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A new proof for some optimal inequalities involving generalized normalized δ-Casorati curvatures

Chul Woo Lee[1], Jae Won Lee[2*] and Gabriel-Eduard Vîlcu[3,4]

*Correspondence:
leejaew@bnue.ac.kr
[2]Department of Mathematics Education, Busan National University of Education, Busan, 611-736, South Korea
Full list of author information is available at the end of the article

Abstract

In this paper we give a new proof for two sharp inequalities involving generalized normalized δ-Casorati curvatures of a slant submanifold in a quaternionic space form. These inequalities were recently obtained in Lee and Vîlcu (Taiwan. J. Math. 19(3):691-702, 2015) using an optimization procedure by showing that a quadratic polynomial in the components of the second fundamental form is parabolic. The new proof is obtained analyzing a suitable constrained extremum problem on submanifold.

MSC: 53A07; 49K35

Keywords: scalar curvature; mean curvature; δ-Casorati curvature; shape operator; quaternionic space form; slant submanifold; optimal inequality

1 Introduction

The most powerful tool to find relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold is provided by Chen's invariants [1]. This theory was initiated in [2]: Chen established a sharp inequality for a submanifold in a real space form using the scalar curvature and the sectional curvature (both being intrinsic invariants) and squared mean curvature (the main extrinsic invariant). On the other hand, it is well known that the Casorati curvature of a submanifold in a Riemannian manifold is an extrinsic invariant defined as the normalized square of the length of the second fundamental form and it was preferred by Casorati over the traditional Gauss curvature because corresponds better with the common intuition of curvature [3–5]. Some optimal Chen-like inequalities involving Casorati curvatures were proved in [6–10] for several submanifolds in real, complex and quaternionic space forms. Recently, two sharp inequalities involving generalized normalized δ-Casorati curvatures of slant submanifolds in quaternionic space forms were obtained in [11] as follows.

Theorem 1.1 *Let M^n be a θ-slant proper submanifold of a quaternionic space form $\overline{M}^{4m}(c)$. Then*:

(i) *The generalized normalized δ-Casorati curvature $\delta_C(r;n-1)$ satisfies*

$$\rho \leq \frac{\delta_C(r;n-1)}{n(n-1)} + \frac{c}{4}\left(1 + \frac{9}{n-1}\cos^2\theta\right) \tag{1}$$

for any real number r such that $0 < r < n(n-1)$.

(ii) *The generalized normalized δ-Casorati curvature $\widehat{\delta}_C(r;n-1)$ satisfies*

$$\rho \leq \frac{\widehat{\delta}_C(r;n-1)}{n(n-1)} + \frac{c}{4}\left(1 + \frac{9}{n-1}\cos^2\theta\right) \tag{2}$$

for any real number $r > n(n-1)$.

Moreover, the equality sign holds in the inequalities (1) *and* (2) *if and only if M^n is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}^{4m}(c)$, such that with respect to suitable orthonormal tangent frame $\{\xi_1,\dots,\xi_n\}$ and normal orthonormal frame $\{\xi_{n+1},\dots,\xi_{4m}\}$, the shape operators $A_r \equiv A_{\xi_r}$, $r \in \{n+1,\dots,4m\}$, take the following forms*:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 & 0 \\ 0 & a & 0 & \cdots & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{n(n-1)}{r}a \end{pmatrix}, \qquad A_{n+2} = \cdots = A_{4m} = 0. \tag{3}$$

The proof given in [11] for these inequalities is based on an optimization procedure by showing that a quadratic polynomial in the components of the second fundamental form is parabolic. In this work we give an alternative proof using Oprea's optimization method on submanifolds [12], namely analyzing a suitable constrained extremum problem (see also [13–17]).

2 Preliminaries

This section gives several basic definitions and notations for our framework based mainly on [18, 19].

