Chen and Lan *Journal of Inequalities and Applications* (2015) 2015:347
DOI 10.1186/s13660-015-0865-3

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Estimates of the modular-type operator norm of the general geometric mean operator

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Abstract

In this paper, the modular-type operator norm of the general geometric mean operator over spherical cones is investigated. We give two applications of a new limit process, introduced by the present authors, to the establishment of Pólya-Knopp-type inequalities. We not only partially generalize the sufficient parts of Persson-Stepanov's and Wedestig's results, but we also provide new proofs to these results.

MSC: 47A30; 26D10; 26D15

Keywords: operator norm; integral operator; Hardy-Knopp-type inequalities; Pólya-Knopp-type inequalities

1 Introduction

Let E be a spherical cone in $\mathbb{R}^n$. By this, we mean that $E = \bigcup_{s>0} sA$ for some Borel measurable subset A of the unit sphere Σ^{n-1}. Let $\|\mathbb{K}\|_{D_{\mathbb{K}}\cap L^p_\Phi(v\,dx)\mapsto L^q_\Phi(u\,dx)}$ (in brief, $\|\mathbb{K}\|_*$) denote the smallest constant C in (1.1):

$$\left\{\int_E \left(\Phi\circ\mathbb{K}f(x)\right)^q u(x)\,dx\right\}^{1/q} \le C\left\{\int_E \left(\Phi\circ f(x)\right)^p v(x)\,dx\right\}^{1/p} \tag{1.1}$$

for all $f \in D_{\mathbb{K}} \cap L^p_\Phi(v\,dx)$, where $p, q > 0$, $u(x) \ge 0$, $v(x) > 0$, $\Phi \in CV^+(I)$, $\Phi \circ f(x) = \Phi(f(x))$, and $\mathbb{K}f(x)$ is of the form

$$\mathbb{K}f(x) := \int_{\tilde{S}_x} k(x,t)f(t)\,dt \quad (x \in E). \tag{1.2}$$

Here $CV^+(I)$ denotes the set of all nonnegative convex functions defined on an open interval I in $\mathbb{R}$, $D_{\mathbb{K}}$ is the space of those f such that $\mathbb{K}f(x)$ is well defined for almost all $x \in E$, and $L^p_\Phi(v\,dx)$ is the set of all real-valued Borel measurable f with

$$\|f\|_{\Phi,p,v} := \left\{\int_E \left(\Phi\circ f(x)\right)^p v(x)\,dx\right\}^{1/p} < \infty.$$

Moreover, $\tilde{S}_x = \bigcup_{0<s\le\|x\|} sA$, $S_x = \tilde{S}_x \setminus \|x\|A$, and $k(x,t) \ge 0$ is locally integrable over $\mathbb{E}\times\mathbb{E}$.

We write $L^p(\nu\,dx)$ and $\|f\|_{p,\nu}$ instead of $L^p_\Phi(\nu\,dx)$ and $\|f\|_{\Phi,p,\nu}$, respectively, for the case $\Phi(s)=|s|$. We also write $L^p(E,\nu\,dx)$ for $L^p(\nu\,dx)$, whenever the integral region E is emphasized.

Clearly,

$$\|\mathbb{K}\|_* = \sup_f \frac{\|\Phi\circ\mathbb{K}f\|_{q,u}}{\|\Phi\circ f\|_{p,\nu}},$$

where the supremum is taken over all $f\in D_{\mathbb{K}}\cap L^p_\Phi(\nu\,dx)$ with $\|\Phi\circ f\|_{p,\nu}\neq 0$. This number reduces to the operator norm of $\mathbb{K}$ for the case $\Phi(s)=|s|$. The investigation of the value $\|\mathbb{K}\|_*$ has a long history in the literature. In [1], the present authors introduced a generalized Muckenhoupt constant $A_M(p,q)$ and established the following Muckenhoupt-type estimate for $\|\mathbb{K}\|_*$:

$$\|\mathbb{K}\|_* \le \left(\frac{q}{p^*}+\frac{q}{\eta}\right)^{1/q}\left(1+\frac{p^*}{\eta}\right)^{\eta^*/(p^*q^*)} A_M(p,q), \tag{1.3}$$

where $1\le p,q\le\infty$, $\eta=\max(p,q)$, and $(\cdot)^*$ is the conjugate exponent of $(\cdot)$ in the sense that $1/(\cdot)+1/(\cdot)^*=1$. For the particular case that

$$\Phi(s)=|s|, \qquad k(x,t)=1, \tag{1.4}$$

there are two other types of estimates. They are

$$\|\mathbb{K}\|_* \le p^* A_{PS}(p,q) \tag{1.5}$$

and

$$\|\mathbb{K}\|_* \le A_W(p,q) := \inf_{1<s<p} A_W(s,p,q)\left(\frac{p-1}{p-s}\right)^{1/p^*}. \tag{1.5a}$$

These two inequalities were proved in [2] and [3], Theorem 3.1 and Lemma 7.4, for the case $1<p\le q<\infty$ (see also [4], Theorem 2.1). We refer the readers to Section 2 for details.

In this paper, we focus on the evaluation of $\|\mathbb{K}\|_*$ for the following case of (1.1):

$$\Phi(s)=e^s, \qquad k(x,t)=g(t)/G(x), \qquad f(t)\longrightarrow \log f(t),$$

where $f(t)>0$, $g(t)>0$, and

$$G(x)=\int_{\tilde{S}_x} g(t)\,dt \quad (x\in E). \tag{1.6}$$

The corresponding inequality to (1.1) takes the form

$$\left(\int_E\left\{\exp\left(\frac{1}{G(x)}\int_{\tilde{S}_x} g(t)\log f(t)\,dt\right)\right\}^q u(x)\,dx\right)^{1/q} \le C\left\{\int_E \bigl(f(x)\bigr)^p \nu(x)\,dx\right\}^{1/p}, \tag{1.7}$$

which is known as the Pólya-Knopp-type inequality.

In [4], Theorem 3.1, [2, 5], and [3], Theorem 7.3, the particular case $g(t) = 1$ of (1.7) was considered. They obtained the following estimates by means of the formula $(G_{\mathbb{K}}f)(x) = \lim_{\epsilon\to 0^+}[\mathbb{K}(f^\epsilon)]^{1/\epsilon}(x)$:

$$\|\mathbb{K}\|_* \le e^{1/p}D^*_{PS} \quad \text{and} \quad \|\mathbb{K}\|_* \le \inf_{s>1} e^{(s-1)/p}D^*_{OG}(s), \tag{1.8}$$

where $0 < p \le q < \infty$. The definitions of D^*_{PS} and $D^*_{OG}(s)$ are given in Section 3.

The purpose of this paper is two-fold. We not only extend the aforementioned sufficient parts of [2, 4, 5], and [3] from $u(x) > 0$ and $g(t) = 1$ to $u(x) \ge 0$ and

$$\min\left(\sup_{x\in E}\bigl|g(x)\bigr|, \sup_{x\in E}\left|\frac{g(x)}{v(x)}\right|\right) < \infty, \tag{1.9}$$

but we also provide a new proof of (1.8) from the viewpoint of (1.10):

$$\|\mathbb{K}\|_* \le \inf_{\epsilon\in\mathfrak{F}^+_\Phi}(A_{p/\epsilon,q/\epsilon})^{1/\epsilon} \le \liminf_{\epsilon\to 0^+}\bigl\{(A_{p/\epsilon,q/\epsilon})^{1/\epsilon}\bigr\}, \tag{1.10}$$

where $0 < p, q < \infty$, $\mathfrak{F}^+_\Phi = \{\epsilon > 0 : \Phi^\epsilon \in CV^+(I)\}$, and $A_{p,q}$ are absolute constants subject to the condition

$$\left(\int_E \bigl|\mathbb{K}f(x)\bigr|^q u(x)\,dx\right)^{1/q} \le A_{p,q}\left(\int_E \bigl|f(x)\bigr|^p v(x)\,dx\right)^{1/p} \quad (f \ge 0). \tag{1.11}$$

It is clear that (1.10) is applicable to the case $\Phi(s) = e^s$. In this case, $\mathfrak{F}^+_\Phi = \{\epsilon > 0\}$ and the second inequality in (1.10) holds. We remark that it may not be an equality (*cf.* [6]). On the other hand, we have $p/\epsilon \to \infty$ and $q/\epsilon \to \infty$ as $\epsilon \to 0^+$. This indicates that the infimum in (1.10) can be estimated by evaluating those $A_{p,q}$ with p, q large enough.

