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Q-Curvature problem on S^n under flatness condition: the case $\beta = n$

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Abstract

In the present work, we consider the prescribed Q-curvature problem on the unit sphere S^n, $n \geq 5$. Under the hypothesis that the prescribed function satisfies a flatness condition of order $\beta = n$, we give a complete description of the lack of compactness of the problem and we provide an existence result in terms of an Euler-Hopf index.

Keywords: Q-curvature; critical exponent; lack of compactness; critical points at infinity

1 Introduction and the main result

On a smooth compact manifold (M^n, g_0) of dimension $n \geq 5$, the Paneitz operator is defined by

$$P^n_{g_0} u = \Delta^2_{g_0} u - \operatorname{div}_{g_0}(a_n S_{g_0} g_0 + b_n \operatorname{Ric}_{g_0})\, du + \frac{n-4}{2} Q^n_{g_0} u,$$

where S_{g_0} denotes the scalar curvature of (M^n, g_0), Ric_{g_0} denotes the Ricci curvature of (M^n, g_0) and

$$a_n = \frac{(n-2)^2 + 4}{2(n-1)(n-2)}, \qquad b_n = -\frac{4}{n-2},$$

$$Q^n_{g_0} = -\frac{1}{2(n-1)} \Delta_{g_0} S_{g_0} + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} S^2_{g_0} - \frac{2}{(n-2)^2} |\operatorname{Ric}_{g_0}|^2.$$

Such a $Q^n_{g_0}$ is a fourth order invariant called the Q-curvature.

The operator $P^n_{g_0}$, is conformally invariant; if $g = u^{\frac{4}{n-4}} g_0$, $u > 0$, is a conformal metric to g_0, then for all $\psi \in C^\infty(M)$ we have

$$P^n_{g_0}(u\psi) = u^{\frac{n+4}{n-4}} P^n_g(\psi).$$

In particular, taking $\psi \equiv 1$, we then have

$$P^n_{g_0}(u) = \frac{n-4}{2} Q^n_g u^{\frac{n+4}{n-4}}. \tag{1.1}$$

The present paper deals with the prescribed Q-curvature problem on the standard sphere (S^n, g_0), $n \geq 5$. According to equation (1.1), this problem can be expressed as follows: Let K :

$S^n \to \mathbb{R}$ be a given smooth function, we look for solutions u on S^n satisfying the nonlinear problem involving the critical exponent

$$\begin{cases} P^n_{g_0} u = \frac{n-4}{2} K u^{\frac{n+4}{n-4}}, \\ u > 0 \quad \text{on } S^n. \end{cases} \tag{1.2}$$

On the Sobolev space $H^2(S^n)$, the operator $P^n_{g_0}$ is coercive and has the following expression:

$$\mathcal{P} := P^n_{g_0} u = \Delta^2_{g_0} u - c_n \Delta_{g_0} u + d_n u,$$

where $c_n = \frac{1}{2}(n^2 - 2n - 4)$ and $d_n = \frac{n-4}{16} n(n^2 - 4)$.

This problem is quite delicate and had drawn a lot of attention from mathematicians because the equation stands for a critical case which generates blow-up and lack of compactness, that the standard analytic machinery cannot apply. Moreover, beside the obvious necessary condition that K be positive somewhere, there are topological obstructions of Kazdan-Warner type to solve (1.2) (see [1]). The determination of the set of all functions K such that problem (1.2) has a solution is still open, although intensive studies were dedicated to this problem trying to understand under what conditions (1.2) is solvable; see [2–8] and the references therein.

One group of existence results for problem (1.2) has been obtained under the following β-flatness condition:

$(f)_\beta$: Assume that $K : S^n \to \mathbb{R}$, $n \geq 5$, is a C^1 positive function such that for any critical point y of K, there exists some real number $\beta = \beta(y)$ such that, in some geodesic normal coordinate system centered at y, we have

$$K(x) = K(0) + \sum_{k=1}^{n} b_k \left|(x)_k\right|^\beta + R(x),$$

where $b_k = b_k(y) \neq 0$, $\forall k = 1, \dots, n$, $\sum_{k=1}^n b_k \neq 0$, and $\sum_{s=0}^{[\beta]} |\nabla^s R(x)||x|^{-\beta+s} = o(1)$ as x tends to zero. Here ∇^s denotes all possible derivatives of order s and $[\beta]$ is the integer part of β.

As far as the authors know, all the existence results dealing with Q-curvature in S^n under $(f)_\beta$ hold when $\beta < n$. The very first result has been handled by V Felli in [8] for $\beta \in]n-4, n[$. Other important results treating β not in the range mentioned above are the following ones: [5] for $\beta \in [n-4, n[$ and [3] for $\beta \in]1, n-4]$. Therefore, only the case β greater than or equal to n has not been addressed until now. The present paper deals with the case $\beta = n$. Our aim is to study the lack of compactness and provide an existence result in terms of the Euler-Hopf index. Let

$$\mathcal{K} = \left\{y \in S^n, \nabla_{g_0} K(y) = 0\right\}, \qquad \mathcal{K}^+ = \left\{y \in \mathcal{K}, -\sum_{k=1}^n b_k > 0\right\} \quad \text{and}$$

$$\tilde{i}(y) = \sharp\left\{b_k = b_k(y), 1 \leq k \leq n \text{ s.t. } b_k < 0\right\}.$$

Our first main result is the following.

Theorem 1.1 *Assume that K satisfies $(f)_\beta$, with*

$$\beta = n.$$

If

$$\sum_{y \in \mathcal{K}^+} (-1)^{n - \tilde{i}(y)} \neq 1,$$

then problem (1.2) *has at least one solution.*

Our method hinges on a readapted characterization of critical points at infinity techniques introduced by Bahri [9] and Bahri-Coron [10] and used in the above mentioned papers [5] and [3] for $\beta < n$. However, there is a serious problem of divergence of the integrals when $\beta = n$. To overcome this challenging problem, we perform a local analysis to give precise estimates to the gradient of the Euler Lagrange functional associated to our problem and identify the critical points at infinity. As we show in Corollary 3.1, we get a new type of critical points at infinity in the space of variation which is different from those of [5] and [3].

In the next section, we recall some preliminary results related to the variational structure of the problem. In Section 3, we study the asymptotic behavior of the gradient flow lines of the Euler-Lagrange functional and we characterize the critical points at infinity. Finally, in Section 4, we state the proof of Theorem 1.1.

2 Variational structure and lack of compactness

Equation (1.2) has a variational structure. Indeed, there is a one to one correspondence between the solutions of (1.2) and the critical points of

$$J(u) = \frac{\int_{S^n} \mathcal{P}uu \, dv_{g_0}}{(\int_{S^n} K|u|^{\frac{2n}{n-4}} \, dv_{g_0})^{\frac{n-4}{n}}}, \quad u \in \Sigma^+,$$

where $\Sigma^+ = \{u \in \Sigma, u > 0\}$ and $\Sigma = \{u \in H^2(S^n), \|u\|^2 = \int_{S^n} \mathcal{P}uu \, dv_{g_0} = 1\}$.

Since the Sobolev embedding $H^2(S^n) \to L^{\frac{2n}{n-4}}(S^n)$ is not compact, the functional J does not satisfy the Palais-Smale condition. To characterize the sequences failing the Palais-Smale condition, we state the following notations.

