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Superstability of the functional equation related to distance measures

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Abstract

The functional equation related to a distance measure

$$f(pr,qs)+f(ps,qr)=f(p,q)f(r,s)$$

can be generalized as follows:

$$\sum_{i=0}^{n-1} f(P\cdot\sigma_i(Q))=f(P)f(Q),$$

where f is an information measure, P and Q are in the set of n-ary discrete complete probability, and σ_i is a permutation for each $i=0,1,\ldots,n-1$.

In this paper, we investigate the superstability of the above functional equation and also four generalized functional equations:

$$\sum_{i=1}^{n-1} f(P\cdot\sigma_i(Q))=f(P)g(Q),\qquad \sum_{i=0}^{n-1} f(P\cdot\sigma_i(Q))=g(P)f(Q),$$

$$\sum_{i=0}^{n-1} f(P\cdot\sigma_i(Q))=g(P)g(Q),\qquad \sum_{i=0}^{n-1} f(P\cdot\sigma_i(Q))=g(P)h(Q).$$

MSC: 39B82; 39B52

Keywords: information measure; distance measure; superstability; multiplicative function; stability of functional equation

1 Introduction

Baker *et al.* in [1] introduced that if f satisfies the stability inequality $|E_1(f)-E_2(f)|\le\varepsilon$, then either f is bounded or $E_1(f)=E_2(f)$. This is now frequently referred to as *superstability*. Baker [2] also proved the superstability of the cosine functional equation (also called the d'Alembert functional equation).

In this paper, let $(G,\cdot)$ be a commutative group and I denote the open unit interval $(0,1)$. Also let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}_+=\{x\in\mathbb{R}\mid x>0\}$ be a set of positive real numbers. Further, let

$$\Gamma_n^0 = \left\{ P = (p_1, p_2, \ldots, p_n) \;\middle|\; 0 < p_k < 1, \sum_{k=1}^{n} p_k = 1 \right\}$$

denote the set of all n-ary discrete complete probability distributions (without zero probabilities), that is, Γ_n^0 is the class of discrete distributions on a finite set Ω of cardinality n with $n \geq 2$. Almost all similarity, affinity or distance measures $\mu_n : \Gamma_n^0 \times \Gamma_n^0 \to \mathbb{R}_+$ that have been proposed between two discrete probability distributions can be represented in the *sum-form*

$$\mu_n(P, Q) = \sum_{k=1}^{n} \phi(p_k, q_k), \tag{1.1}$$

where $\phi : I \times I \to \mathbb{R}$ is a real-valued function on unit square, or a monotonic transformation of the right-hand side of (1.1), that is,

$$\mu_n(P, Q) = \psi\left(\sum_{k=1}^{n} \phi(p_k, q_k) \right), \tag{1.2}$$

where $\psi : \mathbb{R} \to \mathbb{R}_+$ is an increasing function on $\mathbb{R}$. The function ϕ is called a *generating function*. It is also referred to as the *kernel* of $\mu_n(P, Q)$.

In information theory, for P and Q in Γ_n^0, the symmetric divergence of degree α is defined as

$$J_{n,\alpha}(P, Q) = \frac{1}{2^{\alpha-1} - 1} \left[\sum_{k=1}^{n} \left(p_k^\alpha q_k^{1-\alpha} + p_k^{1-\alpha} q_k^\alpha \right) - 2 \right].$$

For all $P, Q \in \Gamma_n^0$, we define the product

$$P \cdot R = (p_1 r_1, p_1 r_2, \ldots, p_1 r_m, p_2 r_1, \ldots, p_2 r_m, \ldots, p_n r_m).$$

In [3], Chung *et al.* characterized all symmetrically compositive sum-form distance measures with a measurable generating function. The following functional equation

$$f(pr, qs) + f(ps, qr) = f(p, q) f(r, s) \tag{DM}$$

holding for all $p, q, r, s \in I$ was instrumental in their characterization of symmetrically compositive sum-form distance measures. They proved the following theorem giving the general solution of this functional equation (DM):

Suppose $f : I^2 \to \mathbb{R}$ satisfies (DM) *for all $p, q, r, s \in I$. Then*

$$f(p, q) = M_1(p) M_2(q) + M_1(q) M_2(p),$$

where $M_1, M_2 : \mathbb{R} \to \mathbb{C}$ are multiplicative functions. Further, either M_1 and M_2 are both real or M_2 is the complex conjugate of M_1. The converse is also true.

