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Extension of some results on univalent functions

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Abstract

In this paper we use a result of Nunokawa to extend some results on univalent functions given by Miller and Mocanu. As a consequence, we get several sufficient conditions for starlikeness over the expression $f(z)f''(z)/f'^2(z)$.

MSC: Primary 30C45; secondary 30C80

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1 Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ that are analytic in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ and are normalized such that $f(0) = f'(0) - 1 = 0$, *i.e.*, $f(z) = z + a_2 z^2 + \cdots$. Functions from $\mathcal{A}$ that are one-to-one are called *normalized univalent functions*. One of the largest classes of univalent functions is the class of *strongly starlike functions of order* α, $0 < \alpha \le 1$, denoted by $\widetilde{S}^*(\alpha)$ and consisting of function $f \in \mathcal{A}$ such that

$$\left|\arg\left[\frac{zf'(z)}{f(z)}\right]\right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{D}).$$

For $\alpha = 1$, we obtain the well-known class of *starlike functions* $f(z)$ that map the unit disk onto a starlike region, *i.e.*, if $\omega \in f(\mathbb{D})$, then $t\omega \in f(\mathbb{D})$ for all $t \in [0,1]$. For details, see [1].

A major contribution in the theory of univalent functions was done by the work of Miller and Mocanu. In the 1980s, in [2, 3], they introduced the concept of differential subordinations, which was a big step forward and led to numerous valuable results by hundreds of mathematicians around the world following their work. More details on this topic can be found in [4].

Using methods from the theory of differential subordinations, Miller and Mocanu proved the following two results.

Theorem 1.1 (Corollary 4.1a.1 from [4], p.189) *Let $B(z)$ and $C(z)$ be complex-valued functions defined in $\mathbb{D}$ with*

$$\left|\operatorname{Im} C(z)\right| \le \operatorname{Re} B(z) \quad (z \in \mathbb{D}).$$

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If $p(z)$ is analytic in $\mathbb{D}$ with $p(0)=1$, and if

$$\operatorname{Re}\bigl[B(z)zp'(z)+C(z)p(z)\bigr]>0 \quad (z\in\mathbb{D}),$$

then

$$\operatorname{Re}p(z)>0 \quad (z\in\mathbb{D}).$$

Theorem 1.2 (Theorem 5 from [5]; Theorem 3.1c from [4], p.73) *Let $\beta_0 = 1.21872\ldots$ be the solution of $\beta\pi+\operatorname{arctg}\beta-3\pi/2=0$, and let $\alpha=\alpha(\beta)=\beta+\frac{2}{\pi}\operatorname{arctg}\beta$ for $0<\beta\le\beta_0$. If p is analytic in $\mathbb{D}$, with $p(0)=1$, then*

$$\bigl|\arg\bigl[zp'(z)+p(z)\bigr]\bigr|<\frac{\alpha\pi}{2} \quad (z\in\mathbb{D})$$

implies

$$\bigl|\arg p(z)\bigr|<\frac{\beta\pi}{2} \quad (z\in\mathbb{D}).$$

These two results are closely related since for $B(z)=C(z)$, bearing in mind that for $\omega = x+iy$ and $x>0$,

$$|\operatorname{Im}\omega|\le\operatorname{Re}\omega \quad\Leftrightarrow\quad \left|\frac{y}{x}\right|\le 1 \quad\Leftrightarrow\quad \bigl|\operatorname{tg}(\arg\omega)\bigr|\le 1 \quad\Leftrightarrow\quad |\arg\omega|\le\frac{\pi}{4},$$

Theorem 1.1 reduces to the following.