For $a \in S^n$ and $\lambda > 0$, let

$$\delta_{(a,\lambda)}(x) = c_n \frac{1}{2^{\frac{n-4}{2}}} \frac{\lambda^{\frac{n-4}{2}}}{(1 + \frac{\lambda^2 - 1}{2}(1 - \cos(d(x,a))))^{\frac{n-4}{2}}},$$

where d is the geodesic distance on (S^n, g_0) and c_n is chosen so that $\delta_{(a,\lambda)}$ is the family of solutions of the problem

$$\mathcal{P}u = u^{\frac{n+4}{n-4}}, \quad u > 0 \text{ on } S^n.$$

We define now the set of potential critical points at infinity associated to the function J. For $\varepsilon > 0$ and $p \in \mathbb{N}^*$, let us define

$$V(p,\varepsilon) = \begin{cases} u \in \Sigma/\exists a_1,\dots,a_p \in S^n, \qquad \exists \lambda_1,\dots,\lambda_p > \varepsilon^{-1}, \\ \exists \alpha_1,\dots,\alpha_p > 0 \quad \text{with } \|u - \sum_{i=1}^p \alpha_i \delta_{(a_i,\lambda_i)}\| < \varepsilon, \\ |J(u)^{\frac{n}{n-4}} \alpha_i^{\frac{8}{n-4}} K(a_i) - 1| < \varepsilon, \quad \forall i \text{ and } \varepsilon_{ij} < \varepsilon, \forall i \neq j, \end{cases}$$

where $\varepsilon_{ij} = [\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \frac{\lambda_i \lambda_j}{2}(1 - \cos d(a_i,a_j))]^{-\frac{n-4}{2}}$.

The failure of the Palais-Smale condition can be described following the idea introduced in [11], pp.325 and 334 as follows.

Proposition 2.1 *Let (u_k) be a sequence in Σ^+ such that $J(u_k)$ is bounded and $\partial J(u_k)$ goes to zero. Then there exist an integer $p \in \mathbb{N}^*$, a sequence $(\varepsilon_k) > 0$, ε_k tends to zero, and an extracted subsequence of u_k's, again denoted (u_k), such that $u_k \in V(p,\varepsilon_k)$.*

If u is a function in $V(p,\varepsilon)$, one can find an optimal representation, following the ideas introduced in Proposition 5.2 of [9] (see also pp.348-350 of [11]). Namely, we have the following proposition.

Proposition 2.2 *For any $p \in \mathbb{N}^*$, there is $\varepsilon_p > 0$ such that if $\varepsilon \leq \varepsilon_p$ and $u \in V(p,\varepsilon)$, then the minimization problem*

$$\min_{\alpha_i>0,\lambda_i>0,a_i\in S^n} \left\| u - \sum_{i=1}^p \alpha_i \delta_{(a_i,\lambda_i)} \right\|$$

has a unique solution (α,λ,a), up to a permutation.

In particular, we can write u as follows:

$$u = \sum_{i=1}^p \alpha_i \delta_{(a_i,\lambda_i)} + v,$$

where v belongs to $H^2(S^n)$ and it satisfies (V_0), (V_0) is the following:

$$(V_0): \quad \langle v, \psi \rangle = 0 \quad \text{for } \psi \in \left\{ \delta_i, \frac{\partial \delta_i}{\partial \lambda_i}, \frac{\partial \delta_i}{\partial a_i}, i = 1,\dots,p \right\},$$

where $\delta_i = \delta_{(a_i,\lambda_i)}$ and $\langle \cdot,\cdot \rangle$ denotes the scalar product defined on $H^2(S^n)$ by

$$\langle u,v \rangle = \int_{S^n} \Delta_{g_0} u \Delta_{g_0} v \, dv_{g_0} + c_n \int_{S^n} \nabla_{g_0} u \nabla_{g_0} v \, dv_{g_0} + d_n \int_{S^n} uv \, dv_{g_0}.$$

In the rest of the paper, we will say that $v \in (V_0)$ if v satisfies (V_0).

The following Morse lemma shows that the v-contributions can be neglected with respect to the concentration phenomenon; see [11] (pp.326, 327 and 334).

Proposition 2.3 *There is a $\mathcal{C}^1$-map which for each $(\alpha_i, a_i, \lambda_i)$ is such that $\sum_{i=1}^p \alpha_i \delta_{(a_i,\lambda_i)}$ belonging to $V(p,\varepsilon)$ associates $\overline{v} = \overline{v}(\alpha, a, \lambda)$ such that $\overline{v}$ is unique and satisfies*

$$J\left(\sum_{i=1}^p \alpha_i \delta_{(a_i,\lambda_i)} + \overline{v}\right) = \min_{v \in (V_0)} \left\{ J\left(\sum_{i=1}^p \alpha_i \delta_{(a_i,\lambda_i)} + v\right)\right\}.$$

Moreover, there exists a change of variables $v - \overline{v} \to V$ such that

$$J\left(\sum_{i=1}^p \alpha_i \delta_{(a_i,\lambda_i)} + v\right) = J\left(\sum_{i=1}^p \alpha_i \delta_{(a_i,\lambda_i)} + \overline{v}\right) + \|V\|^2.$$

In order to define our deformation lemma on the level sets of J, we can work as if V was zero; see [4].

The next definition is extracted from [9] (see Definition 09),

Definition 2.1 A critical point at infinity of J on Σ^+ is a limit of a flow line $u(s)$ of the equation

$$\begin{cases} \frac{\partial u}{\partial s} = -\partial J(u(s)), \\ u(0) = u_0, \end{cases}$$

such that $u(s)$ remains in $V(p, \varepsilon(s))$ for $s \geq s_0$. Here $\varepsilon(s)$ is some positive function tending to zero when $s \to +\infty$. Using Proposition 2.2, $u(s)$ can be written as

$$u(s) = \sum_{i=1}^p \alpha_i(s) \delta_{(a_i(s),\lambda_i(s))} + v(s).$$

Denoting $\tilde{\alpha}_i := \lim_{s\to+\infty} \alpha_i(s)$, $\tilde{y}_i := \lim_{s\to+\infty} a_i(s)$, we denote by

$$\sum_{i=1}^p \tilde{\alpha}_i \delta_{(\tilde{y}_i,\infty)} \quad \text{or} \quad (\tilde{y}_1, \dots, \tilde{y}_p)_\infty$$

such a critical point at infinity.

For such a critical point at infinity there are associated stable and unstable manifolds. These manifolds can easily be described once a Morse type reduction is performed; see [11] (pp.356-357).

3 Characterization of the critical points at infinity

This section will be devoted to a useful expansion of the gradient of J near infinity. Such expansions will be useful for the construction of a suitable pseudo-gradient which allows us to describe the concentration phenomenon of the problem and identify the critical points at infinity. In the following, we will write δ_i instead of $\delta_{(a_i,\lambda_i)}$, we will identify the function K and its composition with the stereographic projection Π_q and we will also identify a point x of S^n and its image by Π_q.

3.1 Expansion of the gradient of the functional

Proposition 3.1 *For any $u=\sum_{j=1}^{p}\alpha_j\delta_j$ in $V(p,\varepsilon)$, the following expansion hold*:

$$\text{(i)}\quad \left\langle \partial J(u),\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\right\rangle = -2c_2J(u)\sum_{i\neq j}\alpha_j\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}+o\left(\sum_{i\neq j}\varepsilon_{ij}\right)+o\left(\frac{1}{\lambda_i}\right),$$

where $c_2=c_0^{\frac{2n}{n-4}}\int_{\mathbb{R}^n}\frac{dy}{(1+|y|^2)^{\frac{n+4}{2}}}$.

