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Compositions involving Schur geometrically convex functions

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Abstract

The decision theorem of the Schur geometric convexity for the compositions involving Schur geometrically convex functions is established and used to determine the Schur geometric convexity of some symmetric functions.

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Keywords: Schur geometrically convex function; geometrically convex function; composite function; symmetric function

1 Introduction

Throughout the article, $\mathbb{R}$ denotes the set of real numbers, $\mathbf{x} = (x_1, x_2, \dots, x_n)$ denotes n-tuple (n-dimensional real vectors), the set of vectors can be written as

$$\begin{aligned}
\mathbb{R}^n &= \{\mathbf{x} = (x_1, \dots, x_n) : x_i \in \mathbb{R}, i = 1, \dots, n\}, \\
\mathbb{R}^n_{++} &= \{\mathbf{x} = (x_1, \dots, x_n) : x_i > 0, i = 1, \dots, n\}, \\
\mathbb{R}^n_{+} &= \{\mathbf{x} = (x_1, \dots, x_n) : x_i \geq 0, i = 1, \dots, n\}.
\end{aligned}$$

In particular, the notations $\mathbb{R}$, $\mathbb{R}_{++}$, and $\mathbb{R}_+$ denote $\mathbb{R}^1$, $\mathbb{R}^1_{++}$and $\mathbb{R}^1_+$, respectively.

The following conclusion is proved in [1], p.91, [2], p.64-65.

Theorem A *Let the interval* $[a,b] \subset \mathbb{R}$, $\varphi : \mathbb{R}^n \to \mathbb{R}$, $f : [a,b] \to \mathbb{R}$, *and* $\psi(x_1, \dots, x_n) = \varphi(f(x_1), \dots, f(x_n)) : [a,b]^n \to \mathbb{R}$.

(i) *If* φ *is increasing and Schur convex and* f *is convex, then* ψ *is Schur convex.*
(ii) *If* φ *is increasing and Schur concave and* f *is concave, then* ψ *is Schur concave.*
(iii) *If* φ *is decreasing and Schur convex and* f *is concave, then* ψ *is Schur convex.*
(iv) *If* φ *is increasing and Schur convex and* f *is increasing and convex, then* ψ *is increasing and Schur convex.*
(v) *If* φ *is decreasing and Schur convex and* f *is decreasing and concave, then* ψ *is increasing and Schur convex.*
(vi) *If* φ *is increasing and Schur convex and* f *is decreasing and convex, then* ψ *is decreasing and Schur convex.*
(vii) *If* φ *is decreasing and Schur convex and* f *is increasing and concave, then* ψ *is decreasing and Schur convex.*
(viii) *If* φ *is decreasing and Schur concave and* f *is decreasing and convex, then* ψ *is increasing and Schur concave.*

Theorem A is very effective for determine of the Schur convexity of the composite functions.

The Schur geometrically convex functions were proposed by Zhang [3] in 2004, and was investigated by Chu *et al.* [4], Guan [5], Sun *et al.* [6], and so on. We also note that some authors use the term 'Schur-multiplicative convexity'. The theory of majorization was enriched and expanded by using these concepts. Regarding the Schur geometrically convex functions, the aim of this paper is to establish the following theorem which is similar to Theorem A.

Theorem 1 *Let the interval* $[a,b] \subset \mathbb{R}_{++}$, $\varphi : \mathbb{R}^n \to \mathbb{R}, f : [a,b] \to \mathbb{R}$, *and* $\psi(x_1,\dots,x_n) = \varphi(f(x_1),\dots,f(x_n)) : [a,b]^n \to \mathbb{R}$.

