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Optimal lower and upper bounds for the geometric convex combination of the error function

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Abstract

For $x \in R$, the error function $\operatorname{erf}(x)$ is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

In this paper, we answer the question: what are the greatest value p and the least value q, such that the double inequality $\operatorname{erf}(M_p(x,y;\lambda)) \le G(\operatorname{erf}(x), \operatorname{erf}(y); \lambda) \le \operatorname{erf}(M_q(x,y;\lambda))$ holds for all $x, y \ge 1$ (or $0 < x, y < 1$) and $\lambda \in (0,1)$? Here, $M_r(x,y;\lambda) = (\lambda x^r + (1-\lambda)y^r)^{1/r}$ ($r \neq 0$), $M_0(x,y;\lambda) = x^{\lambda}y^{1-\lambda}$ and $G(x,y;\lambda) = x^{\lambda}y^{1-\lambda}$ are the weighted power and the weighted geometric mean, respectively.

MSC: Primary 33B20; secondary 26D15

Keywords: error function; power mean; functional inequalities

1 Introduction

For $x \in R$, the error function $\operatorname{erf}(x)$ is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

The most important properties of this function are collected, for example, in [1, 2]. In the recent past, the error function has been a topic of recurring interest, and a great number of results on this subject have been reported in the literature [3–16]. It might be surprising that the error function has application in the field of heat conduction besides probability [17, 18].

In 1933, Aumann [19] introduced a generalized notion of convexity, the so-called MN-convexity, when M and N are mean values. A function $f : [0,\infty) \to [0,\infty)$ is MN-convex if $f(M(x,y)) \le N(f(x), f(y))$ for $x, y \in [0,\infty)$. The usual convexity is the special case when M and N both are arithmetic means. Furthermore, the applications of MN-convexity reveal a new world of beautiful inequalities which involve a broad range of functions from the elementary ones, such as sine and cosine function, to the special ones, such as the Γ

function, the Gaussian hypergeometric function, and the Bessel function. For the details as regards MN-convexity and its applications the reader is referred to [20–25].

Let $\lambda \in (0,1)$, we define $A(x,y;\lambda) = \lambda x + (1-\lambda)y$, $G(x,y;\lambda) = x^{\lambda}y^{1-\lambda}$, $H(x,y;\lambda) = \frac{xy}{\lambda y+(1-\lambda)x}$ and $M_r(x,y;\lambda) = (\lambda x^r + (1-\lambda)y^r)^{1/r}$ $(r \neq 0)$, $M_0(x,y;\lambda) = x^{\lambda}y^{1-\lambda}$. These are commonly known as weighted arithmetic mean, weighted geometric mean, weighted harmonic mean, and weighted power mean of two positive numbers x and y, respectively. Then it is well known that the inequalities

$$H(x,y;\lambda) = M_{-1}(x,y;\lambda) < G(x,y;\lambda) = M_0(x,y;\lambda) < A(x,y;\lambda) = M_1(x,y;\lambda)$$

hold for all $\lambda \in (0,1)$ and $x,y > 0$ with $x \neq y$.

By elementary computations, one has

$$\lim_{r\to-\infty} M_r(x,y;\lambda) = \min(x,y) \tag{1.1}$$

and

$$\lim_{r\to+\infty} M_r(x,y;\lambda) = \max(x,y).$$

In [26], Alzer proved that $c_1(\lambda) = \frac{\lambda+(1-\lambda)\operatorname{erf}(1)}{\operatorname{erf}(1/(1-\lambda))}$ and $c_2(\lambda) = 1$ are the best possible factors such that the double inequality

$$c_1(\lambda)\operatorname{erf}\bigl(H(x,y;\lambda)\bigr) \le A\bigl(\operatorname{erf}(x),\operatorname{erf}(y);\lambda\bigr) \le c_2(\lambda)\operatorname{erf}\bigl(H(x,y;\lambda)\bigr) \tag{1.2}$$

holds for all $x,y \in [1,+\infty)$ and $\lambda \in (0,1/2)$.

Inspired by (1.2), it is natural to ask: does the inequality $\operatorname{erf}(M(x,y)) \le N(\operatorname{erf}(x),\operatorname{erf}(y))$ hold for other means M, N, such as geometric, harmonic or power means?