(ii) *If $a_i\in B(y_{j_i},\rho)$, $y_{j_i}\in\mathcal{K}$, and ρ is a positive constant small enough, we have*

$$\left\langle \partial J(u),\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\right\rangle = 2J(u)\left[(n-4)c_3\frac{\alpha_i}{K(a_i)}\left(\sum_{k=1}^{n}b_k\right)\frac{\ln(\lambda_i)}{\lambda_i^n}-c_2\sum_{j\neq i}\alpha_j\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}\right.$$
$$\left.+o\left(\sum_{j\neq i}\varepsilon_{ij}\right)+O\left(\sum_{j=1}^{p}\frac{1}{\lambda_j^n}\right)+O\left(\sum_{k=1}^{n}\frac{|(a_i-y_{j_i})_k|^2}{\lambda_i^{n-2}}\right)\right], \tag{3.1}$$

where

$$c_3=\begin{cases} c_0^{\frac{2n}{n-4}}\frac{(\frac{n-3}{2})!(\frac{n-1}{2})!}{4(n-1)!} & \text{if } n \text{ is odd},\\ c_0^{\frac{2n}{n-4}}\frac{\prod_{r=1}^{\frac{n-2}{2}}(2r+1)^2}{2^n(n-2)!}\pi & \text{if } n \text{ is even}.\end{cases}$$

Proof Let $u=\sum_{j=1}^{p}\alpha_j\delta_j\in V(p,\varepsilon)$,

$$\left\langle \partial J(u),\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\right\rangle = 2J(u)\left[\left\langle u,\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\right\rangle - J(u)^{\frac{n}{n-4}}\int_{S^n}Ku^{\frac{n+4}{n-4}}\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\right]. \tag{3.2}$$

Following [9], Sections 1 and 2, we have

$$\left\langle u,\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\right\rangle = c_2\sum_{j\neq i}\alpha_j\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}+o\left(\sum_{j\neq i}\varepsilon_{ij}\right) \tag{3.3}$$

and

$$\begin{aligned}\int_{S^n}Ku^{\frac{n+4}{n-4}}\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} &= \alpha_i^{\frac{n+4}{n-4}}\int_{S^n}K\delta_i^{\frac{n+4}{n-4}}\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}+\sum_{j\neq i}\int_{S^n}K(x)(\alpha_j\delta_j)^{\frac{n+4}{n-4}}\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\,dx\\ &\quad+\frac{n+4}{n-4}\sum_{j\neq i}\alpha_i^{\frac{8}{n-4}}\alpha_j\int_{S^n}K(x)\delta_i^{\frac{8}{n-4}}\delta_j\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\,dx+o\left(\sum_{j\neq i}\varepsilon_{ij}\right)\\ &= \alpha_i^{\frac{n+4}{n-4}}\int_{S^n}K\delta_i^{\frac{n+4}{n-4}}\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}+c_2\sum_{j\neq i}\alpha_j^{\frac{n+4}{n-4}}K(a_j)\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}\\ &\quad+c_2\sum_{j\neq i}\alpha_j\alpha_i^{\frac{8}{n-4}}K(a_i)\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}+o\left(\sum_{j\neq i}\varepsilon_{ij}\right)+O\left(\sum_{j=1}^{p}\frac{1}{\lambda_j^n}\right)\\ &= \alpha_i^{\frac{n+4}{n-4}}\int_{S^n}K\delta_i^{\frac{n+4}{n-4}}\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}+2c_2J(u)^{\frac{-n}{n-4}}\sum_{j\neq i}\alpha_j\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}\\ &\quad+o\left(\sum_{j\neq i}\varepsilon_{ij}\right)+O\left(\sum_{j=1}^{p}\frac{1}{\lambda_j^n}\right),\end{aligned} \tag{3.4}$$

since $\alpha_i^{\frac{8}{n-4}} K(a_i) J(u)^{\frac{n}{n-4}} = 1 + o(1), \forall i = 1, \dots, p$. The stereographic projection and the change of variables $y = \lambda_i(x - a_i)$ yield

$$\begin{aligned}\int_{S^n} K\delta_i^{\frac{n+4}{n-4}} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} &= \frac{n-4}{2n} \lambda_i \frac{\partial}{\partial \lambda_i} \left(\int_{\mathbb{R}^n} K(x) \delta_i^{\frac{2n}{n-4}}(x)\, dx \right) \\ &= -\frac{n-4}{2n} c_0^{\frac{2n}{n-4}} \int_{\mathbb{R}^n} DK\left(\frac{y}{\lambda_i} + a_i\right)\left(\frac{y}{\lambda_i}\right) \frac{dy}{(1+|y|^2)^n}.\end{aligned}$$

Let μ be a small positive constant,

$$\begin{aligned}&\int_{\mathbb{R}^n} DK\left(\frac{y}{\lambda_i} + a_i\right)\left(\frac{y}{\lambda_i}\right) \frac{dy}{(1+|y|^2)^n} \\ &\quad = \int_{B(0,\lambda_i\mu)} DK\left(\frac{y}{\lambda_i} + a_i\right)\left(\frac{y}{\lambda_i}\right) \frac{dy}{(1+|y|^2)^n} + O\left(\frac{1}{\lambda_i^n}\right).\end{aligned} \tag{3.5}$$

Using the fact that DK is continuous we get

$$DK\left(\frac{y}{\lambda_i} + a_i\right) = DK(a_i) + o(1), \quad \text{as } \mu \text{ is small enough.}$$

Therefore,

$$\int_{B(0,\lambda_i\mu)} DK\left(\frac{y}{\lambda_i} + a_i\right)\left(\frac{y}{\lambda_i}\right) \frac{dy}{(1+|y|^2)^n} = o\left(\frac{1}{\lambda_i}\right)$$

and thus

$$\int_{\mathbb{R}^n} K\delta_i^{\frac{n+4}{n-4}} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} = o\left(\frac{1}{\lambda_i}\right). \tag{3.6}$$

Collecting (3.2)-(3.6), claim (i) is valid. Now we regard claim (ii). Following the above computation, it remains to expand this integral,

$$\begin{aligned}I &= \int_{S^n} K\delta_i^{\frac{n+4}{n-4}} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \\ &= \int_{B(a_i,\rho)\subset B(y_{j_i},2\rho)} K\delta_i^{\frac{n+4}{n-4}} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} + O\left(\frac{1}{\lambda_i^n}\right).\end{aligned}$$

Using the fact that K satisfies $(f)_\beta$ and the fact that $\int_{S^n} \delta_i^{\frac{n+4}{n-4}} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} = 0$, we get

$$\begin{aligned}I &= \sum_{k=1}^n b_k \int_{B(a_i,\rho)} \left|(x-y_{j_i})_k\right|^n \delta_i^{\frac{n+4}{n-4}} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} + O\left(\frac{1}{\lambda_i^n}\right) \\ &= \frac{n-4}{2n} \sum_{k=1}^n b_k \int_{B(a_i,\rho)} \left|(x-y_{j_i})_k\right|^n \lambda_i \frac{\partial}{\partial \lambda_i}\left(\delta_i^{\frac{2n}{n-4}}\right) dx + O\left(\frac{1}{\lambda_i^n}\right) \\ &= (n-4) c_0^{\frac{2n}{n-4}} \sum_{k=1}^n b_k \int_{B(a_i,\rho)} \frac{|(x-y_{j_i})_k|^n (1-\lambda_i^2|x-a_i|^2)\lambda_i^n}{(1+\lambda_i^2|x-a_i|^2)^{n+1}}\, dx + O\left(\frac{1}{\lambda_i^n}\right).\end{aligned} \tag{3.7}$$