In [4] and [5], Kim (second author) and Sahoo obtained the superstability results of the equation (DM), its stability and four generalizations of (DM), namely

$$f(pr,qs)+f(ps,qr)=f(p,q)g(r,s), \tag{DMfg}$$

$$f(pr,qs)+f(ps,qr)=g(p,q)f(r,s), \tag{DMgf}$$

$$f(pr,qs)+f(ps,qr)=g(p,q)g(r,s), \tag{DMgg}$$

$$f(pr,qs)+f(ps,qr)=g(p,q)h(r,s) \tag{DMgh}$$

for all $p,q,r,s \in G$.

The above equation (DM) characterized by distance measures can be considered by characterization of a symmetrically compositive sum-form information measurable functional equation.

The functional equation (DM) can be generalized as follows. Let $f:\Gamma_n^0 \to R$ be a function and

$$\sum_{i=0}^{n-1} f\bigl(P\cdot\sigma_i(Q)\bigr)=f(P)f(Q) \tag{IM}$$

for all $P=(p_1,p_2,\ldots,p_n), Q=(q_1,q_2,\ldots,q_n)\in\Gamma_n^0$, where $\sigma_i: I^n \to I^n$ is a permutation defined by

$$\sigma_i(x_1,x_2,\ldots,x_n):=(x_{i+1},x_{i+2},\ldots,x_n,x_1,x_2,\ldots,x_i)$$

for each $i\in N$, and define $P\cdot Q:=(p_1q_1,p_2q_2,\ldots,p_nq_n)$.

For other functional equations with the information measure, the interested reader should refer to [6–9] and [10–12].

This paper aims to investigate the superstability of (IM) and also four generalized functional equations of (IM) as well as that of the following type functional equations:

$$\sum_{i=0}^{n-1} f\bigl(P\cdot\sigma_i(Q)\bigr)=f(P)g(Q), \tag{GIMfg}$$

$$\sum_{i=0}^{n-1} f\bigl(P\cdot\sigma_i(Q)\bigr)=g(P)f(Q), \tag{GIMgf}$$

$$\sum_{i=0}^{n-1} f\bigl(P\cdot\sigma_i(Q)\bigr)=g(P)g(Q), \tag{GIMgg}$$

$$\sum_{i=0}^{n-1} f\bigl(P\cdot\sigma_i(Q)\bigr)=g(P)h(Q) \tag{GIMgh}$$

for all $P,Q\in G$.

2 Results

In this section, we investigate the superstability of the pexiderized equation related to (IM).

Theorem 1 *Let $f, g, h : G^n \to \mathbb{R}$ and $\phi : G^n \to \mathbb{R}_+$ be functions satisfying*

$$\left|\sum_{i=0}^{n-1} f\bigl(X \cdot \sigma_i(Y)\bigr) - g(X)h(Y)\right| \le \phi(Y) \tag{2.1}$$

and $|f(X) - g(X)| \le M$ for all $X = (x_1, \dots, x_n), Y = (y_1, \dots, y_n) \in G^n$ and some constant M. Then either g is bounded or h is a solution of (IM).

Proof Let g be an unbounded solution of inequality (2.1). Then there exists a sequence $\{(Z_m) = (z_{1m}, z_{2m}, \dots, z_{nm}) \mid m \in N\}$ in G^n such that $0 \neq |g(Z_m)| \to \infty$ as $m \to \infty$.

Letting $X = Z_m$, *i.e.*, $x_i = z_{im}$ in (2.1) for each i and dividing $|g(Z_m)|$, we have

$$\left|\frac{\sum_{i=0}^{n-1} f(Z_m \cdot \sigma_i(Y))}{g(Z_m)} - h(Y)\right| \le \frac{\phi(Y)}{|g(Z_m)|}.$$

Passing to the limit as $m \to \infty$, we obtain that

$$h(Y) = \lim_{m\to\infty} \frac{\sum_{i=0}^{n-1} f(Z_m \cdot \sigma_i(Y))}{g(Z_m)}. \tag{2.2}$$