(i) *If* φ *is increasing and Schur geometrically convex and* f *is geometrically convex, then* ψ *is Schur geometrically convex.*
(ii) *If* φ *is increasing and Schur geometrically concave and* f *is geometrically concave, then* ψ *is Schur geometrically concave.*
(iii) *If* φ *is decreasing and Schur geometrically convex and* f *is geometrically concave, then* ψ *is Schur geometrically convex.*
(iv) *If* φ *is increasing and Schur geometrically convex and* f *is increasing and geometrically convex, then* ψ *is increasing and Schur geometrically convex.*
(v) *If* φ *is decreasing and Schur geometrically convex and* f *is decreasing and geometrically concave, then* ψ *is increasing and Schur geometrically convex.*
(vi) *If* φ *is increasing and Schur geometrically convex and* f *is decreasing and geometrically convex, then* ψ *is decreasing and Schur geometrically convex.*
(vii) *If* φ *is decreasing and Schur geometrically convex and* f *is increasing and geometrically concave, then* ψ *is decreasing and Schur geometrically convex.*
(viii) *If* φ *is decreasing and Schur geometrically concave and* f *is decreasing and geometrically convex, then* ψ *is increasing and Schur geometrically concave.*

2 Definitions and lemmas

In order to prove our results, in this section we will recall useful definitions and lemmas.

Definition 1 [1, 2] Let $\mathbf{x} = (x_1,\dots,x_n)$ and $\mathbf{y} = (y_1,\dots,y_n) \in \mathbb{R}^n$.

(i) $\mathbf{x} \geq \mathbf{y}$ means $x_i \geq y_i$ for all $i = 1,2,\dots,n$.
(ii) Let $\Omega \subset \mathbb{R}^n$, $\varphi : \Omega \to \mathbb{R}$ is said to be increasing if $\mathbf{x} \geq \mathbf{y}$ implies $\varphi(\mathbf{x}) \geq \varphi(\mathbf{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

Definition 2 [1, 2] Let $\mathbf{x} = (x_1,\dots,x_n)$ and $\mathbf{y} = (y_1,\dots,y_n) \in \mathbb{R}^n$.

We say $\mathbf{y}$ majorizes $\mathbf{x}$ ($\mathbf{x}$ is said to be majorized by $\mathbf{y}$), denoted by $\mathbf{x} \prec \mathbf{y}$, if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1,2,\dots,n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in a descending order.

Definition 3 [1, 2] Let $\mathbf{x} = (x_1,\dots,x_n)$ and $\mathbf{y} = (y_1,\dots,y_n) \in \mathbb{R}^n$.

(i) A set $\Omega \subset \mathbb{R}^n$ is said to be a convex set if

$$\alpha\mathbf{x} + (1-\alpha)\mathbf{y} = \big(\alpha x_1 + (1-\alpha)y_1,\dots,\alpha x_n + (1-\alpha)y_n\big) \in \Omega$$

for all $\mathbf{x},\mathbf{y} \in \Omega$, and $\alpha \in [0,1]$.

(ii) Let $\Omega \subset \mathbb{R}^n$ be a convex set. A function φ: $\Omega \to \mathbb{R}$ is said to be a convex function on Ω if

$$\varphi\bigl(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}\bigr) \le \alpha\varphi(\mathbf{x}) + (1-\alpha)\varphi(\mathbf{y})$$

holds for all $\mathbf{x}, \mathbf{y} \in \Omega$, and $\alpha \in [0,1]$. φ is said to be a concave function on Ω if and only if $-\varphi$ is a convex function on Ω.

(iii) Let $\Omega \subset \mathbb{R}^n$. A function $\varphi : \Omega \to \mathbb{R}$ is said to be a Schur convex function on Ω if $\mathbf{x} \prec \mathbf{y}$ on Ω implies $\varphi(\mathbf{x}) \le \varphi(\mathbf{y})$. A function φ is said to be a Schur concave function on Ω if and only if $-\varphi$ is a Schur convex function on Ω.

Lemma 1 (Schur convex function decision theorem) [1, 2] *Let $\Omega \subset \mathbb{R}^n$ be symmetric and have a nonempty interior convex set. Ω^0 is the interior of Ω. $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then φ is a* Schur convex (*or* Schur concave, *respectively*) function *if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \ge 0 \ (or \le 0, respectively) \tag{1}$$

holds for any $\mathbf{x} \in \Omega^0$.

Definition 4 [3] Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n_{++}$.