The limit process (1.10) differs from the scheme by means of the formula $(G_{\mathbb{K}}f)(x) = \lim_{\epsilon\to 0^+}[\mathbb{K}(f^\epsilon)]^{1/\epsilon}(x)$. It was introduced in [6] to get different types of Pólya-Knopp inequalities, including the n-dimensional extensions of the Levin-Cochran-Lee-type inequalities and Carleson's result. We showed that the infimum in (1.10) can easily be evaluated by applying the following choice of $A_{p,q}$ for $1 < p, q < \infty$:

$$A_{p,q} \le \left(\frac{q}{p^*} + \frac{q}{\eta}\right)^{1/q}\left(1 + \frac{p^*}{\eta}\right)^{\eta^*/(p^*q^*)} A_M(p,q).$$

This choice is due to (1.3). We also pointed out that for some cases, the values of $\|\mathbb{K}\|_*$ obtained from (1.10) are better than the known constants in the literature. In this paper, we consider two other choices of $A_{p,q}$ with $1 < p \le q < \infty$, that is, $A_{p,q} \le p^*\tilde{A}_{PS}(p,q)$ and $A_{p,q} \le \tilde{A}_W(p,q)$, which are general forms of (1.5) and (1.5a). We shall derive them from (1.5) and (1.5a) and relax the conditions on $u(x)$ and $g(t)$ from $u(x) > 0$ and $g(t) = 1$ to $u(x) \ge 0$ and $g(t) > 0$ (*cf.* Section 2). Based on such choices, we prove that (1.8) follows from (1.10). Moreover, (1.8) can be extended from $u(x) > 0$ and $g(t) = 1$ to $u(x) \ge 0$ and $g(t)$ of the form (1.9). This extension gives Persson-Stepanov-type and Opic-Gurka-tpye estimates of the modular-type operator norm of the general geometric mean operator corresponding to $g(t)$. We remark that the particular case $g(t) = |\tilde{S}_t|^{s-1}$ can lead us to the Levin-Cochran-Lee-type inequality (see Section 3 for details).

2 General forms of (1.5) and (1.5a)

Let $1 < p \le q < \infty$, $g(t) > 0$, $u(x) \ge 0$, and $v(x) > 0$. Consider the inequality:

$$\left(\int_E \left\{\frac{1}{G(x)} \int_{\tilde{S}_x} g(t)f(t)\,dt\right\}^q u(x)\,dx\right)^{1/q} \le C\left(\int_E \bigl(f(x)\bigr)^p v(x)\,dx\right)^{1/p} \quad (f \ge 0), \tag{2.1}$$

where $G(x)$ is defined by (1.6). This corresponds to the case $\Phi(s) = |s|$ and $k(x,t) = g(t)/G(x)$ of (1.1). Inequality (2.1) reduces to the form (2.2) for the case $g(t) = 1$:

$$\left(\int_E \left\{\int_{\tilde{S}_x} f(t)\,dt\right\}^q \tilde{u}(x)\,dx\right)^{1/q} \le C\left(\int_E \bigl(f(x)\bigr)^p v(x)\,dx\right)^{1/p} \quad (f \ge 0), \tag{2.2}$$

where $\tilde{u}(x) = u(x)/G(x)^q$. In [4], Theorem 2.1, [2] and [3], Lemma 7.4(a), it was proved that under the conditions $u(x) > 0$ and $A_{PS}(p,q) < \infty$, (1.5) holds, in other words, (2.2) with $\tilde{u}(x)$ replaced by $u(x)$ is true for $C = p^* A_{PS}(p,q)$, where

$$A_{PS}(p,q) := \sup_{x \in E}\left(\int_{\tilde{S}_x} v(t)^{1-p^*}\,dt\right)^{-1/p}\left(\int_{\tilde{S}_x}\left\{\int_{\tilde{S}_t} v(y)^{1-p^*}\,dy\right\}^q u(t)\,dt\right)^{1/q}.$$

This result will be extended below from $g(t) = 1$ and $u(x) > 0$ to $g(t) > 0$ and $u(x) \ge 0$. We shall see its application in the proof of Theorem 3.2.

Theorem 2.1 *Let* $1 < p \le q < \infty$, $u(x) \ge 0$, $v(x) > 0$, $g(t) > 0$, *and* $0 < G(x) < \infty$, *where* $G(x)$ *is defined by* (1.6). *If* $\tilde{A}_{PS}(p,q) < \infty$, *then* (2.1) *holds for* $C \le p^* \tilde{A}_{PS}(p,q)$, *where*

$$\tilde{A}_{PS}(p,q) = \sup_{x \in E}\left(\int_{\tilde{S}_x}\left(\frac{g(t)}{v(t)}\right)^{p*} v(t)\,dt\right)^{\frac{-1}{p}}\left(\int_{\tilde{S}_x}\left\{\frac{1}{G(t)}\int_{\tilde{S}_t}\left(\frac{g(y)}{v(y)}\right)^{p*} v(y)\,dy\right\}^q u(t)\,dt\right)^{\frac{1}{q}}.$$

Proof The case $u(x) > 0$ follows from [4], Theorem 2.1, or [3], Lemma 7.4(a), under the following substitutions:

$$f(t) \longrightarrow g(t)f(t), \qquad u(x) \longrightarrow \frac{u(x)}{(G(x))^q}, \qquad v(x) \longrightarrow \frac{v(x)}{(g(x))^p}. \tag{2.3}$$

As for $u(x) \ge 0$, let $u_\tau(x) = u(x) + \rho_\tau(x)$, where $0 < \tau < 1$ and $\rho_\tau(x) > 0$ is subject to the condition

$$\int_{\tilde{S}_x}\left\{\frac{1}{G(t)}\int_{\tilde{S}_t}\left(\frac{g(y)}{v(y)}\right)^{p^*} v(y)\,dy\right\}^q \rho_\tau(t)\,dt \le \tau\left\{\int_{\tilde{S}_x}\left(\frac{g(t)}{v(t)}\right)^{p^*} v(t)\,dt\right\}^{q/p}. \tag{2.4}$$

Such $\rho_\tau(x)$ exists. We have $u_\tau(x) > 0$ on E. Moreover, the condition $1/q < 1$ implies that $(a+b)^{1/q} \le a^{1/q} + b^{1/q}$ for all $a, b \ge 0$. Putting this together with (2.4) yields

$$\begin{aligned}&\left(\int_{\tilde{S}_x}\left\{\frac{1}{G(t)}\int_{\tilde{S}_t}\left(\frac{g(y)}{v(y)}\right)^{p^*} v(y)\,dy\right\}^q u_\tau(t)\,dt\right)^{1/q}\\ &\quad\le\left(\int_{\tilde{S}_x}\left\{\frac{1}{G(t)}\int_{\tilde{S}_t}\left(\frac{g(y)}{v(y)}\right)^{p^*} v(y)\,dy\right\}^q u(t)\,dt\right)^{\frac{1}{q}} + \tau^{\frac{1}{q}}\left\{\int_{\tilde{S}_x}\left(\frac{g(t)}{v(t)}\right)^{p^*} v(t)\,dt\right\}^{\frac{1}{p}}.\end{aligned}$$

This leads us to

$$\tilde{A}_{PS}(p,q,\tau) \le \tilde{A}_{PS}(p,q) + \tau^{1/q} < \infty, \tag{2.5}$$

where $\tilde{A}_{PS}(p,q,\tau)$ is the number obtained from $\tilde{A}_{PS}(p,q)$ by replacing $u(t)$ by $u_r(t)$. We have $u_\tau(x) > u(x)$ on E. By the result of the case $u(x) > 0$, the following inequality holds for $f \ge 0$:

$$\begin{aligned}
&\left(\int_E \left\{\frac{1}{G(x)}\int_{\tilde{S}_x} g(t)f(t)\,dt\right\}^q u(x)\,dx\right)^{\frac{1}{q}} \\
&\quad \le \left(\int_E \left\{\frac{1}{G(x)}\int_{\tilde{S}_x} g(t)f(t)\,dt\right\}^q u_\tau(x)\,dx\right)^{\frac{1}{q}} \\
&\quad \le p^* \tilde{A}_{PS}(p,q,\tau)\left(\int_E \bigl(f(x)\bigr)^p v(x)\,dx\right)^{\frac{1}{p}}.
\end{aligned} \tag{2.6}$$