In [27, 28], the authors found the greatest values α_1, α_2 and the least values β_1, β_2, such that the double inequalities

$$\operatorname{erf}\bigl(M_{\alpha_1}(x,y;\lambda)\bigr) \le A\bigl(\operatorname{erf}(x),\operatorname{erf}(y);\lambda\bigr) \le \operatorname{erf}\bigl(M_{\beta_1}(x,y;\lambda)\bigr)$$

and

$$\operatorname{erf}\bigl(M_{\alpha_2}(x,y;\lambda)\bigr) \le H\bigl(\operatorname{erf}(x),\operatorname{erf}(y);\lambda\bigr) \le \operatorname{erf}\bigl(M_{\beta_2}(x,y;\lambda)\bigr)$$

hold for all $x,y \ge 1$ (or $0 < x,y < 1$) and $\lambda \in (0,1)$.

In the following we answer the question: what are the greatest value p and the least value q, such that the double inequality

$$\operatorname{erf}\bigl(M_p(x,y;\lambda)\bigr) \le G\bigl(\operatorname{erf}(x),\operatorname{erf}(y);\lambda\bigr) \le \operatorname{erf}\bigl(M_q(x,y;\lambda)\bigr)$$

holds for all $x,y \ge 1$ (or $0 < x,y < 1$) and $\lambda \in (0,1)$?

2 Lemmas

In this section we present two lemmas, which will be used in the proof of our main results.

Lemma 2.1 *Let* $r \neq 0$, $r_0 = -1 - \frac{2}{e\sqrt{\pi}\,\mathrm{erf}(1)} = -1.4926\ldots$, *and* $u(x) = \log \mathrm{erf}(x^{1/r})$. *Then the following statements are true*:

(1) *if* $r < r_0$, *then* $u(x)$ *is strictly convex on* $[1, +\infty)$;

(2) *if* $r_0 \leq r < 0$, *then* $u(x)$ *is strictly concave on* $(0, 1]$;

(3) *if* $r > 0$, *then* $u(x)$ *is strictly concave on* $(0, +\infty)$.

Proof Simple computations lead to

$$u'(x) = \frac{2e^{-x^{2/r}}x^{1/r-1}}{r\sqrt{\pi}\,\mathrm{erf}(x^{1/r})} \tag{2.1}$$

and

$$u''(x) = \frac{2e^{-x^{2/r}}x^{1/r-2}}{r^2\sqrt{\pi}\,\mathrm{erf}^2(x^{1/r})}g(x), \tag{2.2}$$

where

$$g(x) = \left(-2x^{2/r} + 1 - r\right)\mathrm{erf}\left(x^{1/r}\right) - \frac{2}{\sqrt{\pi}}e^{-x^{2/r}}x^{1/r}. \tag{2.3}$$

Then

$$g'(x) = 4x^{2/r-1}g_1(x), \tag{2.4}$$

$$g_1(x) = -\frac{1}{r}\,\mathrm{erf}\left(x^{1/r}\right) - \frac{1}{2\sqrt{\pi}}e^{-x^{2/r}}x^{-1/r}, \tag{2.5}$$

and

$$g_1'(x) = \frac{1}{2r^2\sqrt{\pi}}e^{-x^{2/r}}x^{-1/r-1}\left[(2r-4)x^{2/r} + r\right]. \tag{2.6}$$

We divide the proof into two cases.

Case 1. If $r < 0$, then (2.6), (2.5), and (2.3) lead to

$$g_1'(x) < 0, \tag{2.7}$$

$$\lim_{x\to 0^+} g_1(x) > 0, \qquad \lim_{x\to +\infty} g_1(x) = -\infty, \tag{2.8}$$

$$\lim_{x\to 0^+} g(x) = -\infty, \qquad \lim_{x\to +\infty} g(x) = 0, \tag{2.9}$$

and

$$g(1) = (-1 - r)\,\mathrm{erf}(1) - \frac{2}{e\sqrt{\pi}}. \tag{2.10}$$

Inequality (2.7) implies that $g_1(x)$ is strictly decreasing on $[0, +\infty)$.

It follows from the monotonicity of $g_1(x)$ and (2.8) that there exists $x_1 \in (0, +\infty)$, such that $g(x)$ is strictly increasing on $[0, x_1]$ and strictly decreasing on $[x_1, +\infty)$.