(i) A set $\Omega \subset \mathbb{R}^n_{++}$ is called a geometrically convex set if

$$\mathbf{x}^\alpha\mathbf{y}^{1-\alpha} = \bigl(x_1^\alpha y_1^{1-\alpha}, \ldots, x_n^\alpha y_n^{1-\alpha}\bigr) \in \Omega$$

for all $\mathbf{x}, \mathbf{y} \in \Omega$ and $\alpha \in [0,1]$.

(ii) Let $\Omega \subset \mathbb{R}_{++}$ be geometrically convex set. A function $\varphi : I \to \mathbb{R}_{++}$ is called a geometrically convex(or concave, respectively) function, if

$$\varphi\bigl(\mathbf{x}^\alpha\mathbf{y}^{1-\alpha}\bigr) \le (\text{or} \ge, \text{respectively}) \bigl[\varphi(\mathbf{x})\bigr]^\alpha \bigl[\varphi(\mathbf{y})\bigr]^{1-\alpha}$$

holds for all $\mathbf{x}, \mathbf{y} \in \Omega$ and $\alpha \in [0,1]$.

(iii) Let $\Omega \subset \mathbb{R}^n_{++}$. A function $\varphi : \Omega \to \mathbb{R}_{++}$ is said to be a Schur geometrically convex (or concave, respectively) function on Ω if

$$\log(\mathbf{x}) = (\log x_1, \ldots, \log x_n) \prec \log(\mathbf{y}) = (\log y_1, \ldots, \log y_n)$$

implies

$$\varphi(\mathbf{x}) \le (\text{or} \ge, \text{respectively})\, \varphi(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \Omega$.

By Definition 4(iii), the following is obvious.

Proposition 1 *Let $\Omega \subset \mathbb{R}^n_{++}$ be a set, and let* $\log\Omega = \{(\log x_1, \ldots, \log x_n) : (x_1, \ldots, x_n) \in \Omega\}$. *Then $\varphi : \Omega \to \mathbb{R}_{++}$ is a Schur geometrically convex (or concave, respectively) function on Ω if and only if $\varphi(e^{x_1}, \ldots, e^{x_n})$ is a Schur convex (or concave, respectively) function on* $\log\Omega$.

Lemma 2 (Schur geometrically convex function decision theorem) [3] *Let $\Omega \subset \mathbb{R}^n_{++}$ be a symmetric and geometrically convex set with a nonempty interior Ω^0. Let $\varphi : \Omega \to \mathbb{R}_{++}$ be continuous on Ω and differentiable in Ω^0. If φ is symmetric on Ω and*

$$(\log x_1 - \log x_2)\left(x_1 \frac{\partial \varphi}{\partial x_1} - x_2 \frac{\partial \varphi}{\partial x_2}\right) \geq 0 \ (\leq 0) \tag{2}$$

holds for any $\mathbf{x} = (x_1, \dots, x_n) \in \Omega^0$, then φ is a Schur geometrically convex (Schur geometrically concave) function.

Lemma 3 [7] *If $f : [a, b] \subset \mathbb{R}_{++} \to \mathbb{R}_{++}$ is geometrically convex (or concave, respectively) if and only if $\log f(e^x)$ is convex (or concave, respectively) on $[\log a, \log b]$.*

Lemma 4 [7] *If $f : [a, b] \subset \mathbb{R}_{++} \to \mathbb{R}_{++}$ is a twice differentiable function, then f is a geometrically convex (or concave, respectively) function if and only if*

$$x\left[f''(x)f(x) - \left(f'(x)\right)^2\right] + f(x)f'(x) \geq 0 \ (\text{or} \leq 0, \text{respectively}). \tag{3}$$

3 Proof of main results

Proof of Theorem 1 We only give the proof of Theorem 1(iv) in detail. Similar argument leads to the proof of the rest part.