It follows from (2.5) that $\liminf_{\tau\to 0^+} \tilde{A}_{PS}(p,q,\tau) \le \tilde{A}_{PS}(p,q)$. Putting this together with (2.6) yields the desired inequality. The proof is complete. □

Next, consider (1.5a). The number $A_W(s,p,q)$ in (1.5a) is defined by the formula:

$$A_W(s,p,q) = \sup_{x\in E}\left(\int_{\tilde{S}_x} v(t)^{1-p*}\,dt\right)^{\frac{s-1}{p}} \left(\int_{E\setminus S_x}\left\{\int_{\tilde{S}_t} v(y)^{1-p*}\,dy\right\}^{\frac{q(p-s)}{p}} u(t)\,dt\right)^{\frac{1}{q}}.$$

In [3], Lemma 7.4(b), $A_W(s,p,q)$ is replaced by another notation $A_W^*(s)$. Like (1.5), (1.5a) can be generalized in the following way, in which $g(t) = 1$ and $u(x) > 0$ are relaxed to $g(t) > 0$ and $u(x) \ge 0$. We shall see its application in the proof of Theorem 3.3.

Theorem 2.2 *Let* $1 < p \le q < \infty$, $u(x) \ge 0$, $v(x) > 0$, $g(t) > 0$, *and* $0 < G(x) < \infty$, *where* $G(x)$ *is defined by* (1.6). *If* $\tilde{A}_W(s,p,q) < \infty$ *for some* $1 < s < p$, *then* (2.1) *holds for* $C \le \tilde{A}_W(p,q)$, *where*

$$\tilde{A}_W(p,q) := \inf_{1<s<p} \tilde{A}_W(s,p,q)\left(\frac{p-1}{p-s}\right)^{1/p^*} \tag{2.7}$$

and

$$\begin{aligned}
\tilde{A}_W(s,p,q) = {}& \sup_{x\in E}\left(\int_{\tilde{S}_x}\left(\frac{g(t)}{v(t)}\right)^{p^*} v(t)\,dt\right)^{\frac{s-1}{p}} \\
&\times \left(\int_{E\setminus S_x}\left\{\int_{\tilde{S}_t}\left(\frac{g(y)}{v(y)}\right)^{p^*} v(y)\,dy\right\}^{\frac{q(p-s)}{p}} \frac{u(t)\,dt}{(G(t))^q}\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.8}$$

Proof The case $u(x) > 0$ follows from [3], Lemma 7.4(b), under the substitutions (2.3). For the case $u(x) \ge 0$, we modify the proof of Theorem 2.1 in the following way. Let $1 < s < p$ and $0 < \tau < 1$. Set $u_\tau(x,s) = u(x) + \rho_\tau(x,s)$, where $\rho_\tau(x,s) > 0$ and satisfies the condition

$$\int_{E\setminus S_x}\left\{\int_{\tilde{S}_t}\left(\frac{g(y)}{v(y)}\right)^{p^*} v(y)\,dy\right\}^{\frac{q(p-s)}{p}} \frac{\rho_\tau(t,s)}{(G(t))^q}\,dt \le \tau\left(\frac{p-1}{p-s}\right)^{\frac{-q}{p^*}}\left\{\int_{\tilde{S}_x}\left(\frac{g(t)}{v(t)}\right)^{p^*} v(t)\,dt\right\}^{\frac{q(1-s)}{p}}.$$

Such $\rho_\tau(x,s)$ exists. We have $u_\tau(x,s) > 0$ on $x \in E$. Moreover,

$$\tilde{A}_W^\tau(s,p,q) \le \tilde{A}_W(s,p,q) + \tau^{1/q}\left(\frac{p-1}{p-s}\right)^{-1/p^*}, \tag{2.9}$$

where $\tilde{A}_W^\tau(s,p,q)$ is obtained from $\tilde{A}_W(s,p,q)$ by making the change in (2.8): $u(t) \longrightarrow u_\tau(t,s)$. Obviously, $u_\tau(x,s) > u(x)$. Applying the preceding result of the case $u(x) > 0$ to $u_\tau(x,s)$, we get

$$\begin{aligned}
&\left(\int_E \left\{\frac{1}{G(x)}\int_{\tilde{S}_x} g(t)f(t)\,dt\right\}^q u(x)\,dx\right)^{1/q} \\
&\quad \le \left(\int_E \left\{\frac{1}{G(x)}\int_{\tilde{S}_x} g(t)f(t)\,dt\right\}^q u_\tau(x,s)\,dx\right)^{1/q} \\
&\quad \le \left\{\inf_{1<s'<p} \tilde{A}_W^\tau(s',p,q)\left(\frac{p-1}{p-s'}\right)^{1/p^*}\right\}\left(\int_E \bigl(f(x)\bigr)^p v(x)\,dx\right)^{1/p} \\
&\quad \le \tilde{A}_W^\tau(s,p,q)\left(\frac{p-1}{p-s}\right)^{1/p^*}\left(\int_E \bigl(f(x)\bigr)^p v(x)\,dx\right)^{1/p}.
\end{aligned} \tag{2.10}$$

Taking '$\inf_{1<s<p}$' for both sides of (2.10), we get

$$\left(\int_E \left\{\frac{1}{G(x)}\int_{\tilde{S}_x} g(t)f(t)\,dt\right\}^q u(x)\,dx\right)^{1/q} \le \tilde{A}_W^\tau(p,q)\left(\int_E \bigl(f(x)\bigr)^p v(x)\,dx\right)^{1/p}. \tag{2.11}$$

Here

$$\tilde{A}_W^\tau(p,q) = \inf_{1<s<p} \tilde{A}_W^\tau(s,p,q)\left(\frac{p-1}{p-s}\right)^{1/p^*}.$$

From (2.9), we obtain $\tilde{A}_W^\tau(p,q) \le \tilde{A}_W(p,q) + \tau^{1/q}$. Taking $\tau \to 0^+$ for both sides of (2.11), we get the desired inequality. This completes the proof. □

3 Extensions and new proofs of (1.8)

To derive the extensions of (1.8), we need the following lemma.

Lemma 3.1 *Let* $0 < p < \infty$, $v(x) > 0$, $g(t) > 0$, *and* $0 < G(x) < \infty$, *where* $G(x)$ *is defined by* (1.6). *If* $\sup_{x\in E}\{g(x)/v(x)\} < \infty$, *then, for all* $t \in E$,

$$\lim_{\epsilon\to 0^+}\left(\frac{1}{G(t)}\int_{\tilde{S}_t}\left(\frac{g(y)}{v(y)}\right)^{\frac{\epsilon}{p-\epsilon}} g(y)\,dy\right)^{\frac{1}{\epsilon}} = \left\{\exp\left(\frac{1}{G(t)}\int_{\tilde{S}_t} g(y)\left(\log\frac{g(y)}{v(y)}\right)dy\right)\right\}^{\frac{1}{p}}. \tag{3.1}$$

Proof Let $\alpha \ge \sup_{x\in E}\{g(x)/v(x)\}$. Without loss of generality, we may assume $\alpha > 1$. We first consider the case that $\int_{\tilde{S}_t} g(y)|\log(\frac{g(y)}{v(y)})|\,dy < \infty$. Let

$$h(\epsilon) = \frac{1}{G(t)}\int_{\tilde{S}_t}\left(\frac{g(y)}{v(y)}\right)^{\epsilon/(p-\epsilon)} g(y)\,dy \quad (0 \le \epsilon < p/2).$$