By (2.1), we have

$$\begin{aligned}&\left|\frac{\sum_{i=0}^{n-1} f((Z_m \cdot X) \cdot \sigma_i(Y)) - g(Z_m \cdot X)h(Y)}{g(Z_m)}\right| \\ &\quad \le \frac{\phi(Y)}{|g(Z_m)|} \to 0\end{aligned} \tag{2.3}$$

as $m \to \infty$. Also, for each j,

$$\begin{aligned}&\left|\frac{\sum_{i=0}^{n-1} f(Z_m \cdot \sigma_j(X) \cdot \sigma_i(Y)) - g(Z_m \cdot \sigma_j(X))h(Y)}{g(Z_m)}\right| \\ &\quad \le \frac{\phi(Y)}{|g(Z_m)|} \to 0\end{aligned} \tag{2.4}$$

as $m \to \infty$. Note that $\sigma_i(X \cdot Y) = \sigma_i(X) \cdot \sigma_i(Y)$, $\sigma_i(\sigma_j(Y)) = \sigma_{i+j}(Y)$, $\sigma_{n+j}(Y) = \sigma_j(Y)$ and $\sum_{i=0}^{n-1} f(Z_m \cdot \sigma_1(X) \cdot \sigma_{i+1}(Y)) = \sum_{i=0}^{n-1} f(Z_m \cdot \sigma_1(X) \cdot \sigma_i(Y))$. Thus, from (2.2), (2.3) and (2.4), we obtain

$$\begin{aligned}&\left|\sum_{i=0}^{n-1} h\bigl(X \cdot \sigma_i(Y)\bigr) - h(X)h(Y)\right| \\ &\quad = \lim_{m\to\infty}\left|\sum_{i=0}^{n-1} \frac{\sum_{j=0}^{n-1} f(Z_m \cdot \sigma_j(X \cdot \sigma_i(Y)))}{g(Z_m)} - h(X)h(Y)\right| \\ &\quad = \lim_{m\to\infty}\left|\frac{\sum_{j=0}^{n-1} f(Z_m \cdot \sigma_j(X \cdot \sigma_0(Y))) + \sum_{j=0}^{n-1} f(Z_m \cdot \sigma_j(X \cdot \sigma_1(Y)))}{g(Z_m)}\right. \\ &\qquad \left. + \cdots + \frac{\sum_{j=0}^{n-1} f(Z_m \cdot \sigma_j(X \cdot \sigma_{n-1}(Y)))}{g(Z_m)} - h(X)h(Y)\right|\end{aligned}$$

$$
\begin{aligned}
&= \lim_{m\to\infty}\left|\frac{\sum_{i=0}^{n-1} f(Z_m\cdot\sigma_0(X\cdot\sigma_i(Y))) + \sum_{i=0}^{n-1} f(Z_m\cdot\sigma_1(X\cdot\sigma_i(Y)))}{g(Z_m)}\right. \\
&\quad \left. + \cdots + \frac{\sum_{i=0}^{n-1} f(Z_m\cdot\sigma_{n-1}(X\cdot\sigma_i(Y)))}{g(Z_m)} - h(X)h(Y)\right| \\
&= \lim_{m\to\infty}\left|\frac{\sum_{i=0}^{n-1} f(Z_m\cdot\sigma_0(X)\cdot\sigma_i(Y)) + \sum_{i=0}^{n-1} f(Z_m\cdot\sigma_1(X)\cdot\sigma_i(Y))}{g(Z_m)}\right. \\
&\quad \left. + \cdots + \frac{\sum_{i=0}^{n-1} f(Z_m\cdot\sigma_{n-1}(X)\cdot\sigma_i(Y))}{g(Z_m)} - h(X)h(Y)\right| \\
&\leq \lim_{m\to\infty}\left|\frac{\sum_{i=0}^{n-1} f(Z_m\cdot\sigma_0(X)\cdot\sigma_i(Y)) - g(Z_m\cdot\sigma_0(X))h(Y)}{g(Z_m)}\right| \\
&\quad + \lim_{m\to\infty}\left|\frac{\sum_{i=0}^{n-1} f(Z_m\cdot\sigma_1(X)\cdot\sigma_i(Y)) - g(Z_m\cdot\sigma_1(X))h(Y)}{g(Z_m)}\right| \\
&\quad + \cdots + \lim_{m\to\infty}\left|\frac{\sum_{i=0}^{n-1} f(Z_m\cdot\sigma_{n-1}(X)\cdot\sigma_i(Y)) - g(Z_m\cdot\sigma_{n-1}(X))h(Y)}{g(Z_m)}\right| \\
&\quad + \lim_{m\to\infty}\left|\frac{\sum_{i=0}^{n-1} f(Z_m\cdot\sigma_i(X))\cdot h(Y)}{g(Z_m)} - h(X)h(Y)\right| \\
&\quad + \lim_{m\to\infty}\left|\frac{\sum_{i=0}^{n-1} (g-f)(Z_m\cdot\sigma_i(X))\cdot h(Y)}{g(Z_m)}\right| \\
&= \left|h(X)h(Y) - h(X)h(Y)\right| = 0. \qquad \square
\end{aligned}
$$