If φ is increasing and Schur geometrically convex and f is increasing and geometrically convex, then by Proposition 1, $\varphi(e^{x_1}, \dots, e^{x_n})$ is increasing and Schur convex and by Lemma 3, $g(x) = \log f(e^x)$ is increasing and convex on $[\log a, \log b]$. Then from Theorem A(iv), it follows that $\varphi(e^{\log f(e^{x_1})}, \dots, e^{\log f(e^{x_n})}) = \varphi(f(e^{x_1}), \dots, f(e^{x_n}))$ is increasing and Schur convex. Again by Proposition 1, it follows that $\psi(x_1, \dots, x_n) = \varphi(f(x_1), \dots, f(x_n))$ is increasing and Schur geometrically convex.

The proof of Theorem 1 is completed. □

4 Applications

Let $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. Its elementary symmetric functions are

$$E_r(\mathbf{x}) = E_r(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \cdots < i_r \leq n} \prod_{j=1}^{r} x_{i_j}, \quad r = 1, \dots, n,$$

and we defined $E_0(\mathbf{x}) = 1$, and $E_r(\mathbf{x}) = 0$ for $r < 0$ or $r > n$. The dual form of the elementary symmetric functions are

$$E_r^*(\mathbf{x}) = E_r^*(x_1, \dots, x_n) = \prod_{1 \leq i_1 < \cdots < i_r \leq n} \sum_{j=1}^{r} x_{i_j}, \quad r = 1, \dots, n,$$

and we defined $E_0^*(\mathbf{x}) = 1$, and $E_r^*(\mathbf{x}) = 0$ for $r < 0$ or $r > n$.

It is well known that $E_r(\mathbf{x})$ is an increasing and Schur concave function on $\mathbb{R}^n_+$ [1]. By Lemma 2, it is easy to prove that $E_r(\mathbf{x})$ is a Schur geometrically convex function on $\mathbb{R}_{++}$. In fact, noting that

$$E_r(\mathbf{x}) = x_1 x_2 E_{r-2}(x_3, \dots, x_n) + (x_1 + x_2)E_{r-1}(x_3, \dots, x_n) + E_r(x_3, \dots, x_n),$$

then

$$\begin{aligned}
&(\log x_1 - \log x_2)\left(x_1 \frac{\partial E_r(\mathbf{x})}{\partial x_1} - x_2 \frac{\partial E_r(\mathbf{x})}{\partial x_2}\right) \\
&\quad = (\log x_1 - \log x_2) \\
&\qquad \times \left[x_1\left(x_2 E_{r-2}(x_3,\ldots,x_n) + E_{r-1}(x_3,\ldots,x_n)\right)\right. \\
&\qquad \left. - x_2\left(x_1 E_{r-2}(x_3,\ldots,x_n) + E_{r-1}(x_3,\ldots,x_n)\right)\right] \\
&\quad = (x_1 - x_2)(\log x_1 - \log x_2) E_{r-1}(x_3,\ldots,x_n) \geq 0.
\end{aligned}$$

In [8, 9], Shi proved that $E_r^*(\mathbf{x})$ is an increasing and Schur concave function and Schur geometrically convex function on $\mathbb{R}_+^n$.

For $\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n$, the complete symmetric function $c_n(\mathbf{x}, r)$ is defined as

$$c_n(\mathbf{x}, r) = \sum_{i_1+i_2+\cdots+i_n=r} \prod_{j=1}^{n} x_j^{i_j},$$

where $c_0(\mathbf{x}, r) = 1$, $r \in \{1,2,\ldots,n\}$, $i_1, i_2, \ldots, i_n$ are non-negative integers.

The dual form of the complete symmetric function is

$$c_n^*(\mathbf{x}, r) = \prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j x_j = \prod_{1\leq k_1\leq k_2\leq\cdots\leq k_r\leq n} \sum_{j=1}^{r} x_{k_j},$$

where i_j $(j = 1,2,\ldots,n)$ are non-negative integers.

Guan [10] discussed the Schur convexity of $c_n(\mathbf{x}, r)$ and proved that $c_n(\mathbf{x}, r)$ is increasing and Schur convex on $\mathbb{R}_{++}^n$. Subsequently, Chu *et al.* [11] proved that $c_n(\mathbf{x}, r)$ is Schur geometrically convex on $\mathbb{R}_{++}^n$.