Theorem 1.3 (Corollary 4.1a.1 from [4], p.189) *Let $B(z)$ be a complex-valued function defined in $\mathbb{D}$ with*

$$\bigl|\arg\bigl[B(z)\bigr]\bigr|\le\frac{\pi}{4} \quad (z\in\mathbb{D}).$$

If $p(z)$ is analytic in $\mathbb{D}$ with $p(0)=1$, and if

$$\operatorname{Re}\bigl\{B(z)\bigl[zp'(z)+p(z)\bigr]\bigr\}>0 \quad (z\in\mathbb{D}),$$

then

$$\operatorname{Re}p(z)>0 \quad (z\in\mathbb{D}).$$

Both of these results (Theorem 1.2 and Theorem 1.3) will be extended in this paper. Namely, we will extend the result of Theorem 1.2 for values of β bigger than β_0, and we will obtain a result complementary to Theorem 1.3 (with more flexible conditions). At the end these new results will be applied for obtaining several sufficient conditions for starlikeness over the expression

$$\frac{f(z)f''(z)}{f'^2(z)}.$$

This expression was introduced in [6], and after that it has attracted significant attention (some of those results can be found in [7–11]).

For proving the main result, we will use the following lemma due to Nunokawa which is an extension of the well-known Jack lemma [12].

Lemma 1.4 ([13, 14]) *Let $p(z)$ be a function analytic in $z \in \mathbb{D}$ with $p(0) = 1$ and $p(z) \neq 0$ for all $z \in \mathbb{D}$. If there exists a point $z_0 \in \mathbb{D}$ such that*

$$\left|\arg\left[p(z)\right]\right| < \frac{\varphi\pi}{2} \quad \textit{for } |z| < |z_0|$$

and

$$\left|\arg\left[p(z_0)\right]\right| = \frac{\varphi\pi}{2}$$

for some $\varphi > 0$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = il\varphi,$$

where

$$l \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1 \quad \textit{when } \arg\left[p(z_0)\right] = \frac{\varphi\pi}{2} \tag{1.1}$$

and

$$l \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \leq -1 \quad \textit{when } \arg\left[p(z_0)\right] = -\frac{\varphi\pi}{2}, \tag{1.2}$$

where

$$\left[p(z_0)\right]^{1/\varphi} = \pm ai \quad \textit{and } a > 0.$$

2 Main results and consequences

As we will see later, using Lemma 1.4 we obtain the following result which, we will see later, for some choice of parameters extends the results of Miller and Mocanu given in Theorem 1.2 and Theorem 1.3.

Theorem 2.1 *Let $\beta \in (0, 2]$ and $\beta_0 = 1.21872\ldots$ be the solution of the equation $\beta\pi + \operatorname{arctg}\beta - 3\pi/2 = 0$. Also, let $\alpha \in (0, 2]$ be such that*

(i) $|\beta + \frac{2}{\pi}\operatorname{arctg}\beta - \alpha - 1| \leq 1$ *when $\beta \leq \beta_0$;*

(ii) $|\beta + \alpha - 2| \leq 1$ *when $\beta \geq \beta_0$.*

If $B(z)$ is such that

$$\left|\arg B(z)\right| \leq \begin{Bmatrix} \varphi_1 \equiv \frac{\pi}{2}(\beta - \alpha) + \operatorname{arctg}\beta, & \beta \leq \beta_0 \\ \varphi_2 \equiv \frac{\pi}{2}(3 - \beta - \alpha), & \beta \geq \beta_0 \end{Bmatrix} \equiv \varphi, \tag{2.1}$$

and if $p(z)$ is an analytic function in $\mathbb{D}$ with $p(0) = 1$ and $p(z) \neq 0$ for all $z \in \mathbb{D}$, then

$$\left|\arg\left\{B(z)\left[zp'(z) + p(z)\right]\right\}\right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{D}) \tag{2.2}$$

implies

$$\left|\arg p(z)\right| < \frac{\beta\pi}{2} \quad (z \in \mathbb{D}).$$

Proof In the beginning, let us note a few things that we will use later in the proof. First, the function $g(\beta) \equiv \beta\pi + \operatorname{arctg}\beta - \frac{3\pi}{2}$ is an increasing one on the interval $(0, 2]$, which easily leads to a conclusion that

$$\varphi = \min\{\varphi_1, \varphi_2\},$$

where φ, φ_1 and φ_2 are defined in (2.1). Second, conditions (i) and (ii) are equivalent to

$$0 \le \varphi_1 \le \pi \quad \text{when } \beta \le \beta_0$$

and

$$0 \le \varphi_2 \le \pi \quad \text{when } \beta \ge \beta_0,$$

respectively.

