).
$$

Here $C_{i,\ell}$ depends only on i and ℓ.

Let $c_0 = \|c\|_{\ell^q} = (\sum_{k=1}^{\infty} |c_k|^q)^{1/q}$. Since $0 < \delta < 1/m < 1$, we have

$$\left(\frac{1}{|Q|} \int_Q \left| |T_{\Pi \vec{b}, q} \vec{f}(z)|^\delta - |c_0|^\delta \right| dz \right)^{1/\delta}$$

$$\leq C \left(\frac{1}{|Q|} \int_Q \left\| T_{\Pi \vec{b}}(\vec{f})(z) - c \right\|_{\ell^q}^\delta dz \right)^{1/\delta}$$

$$\leq C \left(\frac{1}{|Q|} \int_Q \left\| \prod_{j=1}^{\ell} |b_j(x) - \lambda_j| T(\vec{f})(z) \right\|_{\ell^q}^\delta dz \right)^{1/\delta}$$

$$+ C \sum_{i=1}^{\ell-1} \sum_{\sigma \in C_i^\ell} C_{i,\ell} \left(\frac{1}{|Q|} \int_Q \prod_{j \in \sigma} (|b_j(z) - \lambda_j| T_{\Pi \vec{b}_{\sigma'}, q} \vec{f}(z))^\delta dz \right)^{1/\delta}$$

$$+ C \left(\frac{1}{|Q|} \int_Q \left\| T \left(\prod_{j=1}^{\ell} (b_j(\cdot_j) - \lambda_j) \vec{f} \right)(z) - c \right\|_{\ell^q}^\delta dz \right)^{1/\delta}$$

$$:= I + II + III.$$

Now exploiting the standard Hölder inequality for finitely many functions with $1 < p < \epsilon/\delta$, it follows that

$$I \leq C \prod_{j=1}^{\ell} \|b_j\|_{BMO} M_\epsilon(T_q \vec{f})(x),$$

$$II \leq C \sum_{i=1}^{\ell-1} \sum_{\sigma \in C_i^\ell} C_{j,\ell} \prod_{j \in \sigma} \|b_j\|_{BMO} M_\epsilon(T_{\Pi \vec{b}_\sigma}, \vec{f})(x).$$

Next let us address part *III*. Set $\vec{f}_j = \vec{f}_j^0 + \vec{f}_j^\infty$, where $\vec{f}_j^0 = \vec{f}_j \chi_{Q^*}$. Let $\vec{f}^\alpha = f_1^{\alpha_1} \cdots f_m^{\alpha_m}$ and $Q^* = (8\sqrt{n} + 4)Q$. Taking $c_0 = \sum_{\alpha_1, \ldots, \alpha_m} \|T((b_1(\cdot_1) - \lambda_1) \cdots (b_\ell(\cdot_\ell) - \lambda_\ell)) f_1^{\alpha_1} \cdots f_m^{\alpha_m}(x)\|_{\ell^q}$, we have

$$\left\| T \left(\prod_{j=1}^{\ell} (b_j(\cdot_j) - \lambda_j) \vec{f} \right)(z) - c \right\|_{\ell^q}$$

$$\leq \left| T_q \left(\prod_{j=1}^{\ell} (b_j(\cdot_j) - \lambda_j) \vec{f}^0 \right)(z) \right|$$

$$+ C \sum_{\alpha_1, \ldots, \alpha_m} \left\| T \left(\prod_{j=1}^{\ell} (b_j(\cdot_j) - \lambda_j) \vec{f}^\alpha \right)(z) - T \left(\prod_{j=1}^{\ell} (b_j(\cdot_j) - \lambda_j) \vec{f}^\alpha \right)(x) \right\|_{\ell^q},$$

where in the last sum each $\alpha_j = 0$ or ∞ and in each term there is at least one $\alpha_j = \infty$.

If $\alpha_1 = \cdots = \alpha_m = 0$, Kolmogorov's inequality and Theorem 6 imply

$$\left(\frac{1}{|Q|} \left| T_q \left(\prod_{j=1}^{\ell} (b_j(\cdot_j) - \lambda_j) \vec{f}^0 \right)(z) \right|^\delta \right)^{1/\delta}$$

$$\leq C \left\| T_q \left(\prod_{j=1}^{\ell} (b_j(\cdot_j) - \lambda_j) \vec{f}^0 \right) \right\|_{L^{1/m, \infty}(Q, \frac{dz}{|Q|})}$$

$$\leq C \prod_{j=1}^{\ell} \|b_j\|_{BMO} \big\| \|f_j\|_{\ell^{q_j}} \big\|_{L(\log L),Q} \prod_{j=\ell+1}^{m} \frac{1}{|Q|} \int_Q \|f_j\|_{\ell^{q_j}} dz$$

$$\leq C \prod_{j=1}^{\ell} \|b_j\|_{BMO} \mathcal{M}_{L(\log L)} \big(\|f_j\|_{\ell^{q_j}} \big)(x).$$