We have

$$\int_{\tilde{S}_t} \left(\frac{g(y)}{\nu(y)}\right)^{\epsilon/(p-\epsilon)} g(y)\,dy \le \alpha^{\epsilon/(p-\epsilon)} G(t) < \infty,$$

so $h(\epsilon)$ is well defined and has a finite value. For $\epsilon \in [0,p/2)$ and $0 < \tau < \min(p/2-\epsilon,\epsilon)$, it follows from the mean value theorem that

$$\begin{aligned}\frac{h(\epsilon+\tau)-h(\epsilon)}{\tau} &= \frac{1}{G(t)}\int_{\tilde{S}_t} \frac{1}{\tau}\left\{\left(\frac{g(y)}{\nu(y)}\right)^{\frac{\epsilon+\tau}{p-\epsilon-\tau}} - \left(\frac{g(y)}{\nu(y)}\right)^{\frac{\epsilon}{p-\epsilon}}\right\} g(y)\,dy \\ &= \frac{p}{G(t)}\int_{\tilde{S}_t} \frac{1}{(p-\epsilon_0)^2}\left(\frac{g(y)}{\nu(y)}\right)^{\epsilon_0/(p-\epsilon_0)} g(y)\left(\log\frac{g(y)}{\nu(y)}\right)dy, \end{aligned} \tag{3.2}$$

where $\epsilon_0 := \epsilon_0(y)$ lies between ϵ and $\epsilon+\tau$. We know that

$$\frac{\chi_{\tilde{S}_t}(y)}{(p-\epsilon_0)^2}\left(\frac{g(y)}{\nu(y)}\right)^{\epsilon_0/(p-\epsilon_0)} g(y)\left|\log\left(\frac{g(y)}{\nu(y)}\right)\right| \le \frac{\alpha\chi_{\tilde{S}_t}(y)g(y)}{(p-\epsilon)^2}\left|\log\left(\frac{g(y)}{\nu(y)}\right)\right| \in L^1(E,dy).$$

By (3.2) and the Lebesgue dominated convergence theorem, h is differentiable on $[0,p/2)$. In addition,

$$h'(\epsilon) = \lim_{\tau\to 0^+}\frac{h(\epsilon+\tau)-h(\epsilon)}{\tau} = \frac{p}{(p-\epsilon)^2 G(t)}\int_{\tilde{S}_t}\left(\frac{g(y)}{\nu(y)}\right)^{\epsilon/(p-\epsilon)} g(y)\left(\log\frac{g(y)}{\nu(y)}\right)dy.$$

Thus,

$$\begin{aligned}&\lim_{\epsilon\to 0^+}\log\left(\frac{1}{G(t)}\int_{\tilde{S}_t}\left(\frac{g(y)}{\nu(y)}\right)^{\epsilon/(p-\epsilon)} g(y)\,dy\right)^{1/\epsilon} \\ &\quad = \lim_{\epsilon\to 0^+}\frac{\log h(\epsilon)-\log h(0)}{\epsilon} \\ &\quad = \frac{d}{d\epsilon}\left(\log h(\epsilon)\right)\Big|_{\epsilon=0} = \frac{h'(0)}{h(0)} = \frac{1}{pG(t)}\int_{\tilde{S}_t} g(y)\left(\log\frac{g(y)}{\nu(y)}\right)dy.\end{aligned}$$

We get the desired result for the case $\int_{\tilde{S}_t} g(y)|\log(\frac{g(y)}{\nu(y)})|\,dy < \infty$. Next, consider the case $\int_{\tilde{S}_t} g(y)|\log(\frac{g(y)}{\nu(y)})|\,dy = \infty$. This implies

$$\infty = \int_{\Omega_1} g(y)\left|\log\left(\frac{g(y)}{\nu(y)}\right)\right|dy + \int_{\Omega_2} g(y)\left|\log\left(\frac{g(y)}{\nu(y)}\right)\right|dy, \tag{3.3}$$

where $\Omega_1 = \{y \in \tilde{S}_t : g(y)/\nu(y) \le 1\}$ and $\Omega_2 = \{y \in \tilde{S}_t : g(y)/\nu(y) > 1\}$. We have

$$\int_{\Omega_2} g(y)\left|\log\left(\frac{g(y)}{\nu(y)}\right)\right|dy \le (\log\alpha)G(t) < \infty.$$

Combining this with (3.3), we find that $\int_{\Omega_1} g(y)|\log(\frac{g(y)}{\nu(y)})|\,dy = \infty$. This leads us to

$$\int_{\tilde{S}_t} g(y)\left(\log\frac{g(y)}{\nu(y)}\right)dy = -\int_{\Omega_1} g(y)\left|\log\left(\frac{g(y)}{\nu(y)}\right)\right|dy + \int_{\Omega_2} g(y)\left|\log\left(\frac{g(y)}{\nu(y)}\right)\right|dy = -\infty.$$

We shall show

$$\lim_{\epsilon\to 0^+}\left(\frac{1}{G(t)}\int_{\tilde{S}_t}\left(\frac{g(y)}{\nu(y)}\right)^{\epsilon/(p-\epsilon)} g(y)\,dy\right)^{1/\epsilon} = 0.$$

If so, the desired equality follows. Let $0 < \epsilon < p/2$ and $y \in \tilde{S}_t$. By the mean value theorem, we get

$$\left(\frac{g(y)}{\nu(y)}\right)^{\epsilon/(p-\epsilon)} - 1 = \frac{\epsilon p}{(p-\epsilon_0)^2}\left(\frac{g(y)}{\nu(y)}\right)^{\epsilon_0/(p-\epsilon_0)}\left(\log\frac{g(y)}{\nu(y)}\right)$$

for some $\epsilon_0 \in (0,\epsilon)$. This implies

$$\begin{aligned}
&\frac{1}{G(t)}\int_{\tilde{S}_t}\left(\frac{g(y)}{\nu(y)}\right)^{\epsilon/(p-\epsilon)} g(y)\,dy \\
&\quad = 1 + \left(\frac{\epsilon p}{G(t)}\int_{\tilde{S}_t}\frac{1}{(p-\epsilon_0)^2}\left(\frac{g(y)}{\nu(y)}\right)^{\epsilon_0/(p-\epsilon_0)} g(y)\left(\log\frac{g(y)}{\nu(y)}\right)dy\right).
\end{aligned} \tag{3.4}$$

By Fatou's lemma, we get

$$\begin{aligned}
&\liminf_{\epsilon\to 0^+}\frac{p}{G(t)}\int_{\tilde{S}_t}\frac{1}{(p-\epsilon_0)^2}\left(\frac{g(y)}{\nu(y)}\right)^{\epsilon_0/(p-\epsilon_0)} g(y)\left|\log\left(\frac{g(y)}{\nu(y)}\right)\right|dy \\
&\quad \geq \frac{1}{pG(t)}\int_{\tilde{S}_t}\left\{\liminf_{\epsilon\to 0^+}\left(\frac{g(y)}{\nu(y)}\right)^{\epsilon_0/(p-\epsilon_0)}\right\} g(y)\left|\log\left(\frac{g(y)}{\nu(y)}\right)\right|dy \\
&\quad = \frac{1}{pG(t)}\int_{\tilde{S}_t} g(y)\left|\log\left(\frac{g(y)}{\nu(y)}\right)\right|dy = \infty.
\end{aligned}$$

Like (3.3), decompose the integral $\int_{\tilde{S}_t}(\cdots)$ as the sum $\int_{\Omega_1}(\cdots) + \int_{\Omega_2}(\cdots)$. For the Ω_2 term, we have

$$\begin{aligned}
&\frac{p}{G(t)}\int_{\Omega_2}\frac{1}{(p-\epsilon_0)^2}\left(\frac{g(y)}{\nu(y)}\right)^{\epsilon_0/(p-\epsilon_0)} g(y)\left|\log\left(\frac{g(y)}{\nu(y)}\right)\right|dy \\
&\quad \leq \frac{4\alpha\log\alpha}{pG(t)}\int_{\Omega_2} g(y)\,dy \leq \frac{4\alpha\log\alpha}{p} < \infty,
\end{aligned}$$

which implies

$$\lim_{\epsilon\to 0^+}\frac{p}{G(t)}\int_{\tilde{S}_t}\frac{1}{(p-\epsilon_0)^2}\left(\frac{g(y)}{\nu(y)}\right)^{\epsilon_0/(p-\epsilon_0)} g(y)\left(\log\frac{g(y)}{\nu(y)}\right)dy = -\infty.$$