Zhang and Shi [12] proved that $c_n^*(\mathbf{x}, r)$ is increasing, Schur concave and Schur geometrically convex on $\mathbb{R}_{++}^n$.

In the following, we prove that the Schur geometric convexity of the composite functions involving the above symmetric functions and their dual form by using Theorem 1.

Let $f(x) = \frac{1+x}{1-x}$, $x \in (0,1)$. Directly calculating yields

$$f'(x) = \frac{2}{(1-x)^2} > 0,$$

$$x\left[f''(x)f(x) - \left(f'(x)\right)^2\right] + f(x)f'(x) = \frac{2(x^2+1)}{(1-x)^4} > 0.$$

That is, f is increasing and geometrically convex on $(0,1)$. Since $E_r(\mathbf{x})$, $E_r^*(\mathbf{x})$, $c_n(\mathbf{x}, r)$, and $c_n^*(\mathbf{x}, r)$ are all increasing and Schur geometrically convex functions on $\mathbb{R}_{++}^n$, and noticing that $f(x) = \frac{1+x}{1-x} > 0$, for $0 < x < 1$, by Theorem 1(iv), the following theorem holds.

Theorem 2 *The following symmetric functions are increasing and Schur geometrically convex on* $(0,1)^n$:

$$E_r\left(\frac{1+\mathbf{x}}{1-\mathbf{x}}\right) = \sum_{1\leq i_1<\cdots<i_r\leq n} \prod_{j=1}^{r} \frac{1+x_{i_j}}{1-x_{i_j}}, \tag{4}$$

$$E_r^*\left(\frac{1+\mathbf{x}}{1-\mathbf{x}}\right)=\prod_{1\le i_1<\cdots<i_r\le n}\sum_{j=1}^{r}\frac{1+x_{i_j}}{1-x_{i_j}},\tag{5}$$

$$c_n\left(\frac{1+\mathbf{x}}{1-\mathbf{x}},r\right)=\sum_{i_1+i_2+\cdots+i_n=r}\prod_{j=1}^{n}\left(\frac{1+x_j}{1-x_j}\right)^{i_j}\tag{6}$$

and

$$c_n^*\left(\frac{1+\mathbf{x}}{1-\mathbf{x}},r\right)=\prod_{i_1+i_2+\cdots+i_n=r}\sum_{j=1}^{n}i_j\left(\frac{1+x_j}{1-x_j}\right).\tag{7}$$

Remark 1 By Lemma 2, Xia and Chu [13] proved that $E_r(\frac{1+\mathbf{x}}{1-\mathbf{x}})$ is Schur geometrically convex on $(0,1)^n$. By the properties of Schur geometrically convex function, Shi and Zhang [14] proved that $E_r^*(\frac{1+\mathbf{x}}{1-\mathbf{x}})$ is Schur geometrically convex on $(0,1)^n$. By Theorem 1, we give a new proof.

For $r\ge 1$, let $g(x)=x^{\frac{1}{r}}$, $x\in\mathbb{R}_{++}$. Directly calculating yields

$$g'(x)=\frac{1}{r}x^{\frac{1}{r}-1}>0,$$
$$x\bigl[g''(x)g(x)-\bigl(g'(x)\bigr)^2\bigr]+g(x)g'(x)=0.$$

That is, g is increasing and geometrically convex (concave) on $\mathbb{R}_{++}^n$. Since $E_r(\mathbf{x})$, $E_r^*(\mathbf{x})$, $c_n(\mathbf{x},r)$, and $c_n^*(\mathbf{x},r)$ are all increasing and Schur geometrically convex function on $\mathbb{R}_{++}^n$, by Theorem 1(iv), the following theorem holds.