Let M^n be an n-dimensional Riemannian submanifold of a Riemannian manifold $(\overline{M},\overline{g})$. Then we denote by g the metric tensor induced on M. Let $K(\pi)$ be the sectional curvature of M associated with a plane section $\pi \subset T_pM$, $p \in M$. If $\{e_1,\dots,e_n\}$ is an orthonormal basis of the tangent space T_pM and $\{e_{n+1},\dots,e_m\}$ is an orthonormal basis of the normal space $T_p^{\perp}M$, then the scalar curvature τ at p is given by

$$\tau(p) = \sum_{1\leq i<j\leq n} K(e_i \wedge e_j)$$

and the normalized scalar curvature ρ of M is defined as

$$\rho = \frac{2\tau}{n(n-1)}.$$

We denote by H the mean curvature vector, that is,

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i),$$

and we also set

$$h_{ij}^{\alpha} = g\big(h(e_i, e_j), e_\alpha\big), \quad i, j \in \{1, \dots, n\}, \alpha \in \{n+1, \dots, m\}.$$

Then it is well known that the squared mean curvature of the submanifold M in $\overline{M}$ is defined by

$$\|H\|^2 = \frac{1}{n^2} \sum_{\alpha=n+1}^{m} \left(\sum_{i=1}^{n} h_{ii}^{\alpha} \right)^2,$$

and the squared norm of h over dimension n is denoted by $\mathcal{C}$ and is called the Casorati curvature of the submanifold M. Therefore we have

$$\mathcal{C} = \frac{1}{n} \sum_{\alpha=n+1}^{m} \sum_{i,j=1}^{n} \big(h_{ij}^{\alpha}\big)^2.$$

The submanifold M is called *invariantly quasi-umbilical* if there exist $m-n$ mutually orthogonal unit normal vectors $\xi_{n+1}, \dots, \xi_m$ such that the shape operators with respect to all directions ξ_α have an eigenvalue of multiplicity $n-1$ and that for each ξ_α the distinguished eigendirection is the same.

Suppose now that L is an s-dimensional subspace of T_pM, $s \geq 2$, and let $\{e_1, \dots, e_s\}$ be an orthonormal basis of L. Then the scalar curvature $\tau(L)$ of the s-plane section L is given by

$$\tau(L) = \sum_{1 \leq \alpha < \beta \leq s} K(e_\alpha \wedge e_\beta)$$

and the Casorati curvature $\mathcal{C}(L)$ of the subspace L is defined as

$$\mathcal{C}(L) = \frac{1}{s} \sum_{\alpha=n+1}^{m} \sum_{i,j=1}^{s} \big(h_{ij}^{\alpha}\big)^2.$$

The generalized normalized δ-Casorati curvatures $\delta_C(r; n-1)$ and $\widehat{\delta}_C(r; n-1)$ of the submanifold M^n are defined for any positive real number $r \neq n(n-1)$ as

$$\big[\delta_C(r; n-1)\big]_p = r\mathcal{C}_p + \frac{(n-1)(n+r)(n^2-n-r)}{rn} \inf\big\{\mathcal{C}(L) | L \text{ a hyperplane of } T_pM\big\},$$

if $0 < r < n^2 - n$, and

$$\big[\widehat{\delta}_C(r; n-1)\big]_p = r\mathcal{C}_p - \frac{(n-1)(n+r)(r-n^2+n)}{rn} \sup\big\{\mathcal{C}(L) | L \text{ a hyperplane of } T_pM\big\},$$

if $r > n^2 - n$.

If $\overline{\nabla}$ is the Levi-Civita connection on $\overline{M}$ and ∇ is the covariant differentiation induced on M, then the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM)$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \quad \forall X \in \Gamma(TM), \forall N \in \Gamma\bigl(TM^{\perp}\bigr),$$

where h is the second fundamental form of M, $\nabla^{\perp}$ is the connection on the normal bundle, and A_N is the shape operator of M with respect to N. If we denote by $\overline{R}$ and R the curvature tensor fields of $\overline{\nabla}$ and ∇, then we have the Gauss equation:

$$\begin{aligned}\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + \overline{g}\bigl(h(X,W),h(Y,Z)\bigr)\\ - \overline{g}\bigl(h(X,Z),h(Y,W)\bigr),\end{aligned} \tag{4}$$

for all $X, Y, Z, W \in \Gamma(TM)$.