Observe that

$$\begin{aligned}
&\int_{B(a_i,\rho)} \frac{|(x-y_{j_i})_k|^n (1-\lambda_i^2|x-a_i|^2)\lambda_i^n}{(1+\lambda_i^2|x-a_i|^2)^{n+1}}\,dx \\
&\quad = \int_{B(a_i,\rho)} \frac{|(x-a_i)_k|^n (1-\lambda_i^2|x-a_i|^2)\lambda_i^n}{(1+\lambda_i^2|x-a_i|^2)^{n+1}}\,dx \\
&\qquad + O\biggl(\int_{B(a_i,\rho)} \frac{|(x-a_i)_k|^{n-2}|(a_i-y_{j_i})_k|^2 (1-\lambda_i^2|x-a_i|^2)\lambda_i^n}{(1+\lambda_i^2|x-a_i|^2)^{n+1}}\,dx\biggr) \\
&\qquad + O\biggl(\int_{B(a_i,\rho)} \frac{|(a_i-y_{j_i})_k|^n (1-\lambda_i^2|x-a_i|^2)\lambda_i^n}{(1+\lambda_i^2|x-a_i|^2)^{n+1}}\,dx\biggr).
\end{aligned} \tag{3.8}$$

It is easy to see that

$$\int_{B(a_i,\rho)} \frac{|(x-a_i)_k|^{n-2}|(a_i-y_{j_i})_k|^2 (1-\lambda_i^2|x-a_i|^2)\lambda_i^n}{(1+\lambda_i^2|x-a_i|^2)^{n+1}}\,dx = O\biggl(\frac{|(a_i-y_{j_i})_k|^2}{\lambda_i^{n-2}}\biggr) \tag{3.9}$$

and

$$\int_{B(a_i,\rho)} \frac{|(a_i-y_{j_i})_k|^n (1-\lambda_i^2|x-a_i|^2)\lambda_i^n}{(1+\lambda_i^2|x-a_i|^2)^{n+1}}\,dx = O\biggl(\frac{|(a_i-y_{j_i})_k|^n}{\lambda_i^n}\biggr). \tag{3.10}$$

So we have to estimate (3.8). Using the change of variables $y = \lambda_i(x-a_i)$, we have

$$\begin{aligned}
&\int_{B(a_i,\rho)} \frac{|(x_i-a_i)_k|^n (1-\lambda_i^2|x-a_i|^2)\lambda_i^n}{(1+\lambda_i^2|x-a_i|^2)^{n+1}}\,dx \\
&\quad = \frac{1}{\lambda_i^n}\int_{B(0,\lambda_i\rho)} \frac{|y_1|^n(1-|y|^2)}{(1+|y|^2)^{n+1}}\,dy \\
&\quad = -\frac{1}{\lambda_i^n}\int_{B(0,\lambda_i\rho)} \frac{|y_1|^n}{(1+|y|^2)^n}\,dy + O\biggl(\frac{1}{\lambda_i^n}\biggr) \\
&\quad = -\frac{1}{\lambda_i^n}\int_{-\lambda_i\rho}^{\lambda_i\rho} |y_1|^n \biggl(\int_{B_{n-1}(0,\sqrt{(\lambda_i\rho)^2-|y_1|^2})} \frac{1}{(1+|y_1|^2+|\tilde{y}|^2)^n}\,d\tilde{y}\biggr)\,dy_1 + O\biggl(\frac{1}{\lambda_i^n}\biggr) \\
&\quad = -\frac{1}{\lambda_i^n}\int_{-\lambda_i\rho}^{\lambda_i\rho} |y_1|^n \biggl(\int_0^{\sqrt{(\lambda_i\rho)^2-|y_1|^2}} \frac{r^{n-2}}{(1+|y_1|^2+r^2)^n}\,dr\biggr)\,dy_1 + O\biggl(\frac{1}{\lambda_i^n}\biggr),
\end{aligned} \tag{3.11}$$

here B_{n-1} is a ball of $\mathbb{R}^{n-1}$ and $\tilde{y} = (y_2, \dots, y_n)$.

Through integrations by parts, we have

$$\begin{aligned}
&\frac{1}{\lambda_i^n}\int_{-\lambda_i\rho}^{\lambda_i\rho} |y_1|^n \biggl(\int_0^{\sqrt{(\lambda_i\rho)^2-|y_1|^2}} \frac{r^{n-2}}{(1+|y_1|^2+r^2)^n}\,dr\biggr)\,dy_1 \\
&\quad = o\biggl(\frac{1}{\lambda_i^n}\biggr) + \begin{cases}
\frac{1}{\lambda_i^n}\frac{(\frac{n-3}{2})!(\frac{n-1}{2})!}{2(n-1)!}\int_{-\lambda_i\rho}^{\lambda_i\rho}\frac{|y_1|^n}{(1+|y_1|^2)^{\frac{n+1}{2}}}\,dy_1 & \text{if } n \text{ is odd,} \\
\frac{1}{\lambda_i^n}\frac{\frac{n}{2}!\prod_{r=1}^{\frac{n-4}{2}}(2r+1)}{2^{\frac{n-2}{2}}(n-1)!} & \\
\quad\times\int_{-\lambda_i\rho}^{\lambda_i\rho}|y_1|^n(\int_0^{\sqrt{(\lambda_i\rho)^2-|y_1|^2}}\frac{1}{(1+r^2+|y_1|^2)^{\frac{n+2}{2}}}\,dr)\,dy_1 & \text{if } n \text{ is even.}
\end{cases}
\end{aligned} \tag{3.12}$$

If n is odd, using integrations by parts

$$\frac{1}{\lambda_i^n}\int_{-\lambda_i\rho}^{\lambda_i\rho}\frac{|y_1|^n}{(1+|y_1|^2)^{\frac{n+1}{2}}}\,dy_1 = O\left(\frac{1}{\lambda_i^n}\right)+\frac{1}{2}\frac{\ln(1+\lambda_i^2)}{\lambda_i^n}. \tag{3.13}$$

And if n is even, using the change of variables $z=\frac{r}{\sqrt{1+|y_1|^2}}$,

$$\begin{aligned}
&\frac{1}{\lambda_i^n}\int_{-\lambda_i\rho}^{\lambda_i\rho}|y_1|^n\left(\int_0^{\sqrt{(\lambda_i\rho)^2-|y_1|^2}}\frac{1}{(1+r^2+|y_1|^2)^{\frac{n+2}{2}}}\,dr\right)dy_1\\
&\quad=\frac{1}{\lambda_i^n}\int_0^{\lambda_i\rho}\frac{|y_1|^n}{(1+|y_1|^2)^{\frac{n+1}{2}}}\left(\int_0^{\frac{\sqrt{(\lambda_i\rho)^2-|y_1|^2}}{\sqrt{1+|y_1|^2}}}\frac{1}{(1+z^2)^{\frac{n+2}{2}}}\,dz\right)dy_1\\
&\quad=O\left(\frac{1}{\lambda_i^n}\right)+\frac{\prod_{r=1}^{\frac{n-2}{2}}(2r+1)}{\prod_{r=1}^{\frac{n}{2}}(2r)}\frac{1}{\lambda_i^n}\\
&\qquad\times\int_0^{\lambda_i\rho}\frac{|y_1|^n}{(1+|y_1|^2)^{\frac{n+1}{2}}}\arctan\left(\frac{\sqrt{(\lambda_i\rho)^2-|y_1|^2}}{\sqrt{1+|y_1|^2}}\right)dy_1.
\end{aligned}\tag{3.14}$$