From the piecewise monotonicity of $g(x)$ and (2.9) we clearly see that there exists $x_2 \in (0, +\infty)$, such that $g(x) < 0$ for $x \in (0, x_2)$ and $g(x) > 0$ for $x \in (x_2, +\infty)$.

Case 1.1. If $r < r_0$, then from (2.10) we know that $g(1) > 0$. This leads to $g(x) > 0$ for $x \in [1, +\infty)$. Therefore (2.2) leads to the conclusion that $u(x)$ is strictly convex on $[1, +\infty)$.

Case 1.2. If $r_0 \leq r < 0$, then (2.10) implies that $g(1) \leq 0$. This leads to $g(x) \leq 0$ for $x \in (0, 1]$. Therefore (2.2) leads to the conclusion that $u(x)$ is strictly concave on $(0, 1]$.

Case 2. If $r > 0$, then (2.5) and (2.3) imply that

$$g_1(x) < 0 \tag{2.11}$$

and

$$\lim_{x\to 0^+} g(x) = 0 \tag{2.12}$$

for $x \in (0, +\infty)$.

It follows from (2.11), (2.4), and (2.12) that $g(x) < 0$. Therefore (2.2) leads to the conclusion that $u(x)$ is strictly concave on $(0, +\infty)$. □

Lemma 2.2 *The function* $h(x) = 2x^2 + \frac{xe^{-x^2}}{\int_0^x e^{-t^2}\,dt}$ *is strictly increasing on* $(0, +\infty)$.

Proof Simple computations lead to

$$h'(x) = \frac{h_1(x)}{(\int_0^x e^{-t^2}\,dt)^2}, \tag{2.13}$$

where

$$h_1(x) = 4x\left(\int_0^x e^{-t^2}\,dt\right)^2 + \left(1 - 2x^2\right)e^{-x^2}\int_0^x e^{-t^2}\,dt - xe^{-2x^2},$$

$$\lim_{x\to 0^+} h_1(x) = 0, \tag{2.14}$$

and

$$h_1'(x) = 4\left(\int_0^x e^{-t^2}\,dt\right)^2 + \left(4x^3 + 2x\right)e^{-x^2}\int_0^x e^{-t^2}\,dt + 2x^2e^{-2x^2} > 0 \tag{2.15}$$

for $x \in (0, +\infty)$.

Hence, $h(x)$ is strictly increasing on $(0, +\infty)$, as follows from (2.15), (2.14), and (2.13). □

3 Main results

Theorem 3.1 *Let* $\lambda \in (0, 1)$ *and* $r_0 = -1 - \frac{2}{e\sqrt{\pi}\,\mathrm{erf}(1)} = -1.4926\ldots$. *Then the double inequality*

$$\mathrm{erf}\big(M_p(x, y; \lambda)\big) \leq G\big(\mathrm{erf}(x), \mathrm{erf}(y); \lambda\big) \leq \mathrm{erf}\big(M_q(x, y; \lambda)\big) \tag{3.1}$$

holds for all $x, y \geq 1$ *if and only if* $p = -\infty$ *and* $q \geq r_0$.

Proof First of all, we prove that inequality (3.1) holds if $p = -\infty$ and $q \geq r_0$. It follows from (1.1) that the first inequality in (3.1) is true if $p = -\infty$. Since the weighted power mean

$M_t(x,y;\lambda)$ is strictly increasing with respect to t on R, thus we only need to prove that the second inequality in (3.1) is true if $r_0 \le q < 0$.

If $r_0 \le q < 0$, $u(z) = \log \operatorname{erf}(z^{1/q})$, then Lemma 2.1(2) leads to

$$\lambda u(s) + (1-\lambda)u(t) \le u\big(\lambda s + (1-\lambda)t\big) \tag{3.2}$$

for $\lambda \in (0,1)$ and $s,t \in (0,1]$.

Let $s = x^q$, $t = y^q$, and $x,y \ge 1$. Then (3.2) leads to the second inequality in (3.1).

Second, we prove that the second inequality in (3.1) implies $q \ge r_0$.