Theorem 3 *The following symmetric functions are increasing and Schur geometrically convex on* $\mathbb{R}_{++}^n$:

$$E_r\bigl(\mathbf{x}^{\frac{1}{r}}\bigr)=\sum_{1\le i_1<\cdots<i_r\le n}\prod_{j=1}^{r}x_{i_j}^{\frac{1}{r}},\tag{8}$$

$$E_r^*\bigl(\mathbf{x}^{\frac{1}{r}}\bigr)=\prod_{1\le i_1<\cdots<i_r\le n}\sum_{j=1}^{r}x_{i_j}^{\frac{1}{r}},\tag{9}$$

$$c_n\bigl(\mathbf{x}^{\frac{1}{r}},r\bigr)=\sum_{i_1+i_2+\cdots+i_n=r}\prod_{j=1}^{n}x_j^{\frac{i_j}{r}}\tag{10}$$

and

$$c_n^*\bigl(\mathbf{x}^{\frac{1}{r}},r\bigr)=\prod_{i_1+i_2+\cdots+i_n=r}\sum_{j=1}^{n}i_jx_j^{\frac{1}{r}}.\tag{11}$$

Remark 2

(i) By Lemma 2, Guan [15] and Jiang [16] proved, respectively, that the Hamy symmetric function $E_r(\mathbf{x}^{\frac{1}{r}})$ and its dual form $E_r^*(\mathbf{x}^{\frac{1}{r}})$ is Schur geometrically convex on $\mathbb{R}_{++}^n$. In contrast, our proof is very simple by Theorem 1.

(ii) Here we prove Schur geometric convexity of $c_n(\mathbf{x}^{\frac{1}{r}},r)$ on $\mathbb{R}_{++}^n$ by Theorem 1. Guan [17] proved that $c_n(\mathbf{x}^{\frac{1}{r}},r)$ is Schur concave on $\mathbb{R}_{++}^n$ by Lemma 1.

Since $f(x) = \frac{1+x}{1-x}$ is increasing and geometrically convex on $(0,1)$, from Theorem 1(iv) and Theorem 3, the following holds.

Theorem 4 *The following symmetric functions are increasing and Schur geometrically convex on* $(0,1)^n$:

$$E_r\left(\left(\frac{1+\mathbf{x}}{1-\mathbf{x}}\right)^{\frac{1}{r}}\right) = \sum_{1\le i_1<\cdots<i_r\le n} \prod_{j=1}^{r}\left(\frac{1+x_{i_j}}{1-x_{i_j}}\right)^{\frac{1}{r}}, \tag{12}$$

$$E_r^*\left(\left(\frac{1+\mathbf{x}}{1-\mathbf{x}}\right)^{\frac{1}{r}}\right) = \prod_{1\le i_1<\cdots<i_r\le n} \sum_{j=1}^{r}\left(\frac{1+x_{i_j}}{1-x_{i_j}}\right)^{\frac{1}{r}}, \tag{13}$$

$$c_n\left(\left(\frac{1+\mathbf{x}}{1-\mathbf{x}}\right)^{\frac{1}{r}}, r\right) = \sum_{i_1+i_2+\cdots+i_n=r} \prod_{j=1}^{n}\left(\frac{1+x_j}{1-x_j}\right)^{\frac{i_j}{r}} \tag{14}$$

and

$$c_n^*\left(\left(\frac{1+\mathbf{x}}{1-\mathbf{x}}\right)^{\frac{1}{r}}, r\right) = \prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j\left(\frac{1+x_j}{1-x_j}\right)^{\frac{1}{r}}. \tag{15}$$

Remark 3 By Lemma 2, Long and Chu [18] proved that $E_r^*((\frac{1+\mathbf{x}}{1-\mathbf{x}})^{\frac{1}{r}})$ is Schur geometrically convex on $(0,1)^n$. By Theorem 1, we give a new proof.

Let $h(x) = \frac{x}{1-x}$, $x \in (0,1)$. Then $h'(x) = \frac{1}{(1-x)^2} > 0$, $h''(x) = \frac{2}{(1-x)^3}$, and

$$x\left[h''(x)h(x) - \left(h'(x)\right)^2\right] + h(x)h'(x) = \frac{x^2}{(1-x)^4} > 0.$$

That is, h is increasing and geometrically convex on $(0,1)$. Since $E_r(\mathbf{x})$, $E_r^*(\mathbf{x})$, $c_n(\mathbf{x}, r)$, and $c_n^*(\mathbf{x}, r)$ are all increasing and Schur geometrically convex function on $\mathbb{R}_{++}^n$, and notice that $h(x) = \frac{x}{1-x} > 0$, for $0 < x < 1$, by Theorem 1(iv), the following holds.