Observing this, using the change of variables $t=\frac{\sqrt{(\lambda_i\rho)^2-|y_1|^2}}{\sqrt{1+|y_1|^2}}$,

$$\begin{aligned}
&\frac{1}{\lambda_i^n}\int_{-\lambda_i\rho}^{\lambda_i\rho}\frac{|y_1|^n}{(1+|y_1|^2)^{\frac{n+1}{2}}}\arctan\left(\frac{\sqrt{(\lambda_i\rho)^2-|y_1|^2}}{\sqrt{1+|y_1|^2}}\right)dy_1\\
&\quad=\frac{1}{\lambda_i^n}\int_{-\lambda_i\rho}^{\lambda_i\rho}\frac{1}{\sqrt{1+|y_1|^2}}\frac{(1+|y_1|^2)^{\frac{n}{2}}+|y_1|^n-(1+|y_1|^2)^{\frac{n}{2}}}{(1+|y_1|^2)^{\frac{n}{2}}}\arctan\left(\frac{\sqrt{(\lambda_i\rho)^2-|y_1|^2}}{\sqrt{1+|y_1|^2}}\right)dy_1\\
&\quad=\frac{1}{\lambda_i^n}\int_{-\lambda_i\rho}^{\lambda_i\rho}\frac{1}{\sqrt{1+|y_1|^2}}\arctan\left(\frac{\sqrt{(\lambda_i\rho)^2-|y_1|^2}}{\sqrt{1+|y_1|^2}}\right)dy_1+O\left(\frac{1}{\lambda_i^n}\right)\\
&\quad=\frac{1}{\lambda_i^n}\int_{-\lambda_i\rho}^{\lambda_i\rho}\frac{t\arctan(t)}{(1+t^2)}\frac{\sqrt{\lambda_i^2+1}}{\sqrt{\lambda_i^2-t^2}}\,dt+O\left(\frac{1}{\lambda_i^n}\right)\\
&\quad=\frac{1}{\lambda_i^n}\left(1+O\left(\frac{1}{\lambda_i^2}\right)\right)\int_{-\lambda_i\rho}^{\lambda_i\rho}\frac{t\arctan(t)}{(1+t^2)}\,dt+O\left(\frac{1}{\lambda_i^n}\right)\\
&\quad=\left(1+O\left(\frac{1}{\lambda_i^2}\right)\right)\left(\frac{\ln(1+\lambda_i^2)\arctan(\lambda_i)}{\lambda_i^n}-\frac{1}{2\lambda_i^n}\int_{-\lambda_i\rho}^{\lambda_i\rho}\frac{\ln(1+t^2)}{(1+t^2)}\,dt\right)+O\left(\frac{1}{\lambda_i^n}\right)\\
&\quad=\pi\frac{\ln(\lambda_i)}{\lambda_i^n}+O\left(\frac{1}{\lambda_i^n}\right).
\end{aligned}\tag{3.15}$$

Combining (3.7)-(3.15), the result follows. □

Proposition 3.2 *Let $u=\sum_{j=1}^p\alpha_j\delta_j\in V(p,\varepsilon)$, we have*

$$\text{(i)}\quad\left\langle\partial J(u),\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i}\right\rangle=-c_5J(u)^{\frac{2(n-2)}{n-4}}\alpha_i^{\frac{n+4}{n-4}}\frac{\nabla K(a_i)}{\lambda_i}+O\left(\sum_{i\neq j}\frac{1}{\lambda_i}\left|\frac{\partial\varepsilon_{ij}}{\partial a_i}\right|\right)+o\left(\sum_{i\neq j}\varepsilon_{ij}+\frac{1}{\lambda_i}\right),$$

where $c_5=\int_{\mathbb{R}^n}\frac{dy}{(1+|y|^2)^n}$.

(ii) *If* $a_i \in B(y_{j_i}, \rho)$, $y_{j_i} \in \mathcal{K}$, *we have*

$$\begin{aligned}
&\Big\langle \partial J(u), \frac{1}{\lambda_i}\frac{\partial \delta_i}{\partial (a_i)_r}\Big\rangle \\
&\quad = 2n(n-4)c_0^{\frac{2n}{n-4}}\alpha_i^{\frac{n+4}{n-4}}\big(J(u)\big)^{\frac{2(n-2)}{n-4}} b_r \frac{(a_i - y_{j_i})_r|(a_i - y_{j_i})_r|^{n-2}}{\lambda_i}\int_{\mathbb{R}^n}\frac{y_1^2}{(1+y^2)^{n+1}}\,dy \\
&\qquad + O\biggl(\frac{|(a_i - y_{j_i})_r|^{n-2}}{\lambda_i^2}\biggr) + o\biggl(\sum_{i\neq j}\varepsilon_{ij}\biggr) + O\biggl(\frac{1}{\lambda_i^n}\biggr) + O\biggl(\sum_{i\neq j}\frac{1}{\lambda_i}\biggl|\frac{\partial \varepsilon_{ij}}{\partial a_i}\biggr|\biggr),
\end{aligned}$$

where $k = 1, \dots, n$ *and* $(a_i)_r$ *is the rth component if* a_i *in some geodesic normal coordinate system.*

Proof Arguing as in the proof of Proposition 3.1, Proposition 3.2 is proved under the following estimates. If $a_i \in B(y_{j_i}, \rho)$, we have

$$\begin{aligned}
&\int_{S^n} K\delta_i^{\frac{n+4}{n-4}}\frac{1}{\lambda_i}\frac{\partial \delta_i}{\partial (a_i)_r} \\
&\quad = \sum_{k=1}^n b_k \int_{B(a_i,\rho)}\big|(x-y_{j_i})_k\big|^n \delta_i^{\frac{n+4}{n-4}}\frac{1}{\lambda_i}\frac{\partial \delta_i}{\partial (a_i)_r} \\
&\qquad + \int_{B(a_i,\rho)^c}\big(K(x)-K(y_i)\big)\delta_i^{\frac{n+4}{n-4}}\frac{1}{\lambda_i}\frac{\partial \delta_i}{\partial (a_i)_r} \\
&\quad = \frac{n-4}{2n}\sum_{k=1}^n b_k \int_{B(a_i,\rho)}\big|(x-y_{j_i})_k\big|^n \frac{1}{\lambda_i}\frac{\partial}{\partial (a_i)_r}\big(\delta_i^{\frac{2n}{n-4}}\big)\,dx \\
&\qquad + \frac{n-4}{2n}\int_{B(a_i,\rho)^c}\big(K(x)-K(y_i)\big)\frac{1}{\lambda_i}\frac{\partial}{\partial (a_i)_r}\big(\delta_i^{\frac{2n}{n-4}}\big)\,dx \\
&\quad = -(n-4)c_0^{\frac{2n}{n-4}} b_r \int_{B(a_i,\rho)}\big|(x-y_{j_i})_r\big|^n \frac{\lambda_i(x-a_i)_r}{(1+\lambda_i^2|x-a_i|^2)^{n+1}}\lambda_i^n\,dx \\
&\qquad - (n-4)c_0^{\frac{2n}{n-4}}\int_{B(a_i,\rho)^c}\big(K(x)-K(y_i)\big)\frac{\lambda_i(x-a_i)_r}{(1+\lambda_i^2|x-a_i|^2)^{n+1}}\lambda_i^n\,dx \\
&\quad = -(n-4)c_0^{\frac{2n}{n-4}} b_r \int_{B(a_i,\rho)}\big|(x-y_{j_i})_r\big|^n \frac{\lambda_i(x-a_i)_r}{(1+\lambda_i^2|x-a_i|^2)^{n+1}}\lambda_i^n\,dx + O\biggl(\frac{1}{\lambda_i^{n+1}}\biggr).
\end{aligned} \tag{3.16}$$