Theorem 2 *Let $f,g,h : G^n \to \mathbb{R}$ and $\phi : G^n \to \mathbb{R}_+$ be functions satisfying*

$$\left|\sum_{i=0}^{n-1} f\bigl(X\cdot\sigma_i(Y)\bigr) - g(X)h(Y)\right| \leq \phi(X) \tag{2.5}$$

and $|f(X) - h(X)| \leq M$ for all $X = (x_1,\ldots,x_n), Y = (y_1,\ldots,y_n) \in G^n$ and some constant M. Then either h is bounded or g is a solution of (IM).

Proof Assume that there exists a sequence $\{(Z_m) = (z_{1m}, z_{2m}, \ldots, z_{nm}) \mid m \in N\}$ in G^n such that $\lim_{mto\infty} |h(Z_m)| = \infty$ with $|h(Z_m)| \neq 0$ for each m.

Letting $Y = Z_m$, *i.e.*, $x_i = z_{im}$ in (2.5) for each i and dividing $|h(Z_m)|$, we have

$$\left|\frac{\sum_{i=0}^{n-1} f(X\cdot\sigma_i(Z_m))}{h(Z_m)} - g(X)\right| \leq \frac{\phi(X)}{|h(Z_m)|}.$$

Passing to the limit as $m \to \infty$, we obtain that

$$g(X) = \lim_{m\to\infty} \frac{\sum_{i=0}^{n-1} f(X\cdot\sigma_i(Z_m))}{h(Z_m)}. \tag{2.6}$$

By (2.5), we have

$$
\begin{aligned}
&\left|\frac{\sum_{i=0}^{n-1} f(X\cdot\sigma_i(Y\cdot Z_m)) - g(X)h(Y\cdot Z_m)}{h(Z_m)}\right| \\
&\quad \leq \frac{\phi(X)}{|h(Z_m)|} \to 0
\end{aligned} \tag{2.7}
$$

as $m \to \infty$. Also, for each j,

$$\begin{aligned}&\left|\frac{\sum_{i=0}^{n-1} f(X\cdot\sigma_i(Y\cdot\sigma_j(Z_m))) - g(X)h(Y\cdot\sigma_j(Z_m))}{h(Z_m)}\right| \\ &\quad\le \frac{\phi(X)}{|h(Z_m)|} \to 0 \qquad (2.8)\end{aligned}$$

as $m \to \infty$.

By using (2.5), (2.6) and (2.7), let us go through the same procedure as in Theorem 1, then we arrive at the required result. □

Corollary 1 *Let $f : G^n \to \mathbb{R}$ and $\phi : G^n \to \mathbb{R}_+$ be functions satisfying*

$$\left|\sum_{i=0}^{n-1} f\bigl(X\cdot\sigma_i(Y)\bigr) - f(X)f(Y)\right| \le \max\bigl\{\phi(X),\phi(Y)\bigr\} \qquad (2.9)$$

for all $X = (x_1,\dots,x_n), Y = (y_1,\dots,y_n) \in G^n$. Then either f is bounded or f is a solution of (IM).

Proof By Theorems 1 and 2, it is trivial. □

Corollary 2 *Let $f : G^n \to \mathbb{R}$ and $\phi : G^n \to \mathbb{R}_+$ be functions satisfying*

$$\left|\sum_{i=0}^{n-1} f\bigl(X\cdot\sigma_i(Y)\bigr) - f(X)g(Y)\right| \le \min\bigl\{\phi(X),\phi(Y)\bigr\} \qquad (2.10)$$

for all $X = (x_1,\dots,x_n), Y = (y_1,\dots,y_n) \in G^n$. Then either f (or g) is bounded or g satisfies (IM). *And also $\{f,g\}$ satisfies* (GIMfg).

Proof By Theorem 1, we have that either f is bounded or g satisfies (IM). Also, it follows from (2.10) that

$$\bigl|g(Y)\bigr| \le \frac{\phi(X) + \sum_{i=0}^{n-1} |f(\sigma_i(Y))|}{|f(X)|}.$$

Thus if f is bounded, then g is bounded. Hence, by Theorem 1, in the case g is unbounded, g also is a solution of (IM).