Theorem 5 *The following symmetric functions are increasing and Schur geometrically convex on* $(0,1)^n$:

$$E_r\left(\frac{\mathbf{x}}{1-\mathbf{x}}\right) = \sum_{1\le i_1<\cdots<i_r\le n} \prod_{j=1}^{r}\frac{x_{i_j}}{1-x_{i_j}}, \tag{16}$$

$$E_r^*\left(\frac{\mathbf{x}}{1-\mathbf{x}}\right) = \prod_{1\le i_1<\cdots<i_r\le n} \sum_{j=1}^{r}\frac{x_{i_j}}{1-x_{i_j}}, \tag{17}$$

$$c_n\left(\frac{\mathbf{x}}{1-\mathbf{x}}, r\right) = \sum_{i_1+i_2+\cdots+i_n=r} \prod_{j=1}^{n}\left(\frac{x_j}{1-x_j}\right)^{i_j} \tag{18}$$

and

$$c_n^*\left(\frac{\mathbf{x}}{1-\mathbf{x}}, r\right) = \prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j\left(\frac{x_j}{1-x_j}\right). \tag{19}$$

Remark 4 By Lemma 2, Guan [19] proved that $E_r(\frac{\mathbf{x}}{1-\mathbf{x}})$ is Schur geometrically convex on $(0,1)^n$. By the judgment theorems of Schur geometric convexity for a class of symmetric functions, Shi and Zhang [20] give another proof. Here by Theorem 1, we give a new proof.

By the properties of Schur geometrically convex function, Shi and Zhang [14] proved that $E_r^*(\frac{\mathbf{x}}{1-\mathbf{x}})$ is Schur geometrically convex on $[\frac{1}{2},1)^n$. By Theorem 1, this conclusion is extended to the collection $(0,1)^n$.

By Lemma 2, Sun *et al.* [21] proved that $c_n(\frac{\mathbf{x}}{1-\mathbf{x}},r)$ is Schur geometrically convex on $[0,1)^n$, here by Theorem 1, we give a new proof.

Since $f(x)=\frac{x}{1-x}$ is increasing and geometrically convex on $(0,1)$, from Theorem 1(iv) and Theorem 3, the following holds.

Theorem 6 *The following symmetric functions are increasing and Schur geometrically convex on* $(0,1)^n$:

$$E_r\left(\left(\frac{\mathbf{x}}{1-\mathbf{x}}\right)^{\frac{1}{r}}\right)=\sum_{1\le i_1<\cdots<i_r\le n}\prod_{j=1}^{r}\left(\frac{x_{i_j}}{1-x_{i_j}}\right)^{\frac{1}{r}},\tag{20}$$

$$E_r^*\left(\left(\frac{\mathbf{x}}{1-\mathbf{x}}\right)^{\frac{1}{r}}\right)=\prod_{1\le i_1<\cdots<i_r\le n}\sum_{j=1}^{r}\left(\frac{x_{i_j}}{1-x_{i_j}}\right)^{\frac{1}{r}},\tag{21}$$

$$c_n\left(\left(\frac{\mathbf{x}}{1-\mathbf{x}}\right)^{\frac{1}{r}},r\right)=\sum_{i_1+i_2+\cdots+i_n=r}\prod_{j=1}^{n}\left(\frac{x_j}{1-x_j}\right)^{\frac{i_j}{r}}\tag{22}$$

and

$$c_n^*\left(\left(\frac{\mathbf{x}}{1-\mathbf{x}}\right)^{\frac{1}{r}},r\right)=\prod_{i_1+i_2+\cdots+i_n=r}\sum_{j=1}^{n}i_j\left(\frac{x_j}{1-x_j}\right)^{\frac{1}{r}}.\tag{23}$$

Remark 5 By Lemma 2, Sun [22] proved that $E_r((\frac{\mathbf{x}}{1-\mathbf{x}})^{\frac{1}{r}})$ is Schur geometrically convex on $[0,1)^n$. Here by Theorem 1, we give a new proof.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by H-NS. This work was carried out in collaboration between both authors. They read and approved the final manuscript.