Assume now that $(\overline{M},\overline{g})$ is a smooth manifold such that there is a rank 3-subbundle σ of $\operatorname{End}(T\overline{M})$ with local basis $\{J_1, J_2, J_3\}$ satisfying for all $\alpha \in \{1,2,3\}$:

$$\overline{g}(J_\alpha \cdot, J_\alpha \cdot) = \overline{g}(\cdot,\cdot)$$

and

$$J_\alpha^2 = -\mathrm{Id}, \qquad J_\alpha J_{\alpha+1} = -J_{\alpha+1}J_\alpha = J_{\alpha+2},$$

where Id denotes the identity tensor field of type $(1,1)$ on $\overline{M}$ and the indices are taken from $\{1,2,3\}$ modulo 3. Then $(\overline{M},\sigma,\overline{g})$ is said to be an almost quaternionic Hermitian manifold. It is easy to see that such a manifold is of dimension $4m$, $m \geq 1$. Moreover, if the bundle σ is parallel with respect to the Levi-Civita connection $\overline{\nabla}$ of $\overline{g}$, then $(\overline{M},\sigma,\overline{g})$ is said to be a quaternionic Kähler manifold.

Let $(\overline{M},\sigma,\overline{g})$ be a quaternionic Kähler manifold and let X be a non-null vector field on $\overline{M}$. Then the 4-plane spanned by $\{X, J_1X, J_2X, J_3X\}$, denoted by $Q(X)$, is called a quaternionic 4-plane. Any 2-plane in $Q(X)$ is called a quaternionic plane. The sectional curvature of a quaternionic plane is called a quaternionic sectional curvature. A quaternionic Kähler manifold is a quaternionic space form if its quaternionic sectional curvatures are equal to a constant, say c. It is well known that a quaternionic Kähler manifold $(\overline{M},\sigma,\overline{g})$ is a quaternionic space form, denoted $\overline{M}(c)$, if and only if its curvature tensor is given by

$$\begin{aligned}\overline{R}(X,Y)Z = \frac{c}{4}\Biggl\{&\overline{g}(Z,Y)X - \overline{g}(X,Z)Y + \sum_{\alpha=1}^{3}\bigl[\overline{g}(Z,J_\alpha Y)J_\alpha X - \\ &- \overline{g}(Z,J_\alpha X)J_\alpha Y + 2\overline{g}(X,J_\alpha Y)J_\alpha Z\bigr]\Biggr\}\end{aligned} \tag{5}$$

for all vector fields X, Y, Z on $\overline{M}$ and any local basis $\{J_1, J_2, J_3\}$ of σ.

A submanifold M of a quaternionic Kähler manifold $(\overline{M},\sigma,\overline{g})$ is said to be a slant submanifold [20] if, for each non-zero vector X tangent to M at p, the angle $\theta(X)$ between $J_\alpha(X)$ and T_pM, $\alpha \in \{1,2,3\}$ is constant, *i.e.* it does not depend on the choice of $p \in M$ and $X \in T_pM$. We can easily see that quaternionic submanifolds are slant submanifolds with $\theta = 0$ and totally real submanifolds are slant submanifolds with $\theta = \frac{\pi}{2}$. A slant submanifold of a quaternionic Kähler manifold is said to be proper (or θ-slant proper) if it is neither quaternionic nor totally real. We recall that every proper slant submanifold of a quaternionic Kähler manifold is of even dimension $n = 2s \geq 2$ and we can choose a canonical orthonormal local frame, called an adapted slant frame, as follows:

$$\{e_1, e_2 = \sec\theta P_\alpha e_1, \ldots, e_{2s-1}, e_{2s} = \sec\theta P_\alpha e_{2s-1}\},$$

where $P_\alpha e_{2k-1}$ denotes the tangential component of $J_\alpha e_{2k-1}$, $k \in \{1, \ldots, s\}$, and α is 1, 2 or 3 (see [21]).

Let $(\overline{M}, \overline{g})$ be a Riemannian manifold, M be a submanifold of $\overline{M}$, g be the induced metric of $\overline{g}$ and $f : M \longrightarrow \mathbb{R}$ be a differentiable function. If we consider the constrained extremum problem

$$\min_{x\in M} f(x), \tag{6}$$

then we have the following result.

Lemma 2.1 [12] *If $x_0 \in M$ is the solution of the problem* (6), *then*

(i) $(\operatorname{grad}(f))(x_0) \in T^\perp_{x_0}M$;

(ii) *the bilinear form*

$$\mathcal{A} : T_{x_0}M \times T_{x_0}M \longrightarrow \mathbb{R},$$
$$\mathcal{A}(X, Y) = \operatorname{Hess}_f(X, Y) + \overline{g}\bigl(h(X, Y), \bigl(\operatorname{grad}(f)\bigr)(x_0)\bigr)$$

is positive semi-definite, where h is the second fundamental form of M in $\overline{M}$ and $\operatorname{grad}(f)$ *is the gradient of f.*

3 New proof of Theorem 1.1

Since M is θ-slant, it is well known from [20] that

$$P_\beta P_\alpha X = -\cos^2\theta X, \quad \forall X \in \Gamma(TM), \alpha, \beta \in \{1, 2, 3\}, \tag{7}$$

where $P_\alpha X$ denotes the tangential component of $J_\alpha X$.