Let g be unbounded. By a similar method as the calculation in Theorem 2 with the unboundedness of g, we have

$$f(X) = \lim_{m\to\infty} \frac{\sum_{i=0}^{n-1} f(X\cdot\sigma_i(Z_m))}{g(Z_m)} \qquad (2.11)$$

for all $X, Z_m \in G^n$ and $0 \ne |g(Z_m)| \to \infty$ as $m \to \infty$.

From a similar calculation as that in Theorem 1 and Theorem 2, we obtain the required result. □

Corollary 3 *Let $f, g : G^n \to \mathbb{R}$ and $\phi : G^n \to \mathbb{R}_+$ be functions satisfying*

$$\left|\sum_{i=0}^{n-1} f\bigl(X \cdot \sigma_i(Y)\bigr) - g(X)f(Y)\right| \le \min\bigl\{\phi(X), \phi(Y)\bigr\} \tag{2.12}$$

for all $X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_n) \in G^n$. Then either f (or g) is bounded or g satisfies (IM). *And also $\{f, g\}$ satisfies* (GIMgf).

Proof By Theorem 2, we have that either f is bounded or g is a solution of (IM). Suppose that g be unbounded, then f is unbounded. Hence, by Theorem 2, g also is a solution of (IM). By a similar method as the calculation in Theorem 1 with the unboundedness of g, we have

$$f(X) = \lim_{m\to\infty} \frac{\sum_{i=0}^{n-1} f(Z_m \cdot \sigma_i(X))}{g(Z_m)} \tag{2.13}$$

for all $X, Z_m \in G^n$ and $0 \neq |g(Z_m)| \to \infty$ as $m \to \infty$.

From a similar calculation as that in Corollary 2 we obtain the required result. □

Corollary 4 *Let $f, g, h : G^n \to \mathbb{R}$ and $\phi : G^n \to \mathbb{R}_+$ be functions satisfying*

$$\left|\sum_{i=0}^{n-1} f\bigl(X \cdot \sigma_i(Y)\bigr) - g(X)h(Y)\right| \le \min\bigl\{\phi(X), \phi(Y)\bigr\} \tag{2.14}$$

and $\max\{|f(X) - g(X)|, |f(X) - h(X)|\} \le M$ *for all $X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_n) \in G^n$ and for some M. Then either f (or g, or h) is bounded or g and h are solutions of* (IM).

Corollary 5 *Let $f, g : G^n \to \mathbb{R}$ and $\phi : G^n \to \mathbb{R}_+$ be functions satisfying*

$$\left|\sum_{i=0}^{n-1} f\bigl(X \cdot \sigma_i(Y)\bigr) - g(X)g(Y)\right| \le \min\bigl\{\phi(X), \phi(Y)\bigr\} \tag{2.15}$$

and $\{|f(X) - g(X)|\} \le M$ *for all $X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_n) \in G^n$ and for some M. Then either f (or g) is bounded or g satisfies* (IM).

3 Discussion

We consider the functional equation

$$\sum_{i=0}^{n-1} f\bigl(X \cdot \sigma_i(Y)\bigr) = f(X)f(Y)$$

for all $X, Y \in G^n$, where $f : G^n \to \mathbb{R}$ is the unknown function to be determined, and $\sigma_i(X) = (x_{i+1}, x_{i+2}, \ldots, x_n, x_1, x_2, \ldots, x_i)$. If $n = 2$, the solution of the above functional equation is known on the semigroup $S = (0, 1)$ when the semigroup operation is multiplication [3]. It is not known when $n \ge 3$, but there is a special solution of it.

For example, let $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$. And define $f(X) = f(x_1, x_2, \ldots, x_n) := \sum_{i=1}^{n} \frac{1}{x_i}$. Then f is a solution of the above equation. Thus our results are not limited. We expect to know the general solution of it.

4 Conclusions

In the present paper we considered generalized functional equations related to distance measures and investigated the stability of them. We extended for two-variables in (DM) to n-variables in (IM). That is, the following functional equation satisfies the property of superstability

$$\sum_{i=0}^{n-1} f\bigl(P \cdot \sigma_i(Q)\bigr) = f(P)f(Q),$$

where f is an information measure, P and Q are in the set of n-ary discrete complete probability, and σ_i is a permutation for each $i = 0, 1, \ldots, n-1$.

Also the pexiderized functional equation of the above equation satisfies the property of superstability.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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