Now, we have

$$\begin{aligned}
&\int_{B(a_i,\rho)}\big|(x-y_{j_i})_r\big|^n \frac{\lambda_i(x-a_i)_r}{(1+\lambda_i^2|x-a_i|^2)^{n+1}}\lambda_i^n\,dx \\
&\quad = \int_{B(a_i,\rho)}\big|(a_i-y_{j_i})_r\big|^n \frac{\lambda_i(x-a_i)_r}{(1+\lambda_i^2|x-a_i|^2)^{n+1}}\lambda_i^n\,dx \\
&\qquad + \beta\int_{B(a_i,\rho)}(a_i-y_{j_i})_r\big|(a_i-y_{j_i})_r\big|^{n-2}\frac{\lambda_i(x-a_i)_r^2}{(1+\lambda_i^2|x-a_i|^2)^{n+1}}\lambda_i^n\,dx \\
&\qquad + O\biggl(\int_{B(a_i,\rho)}\big|(x-a_i)_r\big|^2\big|(a_i-y_{j_i})_r\big|^{n-2}\frac{\lambda_i(x-a_i)_r}{(1+\lambda_i^2|x-a_i|^2)^{n+1}}\lambda_i^n\,dx\biggr) \\
&\qquad + O\biggl(\int_{B(a_i,\rho)}\big|(x-a_i)_r\big|^n\frac{\lambda_i(x-a_i)_r}{(1+\lambda_i^2|x-a_i|^2)^{n+1}}\lambda_i^n\,dx\biggr)
\end{aligned}$$

$$
\begin{aligned}
&= n\frac{(a_i - y_{j_i})_r |(a_i - y_{j_i})_r|^{n-2}}{\lambda_i} \int_{\mathbb{R}^n} \frac{y_1^2}{(1+y^2)^{n+1}}\,dy \\
&\quad + O\left(\frac{|(a_i - y_{j_i})_r|^{n-2}}{\lambda_i^2}\right) + O\left(\frac{1}{\lambda_i^{n+1}}\right). \qquad \square
\end{aligned}
$$

For the whole next construction, we make use of the following notation.

Let $u = \sum_{i=1}^p \alpha_i \delta_{(a_i,\lambda_i)} \in V(p,\varepsilon)$, for simplicity, if a_i is close to a critical point y_{l_i}, we will assume that the critical point is zero, so we will exchange a_i with $(a_i - y_{l_i})$. Now, let $i \in \{1,\dots,p\}$ and let M_1 be a positive large constant. We will say that

$$i \in L_1 \quad \text{if } \lambda_i |a_i| \leq M_1$$

and we will say that

$$i \in L_2 \quad \text{if } \lambda_i |a_i| > M_1.$$

For each $i \in \{1,\dots,p\}$, we define the following vector fields:

$$Z_i(u) = \alpha_i \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \tag{3.17}$$

and

$$X_i = -\alpha_i \sum_{k=1}^n b_k \operatorname{sign}\bigl((a_i)_k\bigr) \frac{1}{\lambda_i} \frac{\partial \tilde{\delta}_{(a_i,\lambda_i)}}{\partial (a_i)_k}, \tag{3.18}$$

where $(a_i)_k$ is the kth component of a_i in some geodesic normal coordinate system.

It is clear that X_i is bounded. Let k_i be an index such that

$$\bigl|(a)_{k_i}\bigr| = \max_{1\leq j\leq n} \bigl|(a_i)_j\bigr|. \tag{3.19}$$

It is easy to see that if $i \in L_2$ then $\lambda_i |(a_i)_{k_i}| > \frac{M_1}{\sqrt{n}}$.

3.2 Critical points at infinity

This subsection is devoted to the characterization of the critical points at infinity in $V(p,\varepsilon)$, $p \geq 1$. First, we will prove that there is no critical points at infinity in $V(p,\varepsilon)$, $p \geq 2$, this result is obtained through the construction of a suitable pseudo-gradient $\widetilde{W}_1$ for which the Palais-Smale condition is satisfied along the decreasing flow lines. Second, we will study the left case. By the construction of a pseudo-gradient $\widetilde{W}_2$, we will give the characterization of the critical points at infinity in $V(1,\varepsilon)$. Now we introduce the following main result.

Theorem 3.1 *For $p \geq 2$, there exists a pseudo-gradient $\widetilde{W}_1$ in $V(p,\varepsilon)$ so that the following holds.*

There exists a constant $c > 0$ independent of $u = \sum_{i=1}^p \alpha_i \delta_i \in V(p,\varepsilon)$ so that

$$\text{(i)} \quad \bigl\langle \partial J(u), \widetilde{W}_1(u)\bigr\rangle \leq -c\left(\sum_{i=1}^p \frac{\ln(\lambda_i)}{\lambda_i^n} + \sum_{i=1}^p \frac{\nabla K(a_i)}{\lambda_i} + \sum_{j\neq i} \varepsilon_{ij}\right),$$

$$\text{(ii)}\quad \left\langle \partial J(u+\overline{v}), \widetilde{W}_1(u) + \frac{\partial \overline{v}}{\partial(\alpha_i, a_i, \lambda_i)}\big(\widetilde{W}_1(u)\big)\right\rangle$$
$$\leq -c\left(\sum_{i=1}^{p}\frac{\ln(\lambda_i)}{\lambda_i^n} + \sum_{i=1}^{p}\frac{\nabla K(a_i)}{\lambda_i} + \sum_{j\neq i}\varepsilon_{ij}\right).$$

Furthermore $|\widetilde{W}_1|$ *is bounded and the* λ_i*'s decrease along the flow lines of* $\widetilde{W}_1$.

Proof We divide $V(p,\varepsilon)$ in two different regions. Let

$$V_1(p,\varepsilon) = \left\{u = \sum_{i=1}^{p}\alpha_i\delta_i \in V(p,\varepsilon) \text{ s.t. } a_i \in B(y_{l_i},\rho), y_{l_i}\in\mathcal{K}, y\forall i\in\{1,\dots,p\}\right\}$$

and

$$V_2(p,\varepsilon) = \left\{u = \sum_{i=1}^{p}\alpha_i\delta_i \in V(p,\varepsilon) \text{ s.t. } \exists i\in\{1,\dots,p\}, a_i \notin \bigcup_{y\in\mathcal{K}} B(y,\rho)\right\}.$$