From (3.4) and the fact that $\lim_{\epsilon\to 0}(1+\epsilon\theta)^{1/\epsilon} = e^\theta$ for any $\theta \in \mathbb{R}$, we get

$$\limsup_{\epsilon\to 0^+}\left(\frac{1}{G(t)}\int_{\tilde{S}_t}\left(\frac{g(y)}{\nu(y)}\right)^{\epsilon/(p-\epsilon)} g(y)\,dy\right)^{1/\epsilon} \leq \limsup_{\epsilon\to 0^+}(1+\epsilon\theta)^{1/\epsilon} = e^\theta$$

for any $\theta < 0$. Letting $\theta \to -\infty$, we get the desired result. □

Lemma 3.1 may be false for the case that $\sup_{x\in E} g(x)/v(x) = \infty$. A counterexample is given as follows. Consider $n = 1$, $t = 1$, $g(t) = 1$, and $v(x) = \sum_{m=2}^{\infty} e^{-m}\chi_{(\frac{1}{m}-\frac{1}{m^3},\frac{1}{m}]}(x) + \chi_{\mathbb{R}\setminus\bigcup_{m\geq 2}(\frac{1}{m}-\frac{1}{m^3},\frac{1}{m}]}(x)$. We have

$$\int_0^1 \left(\frac{g(y)}{v(y)}\right)^{\epsilon/(p-\epsilon)} g(y)\,dy = \int_0^1 v(y)^{\epsilon/(\epsilon-p)}\,dy \geq \sum_{m=2}^{\infty} \frac{1}{m^3} e^{\frac{m\epsilon}{p-\epsilon}} = \infty \quad (0 < \epsilon < p/2)$$

and

$$\int_0^1 g(y)\left(\log\frac{g(y)}{v(y)}\right)dy = \int_0^1 \log\frac{1}{v(y)}\,dy = \sum_{m=2}^{\infty}\frac{1}{m^2} < \infty.$$

From these, we know that (3.1) is false for this example.

Now, we go back to the investigation of the first part of (1.8). Set

$$\tilde{D}_{PS} := \sup_{x\in E} \frac{1}{G(x)^{\frac{1}{p}}}\left(\int_{\tilde{S}_x}\left\{\exp\left(\frac{1}{G(t)}\int_{\tilde{S}_t} g(y)\left(\log\frac{g(y)}{v(y)}\right)dy\right)\right\}^{\frac{q}{p}} u(t)\,dt\right)^{\frac{1}{q}},$$

where $G(x)$ is defined by (1.6). The case $g(t) = 1$ of $\tilde{D}_{PS}$ reduces to D_{PS}^* mentioned in (1.8). We shall establish the following result, which extends the first inequality in (1.8) from $u(x) > 0$ and $g(t) = 1$ to $u(x) \geq 0$ and those $g(t)$ subject to the condition (1.9). This extension gives the Persson-Stepanov-type estimate of the modular-type operator norm of the general geometric mean operator corresponding to $g(t)$. In particular, $g(t)$ can be of the form $g(t) = |\tilde{S}_t|^{s-1}$. An elementary calculation of this case will lead us to the Levin-Cochran-Lee-type inequality. We leave such a calculation to the readers. Our result partially generalizes the sufficient parts of [4], Theorem 3.1, [2], and [3], Theorem 7.3(a).

Theorem 3.2 *Let* $0 < p \leq q < \infty$, $u(x) \geq 0$, $v(x) > 0$, $g(t) > 0$, *and* $0 < G(x) < \infty$, *where* $G(x)$ *is defined by* (1.6). *If* (1.9) *is true and* $\tilde{D}_{PS} < \infty$, *then* (1.7) *holds for* $C \leq e^{1/p}\tilde{D}_{PS}$.

Proof Let $\Phi(s) = e^s$, $k(x,t) = g(t)/G(x)$, and $f(t) \longrightarrow \log f(t)$. The proof is the same as to prove that $\|\mathbb{K}\|_* \leq e^{1/p}\tilde{D}_{PS}$. We first assume that $\sup_{x\in E}\{g(x)/v(x)\} < \infty$. Consider the case that u is bounded on $\tilde{\Omega}_r$ and $u(x) = 0$ on $E\setminus\tilde{\Omega}_r$, where $r \geq 1$ and $\tilde{\Omega}_r = \{x \in E : 1/r \leq \|x\| \leq r\}$. By (1.10)-(1.11) and Theorem 2.1, we know that

$$\|\mathbb{K}\|_* \leq \liminf_{\epsilon\to 0^+}\big((p/\epsilon)^* \tilde{A}_{PS}(p/\epsilon, q/\epsilon)\big)^{1/\epsilon}, \tag{3.5}$$

provided that the term $(\cdots)^{1/\epsilon}$ in (3.5) is finite for all sufficiently small $\epsilon > 0$. By an elementary calculation, we obtain $\lim_{\epsilon\to 0^+}((p/\epsilon)^*)^{1/\epsilon} = \lim_{\epsilon\to 0^+}(\frac{p}{p-\epsilon})^{1/\epsilon} = e^{1/p}$. On the other hand, let $0 < \epsilon < p$. Then $p/\epsilon > 1$ and $q/\epsilon > 1$. Moreover, we have $(p/\epsilon)^* = p/(p-\epsilon)$, so

$$\left(\frac{g(t)}{v(t)}\right)^{(p/\epsilon)^*} v(t) = \left(\frac{g(t)}{v(t)}\right)^{p/(p-\epsilon)} v(t) = \left(\frac{g(t)}{v(t)}\right)^{\epsilon/(p-\epsilon)} g(t).$$

It follows from the definition of $\tilde{A}_{PS}(p/\epsilon, q/\epsilon)$ that

$$\begin{aligned}\left(\tilde{A}_{PS}(p/\epsilon, q/\epsilon)\right)^{1/\epsilon} = \sup_{x\in E}&\left(\int_{\tilde{S}_x}\left(\frac{g(t)}{\nu(t)}\right)^{\epsilon/(p-\epsilon)} g(t)\,dt\right)^{-1/p}\\ &\times\left(\int_{\tilde{S}_x}\left\{\frac{1}{G(t)}\int_{\tilde{S}_t}\left(\frac{g(y)}{\nu(y)}\right)^{\epsilon/(p-\epsilon)} g(y)\,dy\right\}^{q/\epsilon} u(t)\,dt\right)^{1/q}.\end{aligned}\tag{3.6}$$

We have assumed that $u(x) = 0$ on $E\setminus\tilde{\Omega}_r$. Moreover, for $t\in\tilde{S}_x$, we have

$$\begin{aligned}\frac{1}{G(t)}\int_{\tilde{S}_t}\left(\frac{g(y)}{\nu(y)}\right)^{\epsilon/(p-\epsilon)} g(y)\,dy &\le \left\{\sup_{y\in\tilde{S}_x}\left(\frac{g(y)}{\nu(y)}\right)\right\}^{\epsilon/(p-\epsilon)}\left\{\frac{1}{G(t)}\int_{\tilde{S}_t} g(y)\,dy\right\}\\ &= \left\{\sup_{y\in\tilde{S}_x}\left(\frac{g(y)}{\nu(y)}\right)\right\}^{\epsilon/(p-\epsilon)}.\end{aligned}$$

These imply

$$\begin{aligned}\left(\tilde{A}_{PS}(p/\epsilon, q/\epsilon)\right)^{1/\epsilon} \le &\left(\int_{\tilde{B}_{1/r}}\left(\frac{g(t)}{\nu(t)}\right)^{\epsilon/(p-\epsilon)} g(t)\,dt\right)^{-1/p}\\ &\times\left\{\sup_{y\in E}\left(\frac{g(y)}{\nu(y)}\right)\right\}^{1/(p-\epsilon)}\left(\int_{\tilde{\Omega}_r} u(t)\,dt\right)^{1/q} < \infty,\end{aligned}\tag{3.7}$$

where $\tilde{B}_\rho = \{x\in E : \|x\|\le\rho\}$. The above argument guarantees the validity of (3.5). Now, we try to estimate the limit infimum given in (3.5). It suffices to show that

$$\liminf_{\epsilon\to 0^+}\left(\tilde{A}_{PS}(p/\epsilon, q/\epsilon)\right)^{1/\epsilon} \le \tilde{D}_{PS}.\tag{3.8}$$