From (7) it follows immediately that

$$g(P_\alpha X, P_\beta Y) = \cos^2\theta g(X, Y) \tag{8}$$

for $X, Y \in \Gamma(TM)$ and $\alpha, \beta \in \{1, 2, 3\}$.

On the other hand, because $\overline{M}^{4m}(c)$ is a quaternionic space form, from (4) and (5) we derive

$$n^2\|H\|^2 = 2\tau(p) + \|h\|^2 - \frac{n(n-1)c}{4} - \frac{3c}{4}\sum_{\beta=1}^{3}\sum_{i,j=1}^{n} g^2(P_\beta e_i, e_j). \tag{9}$$

Choosing now an adapted slant basis

$$\{e_1, e_2 = \sec\theta P_\alpha e_1, \ldots, e_{2s-1}, e_{2s} = \sec\theta P_\alpha e_{2s-1}\}$$

of T_pM, $p \in M$, where $2s = n$, from (7) and (8), we derive

$$g^2(P_\beta e_i, e_{i+1}) = g^2(P_\beta e_{i+1}, e_i) = \cos^2\theta, \quad \text{for } i = 1, 3, \dots, 2s-1, \tag{10}$$

and

$$g(P_\beta e_i, e_j) = 0, \quad \text{for } (i,j) \notin \{(2l-1, 2l), (2l, 2l-1) | l \in \{1, 2, \dots, s\}\}. \tag{11}$$

By using (10) and (11) in (9) we get

$$2\tau(p) = n^2\|H\|^2 - n\mathcal{C} + \frac{c}{4}\bigl[n(n-1) + 9n\cos^2\theta\bigr]. \tag{12}$$

We consider now the following quadratic polynomial in the components of the second fundamental form:

$$\mathcal{P} = r\mathcal{C} + \frac{(n-1)(n+r)(n^2-n-r)}{rn}\mathcal{C}(L) - 2\tau(p) + \frac{c}{4}\bigl[n(n-1) + 9n\cos^2\theta\bigr],$$

where L is a hyperplane of T_pM. Without loss of generality we can assume that L is spanned by $e_1, \dots, e_{n-1}$. Then we derive

$$\begin{aligned} \mathcal{P} = {} & \frac{r}{n}\sum_{\alpha=n+1}^{4m}\sum_{i,j=1}^{n}\bigl(h_{ij}^\alpha\bigr)^2 + \frac{(n+r)(n^2-n-r)}{rn}\sum_{\alpha=n+1}^{4m}\sum_{i,j=1}^{n-1}\bigl(h_{ij}^\alpha\bigr)^2 \\ & - 2\tau(p) + \frac{c}{4}\bigl[n(n-1) + 9n\cos^2\theta\bigr]. \end{aligned} \tag{13}$$

From (12) and (13), we obtain

$$\begin{aligned} \mathcal{P} = {} & \sum_{\alpha=n+1}^{4m}\sum_{i=1}^{n-1}\left[\frac{n^2+n(r-1)-2r}{r}\bigl(h_{ii}^\alpha\bigr)^2 + \frac{2(n+r)}{n}\bigl(h_{in}^\alpha\bigr)^2\right] \\ & + \sum_{\alpha=n+1}^{4m}\left[\frac{2(n+r)(n-1)}{r}\sum_{i<j=1}^{n-1}\bigl(h_{ij}^\alpha\bigr)^2 - 2\sum_{i<j=1}^{n} h_{ii}^\alpha h_{jj}^\alpha + \frac{r}{n}\bigl(h_{nn}^\alpha\bigr)^2\right] \\ \geq {} & \sum_{\alpha=n+1}^{4m}\left[\sum_{i=1}^{n-1}\frac{n^2+n(r-1)-2r}{r}\bigl(h_{ii}^\alpha\bigr)^2 - 2\sum_{i<j=1}^{n} h_{ii}^\alpha h_{jj}^\alpha + \frac{r}{n}\bigl(h_{nn}^\alpha\bigr)^2\right]. \end{aligned} \tag{14}$$