Pseudo-gradient in $V_1(p,\varepsilon)$. We order the λ_i's for the sake of simplicity, we can assume that $\lambda_1 \leq \cdots \leq \lambda_p$. For each i, $1\leq i\leq p$, we have, by Proposition 3.1,

$$\big\langle \partial J(u), -2^i Z_i(u)\big\rangle \leq c\sum_{j\neq i} 2^i\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} - c\left(\sum_{k=1}^{n} b_k\right)\frac{\ln(\lambda_i)}{\lambda_i^n} + \begin{cases} O(\frac{1}{\lambda_i^n}) & \text{if } i\in L_1,\\ O(\frac{|(a_i)_{k_i}|^2}{\lambda_i^{n-2}}) & \text{if } i\in L_2,\end{cases}$$

and using Proposition 3.2, we get

$$\big\langle \partial J(u), X_i(u)\big\rangle \leq O\left(\sum_{j\neq i}\varepsilon_{ij}\right) - c\frac{|(a_i)_{k_i}|^{n-1}}{\lambda_i} + \begin{cases} O(\frac{1}{\lambda_i^n}) & \text{if } i\in L_1,\\ O(\frac{|(a_i)_{k_i}|^{n-2}}{\lambda_i^2}) & \text{if } i\in L_2.\end{cases}$$

Thus,

$$\left\langle \partial J(u), \sum_{i=1}^{p}\big(m_1X_i - 2^iZ_i\big)(u)\right\rangle$$
$$\leq c\sum_{j\neq i}2^i\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + m_1 O\left(\sum_{i\neq j}\varepsilon_{ij}\right) - c\sum_{i=1}^{p}\frac{|(a_i)_{k_i}|^{n-1}}{\lambda_i} + O\left(\sum_{i=1}^{p}\frac{\ln(\lambda_i)}{\lambda_i^n}\right).$$

Observe now that, for $i<j$, we have

$$2^i\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + 2^j\lambda_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_j} \leq -c\varepsilon_{ij}. \tag{3.20}$$

Let $W_0 = \sum_{i=2}^{p}(m_1X_i - 2^iZ_i)$, taking m_1 positive small enough and using (3.20), we find that

$$\big\langle \partial J(u), W_0(u)\big\rangle \leq -c\left(\sum_{i\neq j}\varepsilon_{ij} + \sum_{i=2}^{p}\frac{|(a_i)_{k_i}|^{n-1}}{\lambda_i}\right) + O\left(\sum_{i=2}^{p}\frac{\ln(\lambda_i)}{\lambda_i^n}\right).$$

Observe that in $V_1(p,\varepsilon)$ we have under the $(f)_\beta$ condition

$$\left|\nabla K(a_i)\right| \sim \sum_{k=1}^{n} |b_k|\left|(a_i)_k\right|^{n-1}, \tag{3.21}$$

this yields $\frac{|\nabla K(a_i)|}{\lambda_i} \leq c\frac{|(a_i)_{k_i}|^{n-1}}{\lambda_i}$. We then have

$$\left\langle \partial J(u), W_0\right\rangle \leq -c\left(\sum_{i=2}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j} \varepsilon_{ij}\right) + O\left(\sum_{i=2}^{p} \frac{\ln(\lambda_i)}{\lambda_i^n}\right). \tag{3.22}$$

Observe that, $\forall j = 2,\ldots,p$, we have

$$\frac{\ln(\lambda_j)}{\lambda_j^n} = o(\varepsilon_{1j}), \tag{3.23}$$

thus we get

$$\left\langle \partial J(u), W_0\right\rangle \leq -c\left(\sum_{i=2}^{p} \frac{\ln(\lambda_i)}{\lambda_i^n} + \sum_{i=2}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j} \varepsilon_{ij}\right). \tag{3.24}$$

We must add the index 1.

If $\lambda_1 \sim \lambda_2$, then we can make $\frac{\ln(\lambda_1)}{\lambda_1^n}$ appear in the above estimates; in this case let

$$W_1 = W_0 + m_1 X_1,$$

we obtain

$$\left\langle \partial J(u), W_1(u)\right\rangle \leq -c\left(\sum_{i\neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{\ln(\lambda_i)}{\lambda_i^n} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i}\right).$$

If $\lambda_1 \ll \lambda_2$, we use the vector field $\widetilde{Z}_1$ defined by

$$\widetilde{Z}_1 = -\left(\sum_{k=1}^{n} b_k\right) Z_1. \tag{3.25}$$

We then have

$$\left\langle \partial J(u), X_1(u) + \widetilde{Z}_1(u)\right\rangle \leq -c\left(\frac{\ln(\lambda_1)}{\lambda_1^n} + \frac{|\nabla K(a_1)|}{\lambda_1}\right) + O\left(\sum_{j\neq 1} \varepsilon_{j1}\right).$$

In this case let

$$W_1 = W_0 + m_1(X_1 + \widetilde{Z}_1).$$

We then have

$$\left\langle \partial J(u), W_1\right\rangle \leq -c\left(\sum_{i=1}^{p} \frac{\ln(\lambda_i)}{\lambda_i^n} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j} \varepsilon_{ij}\right).$$

Pseudo-gradient in $V_2(p,\varepsilon)$. We order the λ_i's in an increasing order, without loss of generality, we suppose that $\lambda_1 \le \cdots \le \lambda_p$. Let i_1 be such that for any $i < i_1$, we have $a_i \in B(y_{l_i},\rho)$, $y_{l_i} \in \mathcal{K}$ and $a_{i_1} \notin \bigcup_{y\in\mathcal{K}} B(y,\rho)$. Let us define

$$u_1 = \sum_{i<i_1} \alpha_i \delta_i.$$

Observe that $u_1 \in V_1(i_1 - 1,\varepsilon)$. We have then the following estimate:

$$\bigl\langle \partial J(u), W_1(u_1)\bigr\rangle \le -c\biggl(\sum_{i<i_1} \frac{\ln(\lambda_i)}{\lambda_i^n} + \sum_{i<i_1} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j, i,j<i_1} \varepsilon_{ij}\biggr) + O\biggl(\sum_{i<i_1, j\ge i_1} \varepsilon_{ij}\biggr).$$

Now, we define the following vector field:

$$W_1' = \frac{1}{\lambda_{i_1}} \frac{\partial \delta_{(a_{i_1}\lambda_{i_1})}}{\partial a_{i_1}} \frac{\nabla K(a_{i_1})}{|\nabla K(a_{i_1})|} - c' \sum_{i\ge i_1} 2^i Z_i.$$

Using Propositions 3.1 and 3.2 and the fact that $|\nabla K(a_{i_1})| \ge c > 0$, we derive

$$\bigl\langle \partial J(u), W_1'(u)\bigr\rangle \le -\frac{c}{\lambda_{i_1}} + O\biggl(\sum_{i\neq i_1} \varepsilon_{ij}\biggr) - c' \sum_{i\ge i_1, j\neq i} \varepsilon_{ij} + o\biggl(\sum_{i\ge i_1} \frac{1}{\lambda_i}\biggr).$$

Taking c' positive large enough, we find

$$\bigl\langle \partial J(u), W_1'(u)\bigr\rangle \le -c\biggl(\sum_{i=i_1}^{p} \frac{\ln(\lambda_i)}{\lambda_i^n} + \sum_{i=i_1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\ge i_1, j\neq i} \varepsilon_{ij}\biggr).$$

Now, let $\overline{W}_1 := W_1' + m_1 W_1$ where m_1 is a small positive constant. We then have

$$\bigl\langle \partial J(u), \overline{W}_1(u)\bigr\rangle \le -c\biggl(\sum_{i=1}^{p} \frac{\ln(\lambda_i)}{\lambda_i^n} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j} \varepsilon_{ij}\biggr).$$

Now, we define the pseudo-gradient $\widetilde{W}_1$ as a convex combination of W_1 and $\overline{W}_1$ The construction of $\widetilde{W}_1$ is completed, it satisfies claim (i) of Theorem 3.1.

From the construction, $\widetilde{W}_1$ is bounded. Observe also that the λ_i's decrease along the flow lines of $\widetilde{W}_1$.