If $\alpha_1 = \cdots = \alpha_m = \infty$, applying Hölder's inequality and Minkowski's inequality, then we get

$$\left(\frac{1}{|Q|} \int_Q \left\| T\left(\prod_{j=1}^{\ell} (b_j(\cdot_j) - \lambda_j) \vec{f}^{\alpha} \right)(z) - T\left(\prod_{j=1}^{\ell} (b_j(y_j) - \lambda_j) \vec{f}^{\alpha} \right)(x) \right\|_{\ell^q}^{\delta} \right)^{1/\delta}$$

$$\leq \frac{C}{|Q|} \int_Q \left(\sum_{k=1}^{\infty} \left(\int_0^{\infty} \left| \int_{(\mathbb{R}^n \setminus Q^*)^m} |K_\nu(z,\vec{y}) - K_\nu(x,\vec{y})| \right. \right. \right.$$

$$\left. \left. \left. \times \prod_{j=1}^{\ell} (b_j(y_j) - \lambda_j) f_{1k}(y_1) \cdots f_{mk}(y_m) \, d\vec{y} \right|^2 \frac{d\nu}{\nu} \right)^{q/2} \right)^{1/q} dz$$

$$\leq \frac{C}{|Q|} \int_Q \left(\sum_{k=1}^{\infty} \left| \int_{(\mathbb{R}^n \setminus Q^*)^m} \left(\int_0^{\infty} |K_\nu(z,\vec{y}) - K_\nu(x,\vec{y})|^2 \frac{d\nu}{\nu} \right)^{1/2} \right. \right.$$

$$\left. \left. \times \prod_{j=1}^{\ell} (b_j(y_j) - \lambda_j) f_{1k}(y_1) \cdots f_{mk}(y_m) \, d\vec{y} \right|^q \right)^{1/q} dz$$

$$\leq \frac{C}{|Q|} \int_Q \left(\sum_{k=1}^{\infty} \left| \int_{(\mathbb{R}^n \setminus Q^*)^m} \left(\int_0^{\infty} |K_\nu(z,\vec{y}) - K_{\nu,t}^0(z,\vec{y})|^2 \frac{d\nu}{\nu} \right)^{1/2} \right. \right.$$

$$\left. \left. \times \prod_{j=1}^{\ell} (b_j(y_j) - \lambda_j) f_{1k}(y_1) \cdots f_{mk}(y_m) \, d\vec{y} \right|^q \right)^{1/q} dz$$

$$+ \frac{C}{|Q|} \int_Q \left(\sum_{k=1}^{\infty} \left| \int_{(\mathbb{R}^n \setminus Q^*)^m} \left(\int_0^{\infty} |K_{\nu,t}^0(z,\vec{y}) - K_{\nu,t}^0(x,\vec{y})|^2 \frac{d\nu}{\nu} \right)^{1/2} \right. \right.$$

$$\left. \left. \times \prod_{j=1}^{\ell} (b_j(y_j) - \lambda_j) f_{1k}(y_1) \cdots f_{mk}(y_m) \, d\vec{y} \right|^q \right)^{1/q} dz$$

$$+ \frac{C}{|Q|} \int_Q \left(\sum_{k=1}^{\infty} \left| \int_{(\mathbb{R}^n \setminus Q^*)^m} \left(\int_0^{\infty} |K_{\nu,t}^0(x,\vec{y}) - K_\nu(x,\vec{y})|^2 \frac{d\nu}{\nu} \right)^{1/2} \right. \right.$$

$$\left. \left. \times \prod_{j=1}^{\ell} (b_j(y_j) - \lambda_j) f_{1k}(y_1) \cdots f_{mk}(y_m) \, d\vec{y} \right|^q \right)^{1/q} dz$$

$$:= III_1 + III_2 + III_3.$$

First we consider III_1. Taking $t = [2\sqrt{n}\ell(Q)]^s$, by Assumption (H2) we have

$$III_1 \leq \frac{C}{|Q|} \int_Q \int_{(\mathbb{R}^n \setminus Q^*)^m} \frac{At^{\epsilon/s}}{(\sum_{j=1}^{m} |z - y_j|)^{mn+\epsilon}} \prod_{j=1}^{\ell} |b_j(y_j) - \lambda_j| \|f_1\|_{\ell^{q_1}} \cdots \|f_m\|_{\ell^{q_m}} \, d\vec{y} \, dz$$