Now, let us assume that there exists a point $z_0 \in \mathbb{D}$ such that

$$\left|\arg\left[p(z)\right]\right| < \frac{\beta\pi}{2} \quad \text{for } |z| < |z_0|$$

and

$$\left|\arg\left[p(z_0)\right]\right| = \frac{\beta\pi}{2}.$$

Then, according to Lemma 1.4,

$$\left[p(z_0)\right]^{1/\beta} = \pm ai, \quad a > 0$$

and

$$\frac{z_0 p'(z_0)}{p(z_0)} = il\beta,$$

where (1.1) and (1.2) hold. Therefore,

$$\begin{aligned} I &\equiv \arg\left\{B(z_0)\left[z_0 p'(z_0) + p(z_0)\right]\right\} = \arg\left\{B(z_0) \cdot p(z_0) \cdot \left[\frac{z_0 p'(z_0)}{p(z_0)} + 1\right]\right\} \\ &= \arg B(z_0) + \frac{\beta\pi}{2} + \arg(il\beta + 1). \end{aligned}$$

Now, let us consider the case when $p(z_0) = ia$, $a > 0$ and $l \ge 1$. For the value of I, condition (2.1) implies

$$\begin{aligned} I &\ge -\left|\arg B(z_0)\right| + \frac{\beta\pi}{2} + \operatorname{arctg}\beta \\ &> -\left|\arg B(z_0)\right| + \underbrace{\frac{\beta\pi}{2} + \operatorname{arctg}\beta - \frac{\alpha\pi}{2}}_{\varphi_1} = -\left|\arg B(z_0)\right| + \varphi_1 \ge 0, \end{aligned}$$

and

$$I < \left|\arg B(z_0)\right| + \frac{\beta\pi}{2} + \frac{\pi}{2} < \left|\arg B(z_0)\right| + \underbrace{\frac{\beta\pi}{2} - \frac{3\pi}{2} + \frac{\alpha\pi}{2}}_{\varphi_2} + 2\pi$$

$$\leq \left|\arg B(z_0)\right| - \varphi_2 + 2\pi \leq 2\pi.$$

Therefore, $I \in (0, 2\pi)$ and

$$\begin{aligned}
|I| &= \begin{Bmatrix} I, & I \in (0,\pi] \\ 2\pi - I, & I \in (\pi, 2\pi) \end{Bmatrix} \\
&\geq \begin{Bmatrix} -|\arg B(z_0)| + \frac{\beta\pi}{2} + \operatorname{arctg}\beta, & I \in (0,\pi] \\ 2\pi - [|\arg B(z_0)| + \frac{\beta\pi}{2} + \frac{\pi}{2}], & I \in (\pi, 2\pi) \end{Bmatrix} \\
&= -\left|\arg B(z_0)\right| + \begin{Bmatrix} \frac{\beta\pi}{2} + \operatorname{arctg}\beta, & I \in (0,\pi] \\ \frac{3\pi}{2} - \frac{\beta\pi}{2}, & I \in (\pi, 2\pi) \end{Bmatrix} \\
&\geq -\left|\arg B(z_0)\right| + \varphi + \frac{\alpha\pi}{2} \geq \frac{\alpha\pi}{2}.
\end{aligned}$$

This is a contradiction to condition (2.2).

In a similar way we come to a contradiction in the remaining case, $p(z_0) = -ia$, $a > 0$ and $l \leq -1$.