Clearly, the term $(\int_{\tilde{S}_x}(\cdots))^{-1/p}$ in (3.6) becomes bigger whenever x with $\|x\| > r$ is replaced by $rx/\|x\|$. Moreover, the term $(\int_{\tilde{S}_x}\{\cdots\}^{q/\epsilon} u(t)\,dt)^{1/q}$ in (3.6) is zero for $\|x\| < 1/r$ and it keeps the same value for the change: x with $\|x\| > r \longrightarrow rx/\|x\|$. Hence, the term '$\sup_{x\in E}$' in (3.6) can be replaced by '$\sup_{x\in\tilde{\Omega}_r}$'. By the Heine-Borel theorem, we can choose $0 < \epsilon_m < p/2$, $\alpha_m > 0$, and $x_0, x_m\in\tilde{\Omega}_r$, such that $\epsilon_m\to 0$, $\alpha_m\to 0$, $x_m\to x_0$, and the following inequality holds for all m:

$$\begin{aligned}&\left(\tilde{A}_{PS}(p/\epsilon_m, q/\epsilon_m)\right)^{1/\epsilon_m}\\ &\quad\le\left(\int_{\tilde{S}_{x_m}}\left(\frac{g(t)}{\nu(t)}\right)^{\epsilon_m/(p-\epsilon_m)} g(t)\,dt\right)^{-1/p}\\ &\quad\times\left(\int_{\tilde{S}_{x_m}}\left\{\frac{1}{G(t)}\int_{\tilde{S}_t}\left(\frac{g(y)}{\nu(y)}\right)^{\epsilon_m/(p-\epsilon_m)} g(y)\,dy\right\}^{q/\epsilon_m} u(t)\,dt\right)^{1/q} + \alpha_m.\end{aligned}\tag{3.9}$$

We have

$$\left|\chi_{\tilde{S}_{x_m}}(t)\left(\frac{g(t)}{\nu(t)}\right)^{\epsilon_m/(p-\epsilon_m)} g(t)\right| \le \chi_{\tilde{B}_r}(t)\left\{\sup_{y\in E}\left(\frac{g(y)}{\nu(y)}\right)+1\right\} g(t)\in L^1(E, dt)\quad (m = 1, 2, \ldots).$$

By the Lebesgue dominated convergence theorem, we infer that

$$\begin{aligned}
&\lim_{m\to\infty}\left(\int_{\tilde{S}_{x_m}}\left(\frac{g(t)}{v(t)}\right)^{\epsilon_m/(p-\epsilon_m)}g(t)\,dt\right)^{-1/p} \\
&\quad=\left(\int_{\tilde{S}_{x_0}}\lim_{m\to\infty}\left\{\left(\frac{g(t)}{v(t)}\right)^{\epsilon_m/(p-\epsilon_m)}\right\}g(t)\,dt\right)^{-1/p}=\bigl(G(x_0)\bigr)^{-1/p}. \qquad (3.10)
\end{aligned}$$

Similarly, the hypotheses on $u(t)$ and $g(t)/v(t)$ imply

$$\begin{aligned}
&\left|\chi_{\tilde{S}_{x_m}}(t)\left\{\frac{1}{G(t)}\int_{\tilde{S}_t}\left(\frac{g(y)}{v(y)}\right)^{\epsilon_m/(p-\epsilon_m)}g(y)\,dy\right\}^{q/\epsilon_m}u(t)\right| \\
&\quad\le\chi_{\tilde{B}_r}(t)\left\{\sup_{y\in E}\left(\frac{g(y)}{v(y)}\right)\right\}^{q/(p-\epsilon_m)}\left\{\frac{1}{G(t)}\int_{\tilde{S}_t}g(y)\,dy\right\}^{q/\epsilon_m}u(t) \\
&\quad\le\chi_{\tilde{B}_r}(t)\left\{\sup_{y\in E}\left(\frac{g(y)}{v(y)}\right)+1\right\}^{2q/p}u(t)\in L^1(E,dt).
\end{aligned}$$

Applying the Lebesgue dominated convergence theorem again, it follows from Lemma 3.1 that

$$\begin{aligned}
&\lim_{m\to\infty}\left(\int_{\tilde{S}_{x_m}}\left\{\frac{1}{G(t)}\int_{\tilde{S}_t}\left(\frac{g(y)}{v(y)}\right)^{\epsilon_m/(p-\epsilon_m)}g(y)\,dy\right\}^{q/\epsilon_m}u(t)\,dt\right)^{1/q} \\
&\quad=\left(\int_{\tilde{S}_{x_0}}\lim_{m\to\infty}\left\{\frac{1}{G(t)}\int_{\tilde{S}_t}\left(\frac{g(y)}{v(y)}\right)^{\epsilon_m/(p-\epsilon_m)}g(y)\,dy\right\}^{q/\epsilon_m}u(t)\,dt\right)^{1/q} \\
&\quad=\left(\int_{\tilde{S}_{x_0}}\left\{\exp\left(\frac{1}{G(t)}\int_{\tilde{S}_t}g(y)\left(\log\frac{g(y)}{v(y)}\right)dy\right)\right\}^{q/p}u(t)\,dt\right)^{1/q}. \qquad (3.11)
\end{aligned}$$

Putting (3.9)-(3.11) together yields (3.8). This finishes the proof for those u and v with the restrictions stated above. Now, we come back to the proof of the case $u \ge 0$ and $\sup_{x\in E}\{g(x)/v(x)\} < \infty$. Let $u_r(x) = \min\{u(x), r\}\chi_{\tilde{\Omega}_r}(x)$, where $r = 1, 2, \ldots$. By the preceding result,

$$\begin{aligned}
&\left(\int_E\left\{\exp\left(\frac{1}{G(x)}\int_{\tilde{S}_x}g(t)\log f(t)\,dt\right)\right\}^q u_r(x)\,dx\right)^{1/q} \\
&\quad\le e^{1/p}\tilde{D}_{PS}(r)\left(\int_E\bigl(f(x)\bigr)^p v(x)\,dx\right)^{1/p}\qquad(f>0), \qquad (3.12)
\end{aligned}$$

where

$$\tilde{D}_{PS}(r)=\sup_{x\in E}\bigl(G(x)\bigr)^{-\frac{1}{p}}\left(\int_{\tilde{S}_x}\left\{\exp\left(\frac{1}{G(t)}\int_{\tilde{S}_t}g(y)\left(\log\frac{g(y)}{v(y)}\right)dy\right)\right\}^{\frac{q}{p}}u_r(t)\,dt\right)^{\frac{1}{q}}.$$

We have $u_r(t) \le u(t)$, so $\tilde{D}_{PS}(r) \le \tilde{D}_{PS}$. Replacing $\tilde{D}_{PS}(r)$ in (3.12) by $\tilde{D}_{PS}$ first and then applying the monotone convergence theorem to (3.12), we get the desired inequality for this case.

Next, we deal with the case $\sup_{x\in E} g(x) < \infty$. Let $\nu_\ell(x) = \nu(x) + 1/\ell$, where $\ell = 1, 2, \ldots$. Then $\sup_{x\in E}\{g(x)/\nu_\ell(x)\} < \infty$ for each ℓ. By the preceding result,

$$\left(\int_E \left\{\exp\left(\frac{1}{G(x)}\int_{\tilde{S}_x} g(t)\log f(t)\,dt\right)\right\}^q u(x)\,dx\right)^{1/q} \le e^{1/p}\tilde{D}^\ell_{PS}\left(\int_E \bigl(f(x)\bigr)^p \nu_\ell(x)\,dx\right)^{1/p} \quad (f > 0), \tag{3.13}$$

where

$$\tilde{D}^\ell_{PS} = \sup_{x\in E}\frac{1}{(G(x))^{\frac{1}{p}}}\left(\int_{\tilde{S}_x}\left\{\exp\left(\frac{1}{G(t)}\int_{\tilde{S}_t} g(y)\log\left(\frac{g(y)}{\nu_\ell(y)}\right)dy\right)\right\}^{\frac{q}{p}} u(t)\,dt\right)^{\frac{1}{q}}.$$

We have $\nu_\ell(x) \ge \nu(x)$, so $\tilde{D}^\ell_{PS} \le \tilde{D}_{PS}$. This says that (3.13) can be replaced by (3.14):

$$\left(\int_E \left\{\exp\left(\frac{1}{G(x)}\int_{\tilde{S}_x} g(t)\log f(t)\,dt\right)\right\}^q u(x)\,dx\right)^{1/q} \le e^{1/p}\tilde{D}_{PS}\left(\int_E \bigl(f(x)\bigr)^p \nu_\ell(x)\,dx\right)^{1/p} \quad (f > 0). \tag{3.14}$$