Let $x \ge 1$ and $y \ge 1$. Then the second inequality in (3.1) leads to

$$D(x,y) =: \operatorname{erf}\big(M_q(x,y;\lambda)\big) - G\big(\operatorname{erf}(x),\operatorname{erf}(y);\lambda\big) \ge 0. \tag{3.3}$$

It follows from (3.3) that

$$D(y,y) = \frac{\partial}{\partial x}D(x,y)|_{x=y} = 0$$

and

$$\frac{\partial^2}{\partial x^2}D(x,y)|_{x=y} = \frac{\lambda(1-\lambda)y}{\operatorname{erf}'(y)}\left[q - 1 + \left(2y^2 + \frac{ye^{-y^2}}{\int_0^y e^{-t^2}\,dt}\right)\right]. \tag{3.4}$$

Therefore,

$$q \ge \lim_{y\to 1^+}\left(1 - 2y^2 - \frac{ye^{-y^2}}{\int_0^y e^{-t^2}\,dt}\right) = r_0$$

follows from (3.3) and (3.4) together with Lemma 2.2.

Finally, we prove that the first inequality in (3.1) implies $p = -\infty$. We distinguish two cases.

Case I. $p \ge 0$. Then for any fixed $y \in [1,+\infty)$ we have

$$\lim_{x\to+\infty} \operatorname{erf}\big(M_p(x,y;\lambda)\big) = 1$$

and

$$\lim_{x\to+\infty} G\big(\operatorname{erf}(x),\operatorname{erf}(y);\lambda\big) = \operatorname{erf}^{1-\lambda}(y) < 1,$$

which contradicts the first inequality in (3.1).

Case II. $-\infty < p < 0$. Let $x \ge 1$, $\alpha = \lambda^{1/p}$ and $y \to +\infty$. Then the first inequality in (3.1) leads to

$$E(x) =: \operatorname{erf}^{\lambda}(x) - \operatorname{erf}(\alpha x) \ge 0. \tag{3.5}$$

It follows from (3.5) that

$$\lim_{x\to+\infty} E(x) = 0 \tag{3.6}$$

and

$$E'(x) = \frac{2\lambda}{\sqrt{\pi}} e^{-x^2} \left[\operatorname{erf}^{\lambda-1}(x) - \frac{\alpha}{\lambda} e^{(1-\alpha^2)x^2} \right]. \tag{3.7}$$

Note that $\alpha > 1$, then

$$\lim_{x \to +\infty} \left[\operatorname{erf}^{\lambda-1}(x) - \frac{\alpha}{\lambda} e^{(1-\alpha^2)x^2} \right] = 1. \tag{3.8}$$

It follows from (3.7) and (3.8) that there exists a sufficiently large $\eta_1 \in [1, +\infty)$, such that $E'(x) > 0$ for $x \in (\eta_1, +\infty)$. Hence $E(x)$ is strictly increasing on $[\eta_1, +\infty)$.

From the monotonicity of $E(x)$ on $[\eta_1, +\infty)$ and (3.6) we conclude that there exists $\eta_2 \in [1, +\infty)$, such that $E(x) < 0$ for $x \in (\eta_2, +\infty)$, this contradicts (3.5). □

Theorem 3.2 *Let* $\lambda \in (0,1)$, *then the double inequality*

$$\operatorname{erf}\bigl(M_\mu(x,y;\lambda)\bigr) \le G\bigl(\operatorname{erf}(x), \operatorname{erf}(y); \lambda\bigr) \le \operatorname{erf}\bigl(M_\nu(x,y;\lambda)\bigr) \tag{3.9}$$

holds for all $0 < x, y < 1$ *if and only if* $\mu \le r_0$ *and* $\nu \ge 0$.

Proof First of all, we prove that (3.9) holds if $\mu \le r_0$ and $\nu \ge 0$.

If $\mu \le r_0$, $u(z) = \log \operatorname{erf}(z^{1/\mu})$, then Lemma 2.1(1) leads to

$$u\bigl(\lambda s + (1-\lambda)t\bigr) \le \lambda u(s) + (1-\lambda)u(t) \tag{3.10}$$

for $\lambda \in (0,1)$, $s, t > 1$.

Let $s = x^\mu$, $t = y^\mu$, and $0 < x, y < 1$. Then (3.10) leads to the first inequality in (3.9).

If $\nu \ge 0$, $u(z) = \log \operatorname{erf}(z^{1/\nu})$, then Lemma 2.1(3) leads to

$$\lambda u(s) + (1-\lambda)u(t) \le u\bigl(\lambda s + (1-\lambda)t\bigr) \tag{3.11}$$

for $\lambda \in (0,1)$, $0 < s, t < 1$.