Thus, the initial assumption is not correct, *i.e.*, $|\arg p(z)| < \frac{\beta\pi}{2}$ for all $z \in \mathbb{D}$. □

Remark 2.2 Theorem 2.1 makes sense, *i.e.*, conditions (2.1) and (2.2) can be true at the same time since $p(0) = 1$ implies that

$$\arg\left\{B(z)\left[zp'(z) + p(z)\right]\right\}\Big|_{z=0} = \arg\left[B(0)\right]$$

can be in $(-\varphi, \varphi)$ and in $(-\alpha\pi/2, \alpha\pi/2)$ at the same time for certain functions $B(z)$ and $p(z)$.

For $\beta = 1$ in Theorem 2.1 we obtain the following.

Corollary 2.3 *Let $\alpha \in (0, \frac{3}{2}]$ and let $B(z)$ be such that*

$$\left|\arg B(z)\right| \leq \frac{\pi}{2}\left(\frac{3}{2} - \alpha\right) \quad (z \in \mathbb{D}).$$

If $p(z)$ is an analytic function in $\mathbb{D}$ with $p(0) = 1$ and $p(z) \neq 0$ for all $z \in \mathbb{D}$, then

$$\left|\arg\left\{B(z)\left[zp'(z) + p(z)\right]\right\}\right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{D})$$

implies

$$\operatorname{Re} p(z) > 0 \quad (z \in \mathbb{D}).$$

For $\alpha = 1$ in Corollary 2.3, we obtain the following.

Corollary 2.4 *Let $B(z)$ be such that*

$$\left|\arg B(z)\right| \le \frac{\pi}{4} \quad (z \in \mathbb{D}).$$

If $p(z)$ is an analytic function in $\mathbb{D}$ with $p(0) = 1$ and $p(z) \neq 0$ for all $z \in \mathbb{D}$, then

$$\operatorname{Re}\left\{B(z)\left[zp'(z) + p(z)\right]\right\} > 0 \quad (z \in \mathbb{D})$$

implies

$$\operatorname{Re} p(z) > 0 \quad (z \in \mathbb{D}).$$

Remark 2.5

(i) In comparison with Theorem 1.2, Corollary 2.3 requires an additional condition ($p(z) \neq 0$ for all $z \in \mathbb{D}$). On the other hand, the remaining conditions are more flexible than the conditions of Theorem 1.2. Namely, Theorem 1.2 requires

$$\left|\arg B(z)\right| \le \frac{\pi}{4} \quad \text{and} \quad \left|\arg\left\{B(z)\left[zp'(z) + p(z)\right]\right\}\right| < \frac{\pi}{2},$$

while Corollary 2.3 requires

$$\left|\arg B(z)\right| \le \frac{3\pi}{4} - \frac{\alpha\pi}{2} \quad \text{and} \quad \left|\arg\left\{B(z)\left[zp'(z) + p(z)\right]\right\}\right| < \frac{\alpha\pi}{2}.$$

Note that the sum of the bounds on the right-hand side in both cases is $3\pi/4$.

(ii) Corollary 2.4 is slightly weaker than Theorem 1.2 because of the extra condition requiring $p(z) \neq 0$ for all $z \in \mathbb{D}$.

Taking $B(z) = 1$ in Theorem 2.1, we obtain the following result.

Corollary 2.6 *Let $\beta \in (0,2]$ and $\beta_0 = 1.21872\ldots$ be the solution of the equation $\beta\pi +$ $\operatorname{arctg}\beta - 3\pi/2 = 0$. Also, let*

$$\alpha \equiv \alpha(\beta) = \begin{cases} \beta + \frac{\pi}{2}\operatorname{arctg}\beta, & \beta \le \beta_0, \\ 3 - \beta, & \beta \ge \beta_0. \end{cases} \tag{2.3}$$

If $p(z)$ is an analytic function in $\mathbb{D}$ with $p(0) = 1$ and $p(z) \neq 0$ for all $z \in \mathbb{D}$, then

$$\left|\arg\left\{B(z)\left[zp'(z) + p(z)\right]\right\}\right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{D})$$

implies

$$\left|\arg p(z)\right| < \frac{\beta\pi}{2} \quad (z \in \mathbb{D}).$$

Proof The conclusion follows directly from Theorem 2.1 since α defined by (2.3) satisfies its conditions (i) and (ii). □

Remark 2.7 In the case when $\beta \le \beta_0$, Corollary 2.6 is the same as Theorem 1.2, *i.e.*, it extends Theorem 1.2 to the values of β in the interval $[\beta_0, 2]$.