$$\leq C \sum_{k=1}^{\infty} \frac{1}{2^{k\epsilon}} \prod_{j=1}^{\ell} \frac{1}{2^{(k+1)n}|Q^*|} \int_{2^{k+1}Q^*} |b_j(y_j) - \lambda_j| \|f_j\|_{\ell^{q_j}} \, dy_j$$

$$\times \prod_{j=\ell+1}^{m} \frac{1}{2^{(k+1)n}|Q^*|} \int_{2^{k+1}Q^*} \|f_j\|_{\ell^{q_j}} \, dy_j$$

$$\leq C \sum_{k=1}^{\infty} \frac{1}{2^{k\epsilon}} \prod_{j=1}^{\ell} \|b_j\|_{BMO} \big\| \|f_j\|_{\ell^{q_j}} \big\|_{L(\log L), 2^{k+1}Q^*}$$

$$\times \prod_{j=\ell+1}^{m} \frac{1}{2^{(k+1)n}|Q^*|} \int_{2^{k+1}Q^*} \|f_j\|_{\ell^{q_j}} \, dy_j$$

$$\leq C \mathcal{M}_{L(\log L)}\big(\|\vec{f}\|_{\ell^q}\big)(x).$$

Similarly, we have $III_2 \leq C\mathcal{M}_{L(\log L)}(\|\vec{f}\|_{\ell^q})(x)$ and $III_3 \leq C\mathcal{M}_{L(\log L)}(\|\vec{f}\|_{\ell^q})(x)$. Now it remains to consider the typical case of III,

$$\left(\frac{1}{|Q|} \int_Q \left\| T\left(\prod_{j=1}^{\ell} (b_j(\cdot_j) - \lambda_j) f_1^{\infty}, \dots, f_{\ell}^{\infty}, f_{\ell+1}^0, \dots, f_m^0 \right)(z) \right. \right.$$

$$\left. \left. - T\left(\prod_{j=1}^{\ell} (b_j(y_j) - \lambda_j) f_1^{\infty}, \dots, f_{\ell}^{\infty}, f_{\ell+1}^0, \dots, f_m^0 \right)(x) \right\|_{\ell^q}^{\delta} \right)^{1/\delta}$$

$$\leq \frac{C}{|Q|} \int_Q \left(\sum_{k=1}^{\infty} \left(\int_0^{\infty} \left| \int_{(\mathbb{R}^n \setminus Q^*)^m} |K_\nu(z, \vec{y}) - K_\nu(x, \vec{y})| \prod_{j=1}^{\ell} (b_j(y_j) - \lambda_j) \right. \right. \right.$$

$$\left. \left. \left. \times f_{1k}^{\infty}(y_1) \cdots f_{\ell k}^{\infty}(y_\ell) f_{\ell+1,k}^0(y_{\ell+1}) \cdots f_{mk}^0(y_m) \, d\vec{y} \right|^2 \frac{d\nu}{\nu} \right)^{q/2} \right)^{1/q} dz$$

$$\leq \left(\int_{(\mathbb{R}^n \setminus Q^*)^\ell} \frac{t^{\epsilon/s} \prod_{j=1}^{\ell} (b_j(y_j) - \lambda_j) \|f_j(y_j)\|_{\ell^{q_j}} \, dy_j}{(\sum_{j=1}^{\ell} |z - y_j|)^{mn+\epsilon}} \right.$$

$$\left. + \int_{(\mathbb{R}^n \setminus Q^*)^\ell} \frac{\prod_{j=1}^{\ell} (b_j(y_j) - \lambda_j) \|f_j(y_j)\|_{\ell^{q_j}} \, dy_j}{(\sum_{j=1}^{\ell} |z - y_j|)^{mn}} \right) \prod_{j=\ell+1}^{m} \int_{Q^*} \|f(y_j)\|_{\ell^{q_j}} \, dy_j$$

$$\leq C \prod_{j=1}^{\ell} \|b_j\|_{BMO} \mathcal{M}_{L(\log L)}\big(\|\vec{f}\|_{\ell^q}\big)(x). \qquad \square$$

Lemma 16 *Let $0 < p < \infty$, $1/m < q < \infty$, and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$ with $1 < q_1, \dots, q_m < \infty$ and let $\omega \in A_\infty$. Suppose that $\vec{b} \in (BMO)^\ell$. Then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^n} |T_{\Pi \vec{b}, q} \vec{f}|^p \omega(x) \, dx \leq C \prod_{j=1}^{\ell} \|b_j\|_{BMO}^p \int_{\mathbb{R}^n} \big(\mathcal{M}_{L(\log L)}\big(\|\vec{f}\|_{\ell^q}\big)(x) \big)^p \omega(x) \, dx. \tag{6.8}$$

The proof is similar to [16], so we omit it here.

Based on the above lemmas, the proof of Theorem 1.3 in [16] provides the main ideas for the proof of Theorem 7.

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