We shall claim that $\nu_\ell(x)$ in (3.14) can be replaced by $\nu(x)$. Without loss of generality, we may assume $\int_E (f(x))^p \nu(x)\,dx < \infty$. Set

$$f_r(x) = \chi_{\tilde{B}_r}(x)\min\bigl(f(x), r\bigr) + \chi_{E\setminus\tilde{B}_r}(x)h(x) \quad (r = 1, 2, \ldots),$$

where $\tilde{B}_\rho$ is defined before and $h : E \mapsto (0, \infty)$ is chosen so that

$$h(x) \le \min\bigl(f(x), 1\bigr) \quad \text{and} \quad \int_E \bigl(h(x)\bigr)^p \nu_1(x)\,dx < \infty.$$

Replacing f in (3.14) by f_r, we get

$$\left(\int_E \left\{\exp\left(\frac{1}{G(x)}\int_{\tilde{S}_x} g(t)\log f_r(t)\,dt\right)\right\}^q u(x)\,dx\right)^{1/q} \le e^{1/p}\tilde{D}_{PS}\left(\int_E \bigl(f_r(x)\bigr)^p \nu_\ell(x)\,dx\right)^{1/p}. \tag{3.15}$$

For each r, we have

$$\begin{aligned}\int_E \bigl(f_r(x)\bigr)^p \nu_1(x)\,dx &= \int_{\tilde{B}_r}\bigl(\min\bigl(f(x), r\bigr)\bigr)^p \nu_1(x)\,dx + \int_{E\setminus\tilde{B}_r}\bigl(h(x)\bigr)^p \nu_1(x)\,dx \\ &\le \int_E \bigl(f(x)\bigr)^p \nu(x)\,dx + \int_{\tilde{B}_r} r^p\,dx + \int_E \bigl(h(x)\bigr)^p \nu_1(x)\,dx < \infty\end{aligned}$$

and $|f_r(x)|^p\nu_\ell(x) \le (f_r(x))^p\nu_1(x)$ for $\ell = 1, 2, \ldots$. Applying the Lebesgue dominated convergence theorem to the right hand side of (3.15), we get

$$\begin{aligned}&\left(\int_E \left\{\exp\left(\frac{1}{G(x)}\int_{\tilde{S}_x} g(t)\log f_r(t)\,dt\right)\right\}^q u(x)\,dx\right)^{1/q}\\&\quad\le e^{1/p}\tilde{D}_{PS}\left(\int_E (f_r(x))^p \nu(x)\,dx\right)^{1/p}.\end{aligned}\tag{3.16}$$

By definition, $f_r(x) \uparrow f(x)$ as $r \to \infty$. Applying the monotone convergence theorem to both sides of (3.16), the right hand side tends to

$$e^{1/p}\tilde{D}_{PS}\left(\int_E (f(x))^p \nu(x)\,dx\right)^{1/p} \quad (\text{as } r \to \infty)$$

and the left hand side has the limit

$$\left(\int_E \left\{\exp\left(\frac{1}{G(x)}\lim_{r\to\infty}\int_{\tilde{S}_x} g(t)\log f_r(t)\,dt\right)\right\}^q u(x)\,dx\right)^{1/q}.\tag{3.17}$$

Let $x \in E$. Since $\int_{\tilde{S}_x} g(t)\log f(t)\,dt$ is well defined, the following equality makes sense:

$$\int_{\tilde{S}_x} g(t)\log f(t)\,dt = \int_{\tilde{S}_x} g(t)\bigl(\log f(t)\bigr)^+\,dt - \int_{\tilde{S}_x} g(t)\bigl(\log f(t)\bigr)^-\,dt,$$

where $\xi^+ = \max(\xi, 0)$ and $\xi^- = \min(-\xi, 0)$. Consider $r \ge \max(\|x\|, 1)$. By the monotone convergence theorem,

$$\begin{aligned}\int_{\tilde{S}_x} g(t)\log f_r(t)\,dt &= \int_{\tilde{S}_x} g(t)\log\bigl\{\min\bigl(f(t), r\bigr)\bigr\}\,dt\\&= \int_{\tilde{S}_x} g(t)\min\bigl(\bigl(\log f(t)\bigr)^+, \log r\bigr)\,dt - \int_{\tilde{S}_x} g(t)\bigl(\log f(t)\bigr)^-\,dt\\&\longrightarrow \int_{\tilde{S}_x} g(t)\bigl(\log f(t)\bigr)^+\,dt - \int_{\tilde{S}_x} g(t)\bigl(\log f(t)\bigr)^-\,dt = \int_{\tilde{S}_x} g(t)\log f(t)\,dt.\end{aligned}$$

Inserting this limit in (3.17) yields the desired inequality. This finishes the proof. □

Theorem 3.2 gives a new proof of [3], Theorem 7.3(a). In the following, we shall display another example to show how (1.10) works well for the estimate of Opic-Gurka type. Set

$$\begin{aligned}\tilde{D}_{OG}(s) := {}&\sup_{x\in E}\bigl(G(x)\bigr)^{\frac{s-1}{p}}\\&\times\left(\int_{E\setminus S_x}\bigl(G(t)\bigr)^{\frac{-sq}{p}}\left\{\exp\left(\frac{1}{G(t)}\int_{\tilde{S}_t} g(y)\left(\log\frac{g(y)}{\nu(y)}\right)dy\right)\right\}^{\frac{q}{p}} u(t)\,dt\right)^{\frac{1}{q}},\end{aligned}$$

where $G(x)$ is defined by (1.6). The number $D^*_{OG}(s)$ in (1.8) is just the case $g(t) = 1$ of $\tilde{D}_{OG}(s)$. In the following, we shall extend the second inequality in (1.8) from $u(x) > 0$ and $g(t) = 1$ to $u(x) \ge 0$ and those $g(t)$ subject to the condition (1.9). This extension gives the Opic-Gurka-type estimate of the modular-type operator norm of the general geometric mean operator corresponding to $g(t)$. In particular, $g(t)$ can be of the form $g(t) = |\tilde{S}_t|^{s-1}$, which leads us to the Levin-Cochran-Lee-type inequality. Our result partially generalizes the sufficient parts of [5] and [3], Theorem 7.3(b).

Theorem 3.3 *Let* $0 < p \leq q < \infty$, $u(x) \geq 0$, $v(x) > 0$, $g(t) > 0$, *and* $0 < G(x) < \infty$, *where* $G(x)$ *is defined by* (1.6). *If* (1.9) *is true and* $\tilde{D}_{OG}(s) < \infty$ *for some* $s > 1$, *then* (1.7) *holds for* $C \leq \inf_{s>1} e^{(s-1)/p}\tilde{D}_{OG}(s)$.

Proof Let $\Phi(s) = e^s$, $k(x,t) = g(t)/G(x)$, and $f(t) \longrightarrow \log f(t)$. The proof is similar to Theorem 3.2. We shall show that $\|\mathbb{K}\|_* \leq \inf_{s>1} e^{(s-1)/p}\tilde{D}_{OG}(s)$. To observe the proof of Theorem 3.2, we find that it suffices to prove this inequality for the case: u is bounded on $\tilde{\Omega}_r$, $u(x) = 0$ on $E \setminus \tilde{\Omega}_r$, and $\sup_{x\in E}\{g(x)/v(x)\} < \infty$, where $\tilde{\Omega}_r$ is defined in the proof of Theorem 3.2. It follows from (1.10)-(1.11) and Theorem 2.2 that

$$\begin{aligned} \|\mathbb{K}\|_* &\leq \inf_{0<\epsilon<p} \big(\tilde{A}_W(p/\epsilon, q/\epsilon)\big)^{1/\epsilon} \\ &= \inf_{0<\epsilon<p}\left\{\inf_{1<s<p/\epsilon}\left(\frac{p-\epsilon}{p-\epsilon s}\right)^{1/\epsilon-1/p}\big(\tilde{A}_W(s,p/\epsilon,q/\epsilon)\big)^{1/\epsilon}\right\} \\ &\leq \inf_{s>1}\left\{\liminf_{\epsilon\to 0^+}\left(\frac{p-\epsilon}{p-\epsilon s}\right)^{1/\epsilon-1/p}\big(\tilde{A}_W(s,p/\epsilon,q/\epsilon)\big)^{1/\epsilon}\right\}. \end{aligned} \tag{3.18}$$