Therefore, the second inequality in (3.9) follows from $s = x^\nu$, $t = y^\nu$, and $0 < x, y < 1$ together with (3.11).

Second, we prove that the second inequality in (3.9) implies $\nu \ge 0$.

Let $0 < x, y < 1$. Then the second inequality in (3.9) leads to

$$J(x,y) =: \operatorname{erf}\bigl(M_\nu(x,y;\lambda)\bigr) - G\bigl(\operatorname{erf}(x), \operatorname{erf}(y); \lambda\bigr) \ge 0. \tag{3.12}$$

It follows from (3.12) that

$$J(y,y) = \frac{\partial}{\partial x} J(x,y)|_{x=y} = 0$$

and

$$\frac{\partial^2}{\partial x^2} J(x,y)|_{x=y} = \frac{\lambda(1-\lambda)y}{\operatorname{erf}'(y)} \left[\nu - 1 + \left(2y^2 + \frac{y e^{-y^2}}{\int_0^y e^{-t^2}\, dt} \right) \right]. \tag{3.13}$$

Hence, from (3.12) and (3.13) together with Lemma 2.2 we know that

$$\nu \geq \lim_{y\to 0^+}\left[1-\left(2y^2+\frac{ye^{-y^2}}{\int_0^y e^{-t^2}\,dt}\right)\right]=0.$$

Finally, we prove that the first inequality in (3.9) implies $\mu \leq r_0$.

Let $y \to 1$. Then the first inequality in (3.9) leads to

$$L(x) =: G\bigl(\operatorname{erf}(x), \operatorname{erf}(1); \lambda\bigr) - \operatorname{erf}\bigl(M_\mu(x, 1; \lambda)\bigr) \geq 0 \tag{3.14}$$

for $0 < x < 1$.

It follows from (3.14) that

$$L(1) = 0 \tag{3.15}$$

and

$$L'(x) = \frac{2\lambda e^{-x^2}}{\sqrt{\pi}}\left[\operatorname{erf}^{1-\lambda}(1)\operatorname{erf}^{\lambda-1}(x) - x^{\mu-1}\bigl(\lambda x^\mu + 1 - \lambda\bigr)^{1/\mu-1} e^{x^2-(\lambda x^\mu+1-\lambda)^{2/\mu}}\right]. \tag{3.16}$$

Let

$$L_1(x) = \log\bigl[\operatorname{erf}^{1-\lambda}(1)\operatorname{erf}^{\lambda-1}(x)\bigr] - \log\bigl[x^{\mu-1}\bigl(\lambda x^\mu + 1 - \lambda\bigr)^{1/\mu-1} e^{x^2-(\lambda x^\mu+1-\lambda)^{2/\mu}}\bigr]. \tag{3.17}$$

Then

$$\lim_{x\to 1^-} L_1(x) = 0, \tag{3.18}$$

$$L_1'(x) = (\lambda - 1)\frac{\operatorname{erf}'(x)}{\operatorname{erf}(x)} - \frac{(\mu-1)(1-\lambda)}{x(\lambda x^\mu + 1 - \lambda)} - 2x + 2\lambda x^{\mu-1}\bigl(\lambda x^\mu + 1 - \lambda\bigr)^{2/\mu-1},$$

and

$$\lim_{x\to 1^-} L_1'(x) = (1-\lambda)\left[-\mu - 1 - \frac{2}{e\sqrt{\pi}\operatorname{erf}(1)}\right]. \tag{3.19}$$

If $\mu > r_0$, then from (3.19) we clearly see that there exists a small $\delta_1 > 0$, such that $L_1'(x) < 0$ for $x \in (1-\delta_1, 1)$. Therefore, $L_1(x)$ is strictly decreasing on $[1-\delta_1, 1]$.

The monotonicity of $L_1(x)$ on $[1-\delta_1, 1]$ and (3.18) imply that there exists $\delta_2 > 0$, such that $L_1(x) > 0$ for $x \in (1-\delta_2, 1)$.

Hence, (3.16) and (3.17) lead to $L(x)$ being strictly increasing on $[1-\delta_2, 1]$. It follows from the monotonicity of $L(x)$ and (3.15) that there exists $\delta_3 > 0$, such that $L(x) < 0$ for $x \in (1-\delta_3, 1)$, this contradicts (3.14). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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