3 Sufficient conditions for starlikeness

Theorem 2.1 in the case when $p(z) = \frac{f(z)}{zf'(z)}$ ($\Rightarrow zp'(z) + p(z) = 1 - f(z)f''(z)/f'^2(z)$) and $B(z) = e^{i\theta\pi/2}$ brings the following sufficient condition for starlikeness.

Corollary 3.1 *Let $\beta \in (0,2]$ and $\beta_0 = 1.21872\ldots$ be the solution of the equation $\beta\pi + \operatorname{arctg}(\beta) - \frac{3\pi}{2} = 0$. Also, let $\alpha \in (0,2]$ be such that*

(i) $|\beta + \frac{2}{\pi}\operatorname{arctg}\beta - \alpha - 1| \le 1$ *when* $\beta \le \beta_0$;

(ii) $|\beta + \alpha - 2| \le 1$ *when* $\beta \ge \beta_0$.

If $\theta \in [-2,2]$ is such that

$$|\theta| \le \begin{cases} \beta - \alpha + \frac{2}{\pi}\operatorname{arctg}\beta, & \beta \le \beta_0, \\ 3 - \beta - \alpha, & \beta \ge \beta_0, \end{cases} \tag{3.1}$$

and if $f \in \mathcal{A}$ is such that $\frac{f(z)}{zf'(z)} \neq 0$ for all $z \in \mathbb{D}$, then

$$\left|\arg\left\{e^{i\theta\pi/2} \cdot \left[1 - \frac{f(z)f''(z)}{f'^2(z)}\right]\right\}\right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{D}) \tag{3.2}$$

implies that the function $f(z)$ is strongly starlike of order β.

Proof It is easy to check that (3.1) implies inequality (2.1) and the rest follows from Theorem 2.1. □

Now we will apply part (i) from the previous corollary on a specific function from the class $\mathcal{A}$ and get that it is strongly starlike of certain order. It is much more difficult to obtain this conclusion directly.

Example

(i) Let $a \in (0,1)$ and let β_* be the unique real root of the equation

$$\frac{\beta\pi}{2} + \operatorname{arctg}\beta = \arcsin a \tag{3.3}$$

on the interval $(0,1]$. Then the function $f(z) = z(1 + \frac{a}{3}z^2)^{-1/2}$ is strongly starlike of order β_*.

(ii) If we choose $a = 1$ in the previous example, we obtain that the function $f(z) = z(1 + \frac{1}{3}z^2)^{-1/2}$ is strongly starlike of order $\overline{\beta} = 0.6383\ldots$.

(iii) Example (i) can be rewritten in the following form: if $\beta \in (0, \overline{\beta}]$ and $a = \sin(\frac{\beta\pi}{2} + \operatorname{arctg}\beta)$, then the function $f(z) = z(1 + \frac{a}{3}z^2)^{-1/2}$ is strongly starlike of order β.

(iv) Choosing $\beta = 1/2$ in the previous example, we obtain $a = 0.94868\ldots$ and $f(z) = z(1 + \frac{a}{3}z^2)^{-1/2} \in \widetilde{\mathcal{S}}^*(1/2)$.

Proof (i) First, let us note that $\frac{\beta\pi}{2} + \operatorname{arctg}\beta$ is an increasing function on the interval $[0,1]$ with minimal and maximal value, 0 and $\frac{3\pi}{4}$, respectively. Therefore, equation (3.3) has, indeed, a unique real solution on the interval $(0,1]$.