Now, we argue as in [11], Appendix 2, claim (ii) holds under claim (i) and the following lemma which proves that the norm of $\|\overline{v}\|^2$ is small with respect to the absolute value of the upper bound of claim (i).

Lemma 3.1 *Let* $u = \sum_{i=1}^{p} \alpha_i \delta_i + \alpha_0(w+h) \in V(p,\varepsilon,w)$ *and let* $\overline{v}$ *be defined as in Proposition* 2.3. *We have the following estimates*: *there exists* $c > 0$ *independent of* u *such that the following holds*:

$$\|\overline{v}\| \le c\sum_{i=1}^{p}\biggl[\frac{1}{\lambda_i^{\frac{n}{2}}} + \frac{|\nabla K(a_i)|}{\lambda_i}\biggr] + c\begin{cases} \sum_{k\neq r} \varepsilon_{kr}^{\frac{n+4}{2(n-4)}} (\log \varepsilon_{kr}^{-1})^{\frac{n+4}{2n}} & \textit{if } n \ge 12, \\ \sum_{k\neq r} \varepsilon_{kr} (\log \varepsilon_{kr}^{-1})^{\frac{n-4}{n}} & \textit{if } n < 12. \end{cases}$$

Proof Arguing as in the proof of Lemma 3.1 of [5], the proof of Lemma 3.1 follows. □

This concludes the proof of Theorem 3.1. □

Theorem 3.2 *There exists a pseudo-gradient $\widetilde{W}_2$ in $V(1,\varepsilon)$ so that the following holds. There is a positive constant $c > 0$ independent of $u = \alpha_1\delta_{a_1\lambda_1} \in V(1,\varepsilon)$ such that*

$$\text{(i)}\quad \langle \partial J(u), \widetilde{W}_2(u)\rangle \le -c\left(\frac{\ln(\lambda_1)}{\lambda_1^n} + \frac{|\nabla K(a_1)|}{\lambda_1}\right),$$

$$\text{(ii)}\quad \left\langle \partial J(u+\overline{v}), \widetilde{W}_2(u) + \frac{\partial \overline{v}}{\partial(\alpha_1, a_1, \lambda_1)}\big(\widetilde{W}_2(u)\big)\right\rangle \le -c\left(\frac{\ln(\lambda_1)}{\lambda_1^n} + \frac{|\nabla K(a_1)|}{\lambda_1}\right).$$

Furthermore $|\widetilde{W}_2|$ is bounded and the only case where λ_1 is not bounded is where $a_1 \in B(y,\rho)$, $y \in \mathcal{K}^+$.

Proof Let $u = \alpha_1\delta_{a_1\lambda_1} \in V(1,\varepsilon)$.

Case 1: If $a_1 \in B(y,\rho)$, $y \in \mathcal{K}$, we define

$$W_2 = \widetilde{Z}_1 + X_1,$$

here X_1 is defined by (3.18) and $\widetilde{Z}_1$ by (3.25). Using (2.1), and Propositions 3.1 and 3.2, we have

$$\begin{aligned}\langle \partial J(u), W_2(u)\rangle &\le -c_3\left(\sum_{k=1}^n b_k\right)^2 \frac{\ln(\lambda_1)}{\lambda_1^n} - c\frac{|(a_1)_{k_1}|^{n-1}}{\lambda_1} \\ &\quad + \begin{cases} O(\frac{1}{\lambda_1^n}) & \text{if } 1 \in L_1, \\ O(\frac{|(a_1)_{k_1}|^{n-2}}{\lambda_1^2}) & \text{if } 1 \in L_2. \end{cases}\end{aligned} \tag{3.26}$$

Using (3.21), we derive

$$\langle \partial J(u), W_2(u)\rangle \le -c\left(\frac{\ln(\lambda_1)}{\lambda_1^n} + \frac{|\nabla K(a_1)|}{\lambda_1}\right). \tag{3.27}$$

Case 2: If $a_1 \notin \bigcup_{y\in\mathcal{K}} B(y,\rho)$, we define

$$\overline{W}_2 = \frac{1}{\lambda_1}\frac{\partial \delta_{a_1\lambda_1}}{\partial a_1}\frac{\nabla K(a_1)}{|\nabla K(a_1)|}.$$

Using Proposition 3.2 and the fact that $|\nabla K(a_1)| \ge c > 0$, we derive

$$\langle \partial J(u), \overline{W}_2(u)\rangle \le -c\left(\frac{\ln(\lambda_1)}{\lambda_1^n} + \frac{|\nabla K(a_1)|}{\lambda_1}\right).$$

The required pseudo-gradient $\widetilde{W}_2$ will be defined by convex combination of W_2 and $\overline{W}_2$. □

Corollary 3.1 *The only critical point at infinity of J in $V(1,\varepsilon)$ corresponds to*

$$\frac{1}{K(y)^{\frac{n-4}{2}}}\delta_{(y,\infty)}, \quad y \in \mathcal{K}^+;$$

such a critical point has an index equal to $n - \tilde{i}(y)$.

Proof Observe from Theorem 3.2 that the Palais-Smale condition is satisfied along each flow line of $\widetilde{W}_2$, until the concentration points of the flow do not enter some neighborhood of y such that $y \in \mathcal{K}^+$; we observe that $\sup\lambda$ has to increase and go to $+\infty$ as well as $\inf\lambda$. Thus we obtain a critical point at infinity. In this region arguing as in the proof of Proposition 3.1 of [4], we can find the change of variable

$$(a,\lambda) \mapsto (\tilde{a},\tilde{\lambda}) := (\tilde{a},\tilde{\lambda})$$

such that

$$J(\alpha\delta_{a\lambda} + \overline{v}) = \psi(\alpha,\tilde{a},\tilde{\lambda}) := \frac{\alpha^2 S_n}{(S_n\alpha^{\frac{2n}{n-4}}K(\tilde{a}))^{\frac{n-4}{n}}}\bigl[1 + o(1)\bigr].$$

Since K satisfy the $(f)_\beta$ condition, then the index of such a critical point at infinity is equal to $n - \tilde{i}(y)$. The result of Corollary 3.1 follows. □

4 Proof of Theorem 1.1

We argue by contradiction. Assume that J has no critical points at Σ^+. By Corollary 3.1, the only critical points at infinity of the associated variational problem are

$$(y)_\infty := \frac{1}{K(y)^{\frac{n-4}{2}}}\delta_{(y,\infty)}, \quad y \in \mathcal{K}^+.$$

The indices of such critical points at infinity are

$$i(y)_\infty := n - \tilde{i}(y).$$

For each $(y)_\infty$, we denote by $W_u^\infty(y)_\infty$ its unstable manifold. By using a deformation lemma, see [10], we have

$$\Sigma^+ \text{ retracts by deformation on } \bigcup_{y\in\mathcal{K}^+} W_u^\infty(y)_\infty. \tag{4.1}$$

It is well known that if M is a finite cw complex in dimension k, its Euler-Poincaré characteristic denoted $\chi(M)$ is given by

$$\chi(M) = \sum_{j=0}^{k}(-1)^j n(j), \tag{4.2}$$

where $n(j)$ is the number of cells of dimension j in M (see [12]). We apply this to both sides of (4.1), we obtain

$$1 = \chi\bigl(\Sigma^+\bigr) = \sum_{y\in\mathcal{K}^+}(-1)^{i(y)_\infty}.$$

Such an equality contradicts the assumption of Theorem 1.1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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