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References
1. Marshall, AW, Olkin, I, Arnold, BC: Inequalities: Theory of Majorization and Its Applications, 2nd edn. Springer, New York (2011)
2. Wang, BY: Foundations of Majorization Inequalities. Beijing Normal University Press, Beijing (1990) (in Chinese)
3. Zhang, XM: Geometrically Convex Functions. An'hui University Press, Hefei (2004) (in Chinese)

4. Chu, YM, Zhang, XM, Wang, GD: The Schur geometrical convexity of the extended mean values. J. Convex Anal. **15**(4), 707-718 (2008)
5. Guan, K-Z: A class of symmetric functions for multiplicatively convex function. Math. Inequal. Appl. **10**(4), 745-753 (2007)
6. Sun, T-C, Lv, Y-P, Chu, Y-M: Schur multiplicative and harmonic convexities of generalized Heronian mean in *n* variables and their applications. Int. J. Pure Appl. Math. **55**(1), 25-33 (2009)
7. Niculescu, CP: Convexity according to the geometric mean. Math. Inequal. Appl. **3**(2), 155-167 (2000)
8. Shi, H-N: Schur-concavity and Schur-geometrically convexity of dual form for elementary symmetric function with applications. RGMIA Res. Rep. Collect. **10**(2), Article 15 (2007). http://rgmia.org/papers/v10n2/hnshi.pdf
9. Shi, H-N: Theory of Majorization and Analytic Inequalities. Harbin Institute of Technology Press, Harbin (2012) (in Chinese)
10. Guan, K-Z: Schur-convexity of the complete symmetric function. Math. Inequal. Appl. **9**(4), 567-576 (2006)
11. Chu, Y-M, Wang, G-D, Zhang, X-H: The Schur multiplicative and harmonic convexities of the complete symmetric function. Math. Nachr. **284**(5-6), 653-663 (2011)
12. Zhang, K, Shi, H-N: Schur convexity of dual form of the complete symmetric function. Math. Inequal. Appl. **16**(4), 963-970 (2013)
13. Xia, W-F, Chu, Y-M: On Schur convexity of some symmetric functions. J. Inequal. Appl. **2010**, Article ID 543250 (2010)
14. Shi, H-N, Zhang, J: Schur-convexity, Schur geometric and Schur harmonic convexities of dual form of a class symmetric functions. J. Math. Inequal. **8**(2), 349-358 (2014)
15. Guan, K-Z: A class of symmetric functions for multiplicatively convex function. Math. Inequal. Appl. **10**(4), 745-753 (2007)
16. Jiang, W-D: Some properties of dual form of the Hamy's symmetric function. J. Math. Inequal. **1**(1), 117-125 (2007)
17. Guan, K-Z: The Hamy symmetric function and its generalization. Math. Inequal. Appl. **9**(4), 797-805 (2006)
18. Long, BY, Chu, YM: The Schur convexity and inequalities for a class of symmetric functions. Acta Math. Sci. Ser. A **32**(1), 80-89 (2012) (in Chinese)
19. Guan, K-Z: Some properties of a class of symmetric functions. J. Math. Anal. Appl. **336**, 70-80 (2007)
20. Shi, H-N, Zhang, J: Some new judgement theorems of Schur geometric and Schur harmonic convexities for a class of symmetric functions. J. Inequal. Appl. **2013**, 527 (2013). doi:10.1186/1029-242X-2013-527
21. Sun, M-B, Chen, N-B, Li, S-H: Some properties of a class of symmetric functions and its applications. Math. Nachr. **287**(13), 1530-1544 (2014). doi:10.1002/mana.201300073
22. Sun, MB: The Schur convexity for two classes of symmetric functions. Sci. Sin., Math. **44**(6), 633-656 (2014). doi:10.1360/N012013-00157 (in Chinese)