Further, it can be easily verified that $f(0) = 0$ and that $f'(z) = 3\sqrt{3}(3 + az^2)^{-3/2}$, $f'(0) = 1$, *i.e.*, $f(z) \in \mathcal{A}$. Also, $\frac{f(z)}{zf'(z)} = \frac{1+az^2/3}{z} \neq 0$ for all $z \in \mathbb{D}$ and

$$1 - \frac{f(z)f''(z)}{f'^2(z)} = 1 + az^2,$$

i.e.,

$$\sup_{z\in\mathbb{D}} \left| \arg\left[1 - \frac{f(z)f''(z)}{f'^2(z)}\right]\right| = \arcsin a = \frac{\alpha\pi}{2}.$$

Therefore, choosing in Corollary 3.1(i): $\beta = \beta_*$, $\alpha = \beta + \frac{2}{\pi}\operatorname{arctg}\beta$ and $\theta = 0$, we realize that all its conditions are satisfied. So, $f(z) \in \widetilde{\mathcal{S}}^*(\beta)$.

This is more difficult to obtain from the definition of strong starlikeness of order β since $\frac{zf'(z)}{f(z)} = \frac{z}{1+az^2/3}$. □

Remark 3.2 For the function $F(z) = \frac{f(z)f''(z)}{f'^2(z)}$, condition (3.2) is equivalent to the condition: $F(\mathbb{D})$ lies in the sector $\{\omega : |\arg\omega| < \frac{\alpha\pi}{2}\}$ rotated by angle $(2 - \theta)\pi/2$ and translated by 1.

Proof Condition (3.2) is equivalent to

$$e^{i\theta\pi/2} \cdot \left[1 - F(\mathbb{D})\right] \subseteq \left\{\omega \in \mathbb{C} : |\arg\omega| < \frac{\alpha\pi}{2}\right\},$$

i.e., to

$$-e^{i\theta\pi/2} \cdot F(\mathbb{D}) \subseteq -e^{i\theta\pi/2} + \left\{\omega \in \mathbb{C} : |\arg\omega| < \frac{\alpha\pi}{2}\right\},$$

$$F(\mathbb{D}) \subseteq 1 - e^{-i\theta\pi/2} \cdot \left\{\omega \in \mathbb{C} : |\arg\omega| < \frac{\alpha\pi}{2}\right\}$$

and

$$F(\mathbb{D}) \subseteq 1 + e^{i(2-\theta)\pi/2} \cdot \left\{\omega : |\arg\omega| < \frac{\alpha\pi}{2}\right\}.$$

□

If we choose $\beta = 1$, $\alpha = 3/2$ and $\theta = 0$ in Corollary 3.1, we have the following.

Corollary 3.3 *Let $f \in \mathcal{A}$ and $\frac{f(z)}{zf'(z)} \neq 0$ for all $z \in \mathbb{D}$. If*

$$\left|\arg\left[1 - \frac{f(z)f''(z)}{f'^2(z)}\right]\right| < \frac{3\pi}{4} \quad (z \in \mathbb{D}),$$

i.e.,

$$F(\mathbb{D}) \subseteq 1 - \left\{\omega : |\arg\omega| < \frac{3\pi}{4}\right\},$$

where $F(z) = \frac{f(z)f''(z)}{f'^2(z)}$, then $f(z)$ is a starlike function.

Remark 3.4 Corollary 2.3(i) from [6] says that starlikeness of $f(z) \in \mathcal{A}$ follows from

$$F(\mathbb{D}) \subset \{x + iy : x \leq 1.5\} \cup \{x + iy : y^2 > -3 + 2x\} \equiv \Sigma.$$

The boundary of the region Σ is the curve $y = \pm\sqrt{-3 + 2x}$ which for $x = 2$ has two tangents $y = \pm(x-1)$ that, for $x \geq 1$, are boundary of the region $\Omega = \{\omega : |\arg \omega| < 3\pi/4\}$. This implies that $\Omega \subset \Sigma$, *i.e.*, that the result from Corollary 3.3 follows from Corollary 2.3(i) [6].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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