For $s > 1$, we have $\lim_{\epsilon\to 0^+}(\frac{p-\epsilon}{p-\epsilon s})^{1/\epsilon-1/p} = e^{(s-1)/p}$. We shall prove

$$\liminf_{\epsilon\to 0^+}\big(\tilde{A}_W(s,p/\epsilon,q/\epsilon)\big)^{1/\epsilon} \leq \tilde{D}_{OG}(s).$$

If so, the desired inequality follows from (3.18). Let $0 < \epsilon < p/s$. We have

$$\begin{aligned} \big(\tilde{A}_W(s,p/\epsilon,q/\epsilon)\big)^{1/\epsilon} &= \sup_{x\in E}\left(\int_{\tilde{S}_x}\left(\frac{g(t)}{v(t)}\right)^{\frac{\epsilon}{p-\epsilon}} g(t)\,dt\right)^{\frac{s-1}{p}} \\ &\quad \times \left(\int_{E\setminus S_x}\left\{\int_{\tilde{S}_t}\left(\frac{g(y)}{v(y)}\right)^{\frac{\epsilon}{p-\epsilon}} g(y)\,dy\right\}^{\frac{q(p-\epsilon s)}{\epsilon p}} \frac{u(t)\,dt}{(G(t))^{q/\epsilon}}\right)^{1/q}. \end{aligned} \tag{3.19}$$

The term '$(\int_{\tilde{S}_x}(\cdots))^{\frac{s-1}{p}}$' in (3.19) increases in $\|x\|$. On the other hand, the term '$(\int_{E\setminus S_x}\{\cdots\}^{q(p-\epsilon s)/(\epsilon p)}\frac{u(t)\,dt}{(G(t))^{q/\epsilon}})^{1/q}$' in (3.19) is zero for $\|x\| > r$ and it keeps the same value for the change: x with $\|x\| < 1/r \longrightarrow (1/r)x/\|x\|$. These imply that the term '$\sup_{x\in E}$' in (3.19) can be replaced by '$\sup_{x\in\tilde{\Omega}_r}$'. By the Heine-Borel theorem, we can choose $0 < \epsilon_m < p/s$, $\alpha_m > 0$, and $x_0, x_m \in \tilde{\Omega}_r$ such that $\epsilon_m \to 0$, $\alpha_m \to 0$, $x_m \to x_0$, and the following inequality holds for all m:

$$\begin{aligned} &\big(\tilde{A}_W(s,p/\epsilon_m,q/\epsilon_m)\big)^{1/\epsilon_m} \\ &\quad\leq \left(\int_{\tilde{S}_{x_m}}\left(\frac{g(t)}{v(t)}\right)^{\frac{\epsilon_m}{p-\epsilon_m}} g(t)\,dt\right)^{\frac{s-1}{p}} \\ &\qquad \times \left(\int_{E\setminus S_{x_m}}\left\{\int_{\tilde{S}_t}\left(\frac{g(y)}{v(y)}\right)^{\frac{\epsilon_m}{p-\epsilon_m}} g(y)\,dy\right\}^{\frac{q(p-\epsilon_m s)}{\epsilon_m p}} \frac{u(t)\,dt}{(G(t))^{q/\epsilon_m}}\right)^{1/q} + \alpha_m. \end{aligned} \tag{3.20}$$

For the first integral in (3.20), we have

$$\begin{aligned}
&\lim_{m\to\infty}\left(\int_{\tilde{S}_{x_m}}\left(\frac{g(t)}{\nu(t)}\right)^{\frac{\epsilon_m}{p-\epsilon_m}} g(t)\,dt\right)^{\frac{s-1}{p}} \\
&\quad=\left(\int_{\tilde{S}_{x_0}}\lim_{m\to\infty}\left\{\left(\frac{g(t)}{\nu(t)}\right)^{\frac{\epsilon_m}{p-\epsilon_m}}\right\}g(t)\,dt\right)^{\frac{s-1}{p}} = \bigl(G(x_0)\bigr)^{\frac{s-1}{p}}.
\end{aligned}\tag{3.21}$$

As for the second integral, it follows from Lemma 3.1 that

$$\begin{aligned}
&\left(\int_{\tilde{S}_t}\left(\frac{g(y)}{\nu(y)}\right)^{\frac{\epsilon_m}{p-\epsilon_m}} g(y)\,dy\right)^{\frac{q(p-\epsilon_m s)}{\epsilon_m p}}\frac{1}{(G(t))^{q/\epsilon_m}} \\
&\quad=\left(\int_{\tilde{S}_t}\left(\frac{g(y)}{\nu(y)}\right)^{\frac{\epsilon_m}{p-\epsilon_m}} g(y)\,dy\right)^{-qs/p}\left(\frac{1}{G(t)}\int_{\tilde{S}_t}\left(\frac{g(y)}{\nu(y)}\right)^{\frac{\epsilon_m}{p-\epsilon_m}} g(y)\,dy\right)^{q/\epsilon_m} \\
&\quad\longrightarrow \bigl(G(t)\bigr)^{\frac{-qs}{p}}\left\{\exp\left(\frac{1}{G(t)}\int_{\tilde{S}_t} g(y)\left(\log\frac{g(y)}{\nu(y)}\right)dy\right)\right\}^{\frac{q}{p}} \quad \text{as } m\to\infty.
\end{aligned}\tag{3.22}$$

Moreover, for m large enough,

$$\begin{aligned}
&\left|\chi_{E\setminus S_{x_m}(t)}\left(\int_{\tilde{S}_t}\left(\frac{g(y)}{\nu(y)}\right)^{\frac{\epsilon_m}{p-\epsilon_m}} g(y)\,dy\right)^{\frac{q(p-\epsilon_m s)}{\epsilon_m p}}\frac{u(t)}{(G(t))^{q/\epsilon_m}}\right| \\
&\quad\le\left\{\sup_{x\in E}\left(\frac{g(y)}{\nu(y)}\right)+1\right\}^{q/p}\chi_{\tilde{\Omega}_r}(t)G(t)^{-qs/p}u(t)\in L^1(E,dt).
\end{aligned}$$

Integrating the left hand side of (3.22) with respect to $u(t)\,dt$ first and then applying the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
&\lim_{m\to\infty}\left(\int_{E\setminus S_{x_m}}\left(\int_{\tilde{S}_t}\left(\frac{g(y)}{\nu(y)}\right)^{\frac{\epsilon_m}{p-\epsilon_m}} g(y)\,dy\right)^{\frac{q(p-\epsilon_m s)}{\epsilon_m p}}\frac{u(t)\,dt}{(G(t))^{q/\epsilon_m}}\right)^{1/q} \\
&\quad=\left(\int_{E\setminus S_{x_0}}\lim_{m\to\infty}\left\{\frac{1}{(G(t))^{q/\epsilon_m}}\left(\int_{\tilde{S}_t}\left(\frac{g(y)}{\nu(y)}\right)^{\frac{\epsilon_m}{p-\epsilon_m}} g(y)\,dy\right)^{\frac{q(p-\epsilon_m s)}{\epsilon_m p}}\right\}u(t)\,dt\right)^{1/q} \\
&\quad=\left(\int_{E\setminus S_{x_0}}\frac{1}{(G(t))^{qs/p}}\left\{\exp\left(\frac{1}{G(t)}\int_{\tilde{S}_t} g(y)\left(\log\frac{g(y)}{\nu(y)}\right)dy\right)\right\}^{\frac{q}{p}}u(t)\,dt\right)^{\frac{1}{q}}.
\end{aligned}\tag{3.23}$$

Putting (3.20), (3.21), and (3.23) together yields the desired inequality. This finishes the proof. □

For other estimates of Hardy-type inequalities, we may use a similar limit process to Theorems 3.2 and 3.3 to get the corresponding Pólya-Knopp inequalities.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in drafting this manuscript and giving the main proofs. All authors read and approved the final manuscript.

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Acknowledgements
The first author was supported in part by the Ministry of Science and Technology, Taipei, ROC, under Grants Most103-2115-M-364-001 and Most104-2115-M-364-001. We express our gratitude to Professor Lars-Erik Persson and the reviewers for their valued comments in developing the final version of the article.

Received: 18 May 2015 Accepted: 12 October 2015 Published online: 29 October 2015

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