For $r = n+1, \dots, 4m$, let us consider the quadratic form $f_\alpha : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$$f_\alpha\bigl(h_{11}^\alpha, \dots, h_{nn}^\alpha\bigr) = \sum_{i=1}^{n-1}\frac{n^2+n(r-1)-2r}{r}\bigl(h_{ii}^\alpha\bigr)^2 - 2\sum_{i<j=1}^{n} h_{ii}^\alpha h_{jj}^\alpha + \frac{r}{n}\bigl(h_{nn}^\alpha\bigr)^2,$$

and the constrained extremum problem

$$\begin{aligned} & \min f_\alpha \\ & \text{subject to } F : h_{11}^\alpha + \cdots + h_{nn}^\alpha = c^\alpha, \end{aligned}$$

where c^α is a real constant.

The partial derivatives of f_α are

$$\begin{cases} \frac{\partial f_\alpha}{\partial h_{ii}^\alpha} = \frac{2(n+r)(n-1)}{r} h_{ii}^\alpha - 2\sum_{k=1}^n h_{kk}^\alpha = 0, \\ \frac{\partial f_\alpha}{\partial h_{nn}^\alpha} = \frac{2r}{n} h_{nn}^\alpha - 2\sum_{k=1}^{n-1} h_{kk}^\alpha = 0, \end{cases} \tag{15}$$

with $i \in \{1,\dots,n-1\}$, $i \neq j$, and $\alpha \in \{n+1,\dots,4m\}$.

For an optimal solution $(h_{11}^\alpha,\dots,h_{nn}^\alpha)$ of the problem, the vector $\operatorname{grad}(f_\alpha)$ is normal at F. That is, it is collinear with the vector $(1,1,\dots,1)$. From (15), it follows that a critical point of the corresponding problem has the form

$$\begin{cases} h_{ii}^\alpha = \frac{r}{n(n-1)} t^\alpha, \quad i \in \{1,\dots,n-1\}, \\ h_{nn}^\alpha = t^\alpha. \end{cases} \tag{16}$$

Using (16) and $\sum_{i=1}^n h_{ii}^\alpha = c^\alpha$, we derive

$$\begin{cases} h_{ii}^\alpha = \frac{r}{(n+r)(n-1)} c^\alpha, \quad i \in \{1,\dots,n-1\}, \\ h_{nn}^\alpha = \frac{n}{n+r} c^\alpha. \end{cases} \tag{17}$$

We fix an arbitrary point $x \in F$. The 2-form $\mathcal{A} : T_xF \times T_xF \longrightarrow \mathbb{R}$ has the form

$$\mathcal{A}(X,Y) = \operatorname{Hess}(f_\alpha)(X,Y) + \big\langle h(X,Y), \big(\operatorname{grad}(f)\big)(x_0)\big\rangle,$$

where h is the second fundamental form of F in $\mathbb{R}^n$ and $\langle\,,\,\rangle$ is the standard inner product on $\mathbb{R}^n$. Moreover, it is easy to see that the Hessian matrix of f_α has the form

$$\operatorname{Hess}(f_\alpha) = \begin{pmatrix} \frac{2(n+r)(n-1)}{r} - 2 & -2 & \cdots & -2 & -2 \\ -2 & \frac{2(n+r)(n-1)}{r} - 2 & \cdots & -2 & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & \cdots & \frac{2(n+r)(n-1)}{r} - 2 & -2 \\ -2 & -2 & \cdots & -2 & \frac{2r}{n} \end{pmatrix}.$$

As F is totally geodesic in $\mathbb{R}^n$, considering a vector $X = (X_1, X_2, \dots, X_n)$ tangent to F at the arbitrary point x on F (that is, verifying the relation $\sum_{i=1}^n X_i = 0$), we obtain

$$\begin{aligned} \mathcal{A}(X,X) &= \frac{2(n^2 - n + rn - 2r)}{r} \sum_{i=1}^{n-1} X_i^2 + \frac{2r}{n} X_n^2 - 2\left(\sum_{i=1}^n X_i\right)^2 \\ &= \frac{2(n^2 - n + rn - 2r)}{r} \sum_{i=1}^{n-1} X_i^2 + \frac{2r}{n} X_n^2 \\ &\geq 0. \end{aligned}$$

Hence the point $(h_{11}^\alpha,\dots,h_{nn}^\alpha)$ from (16) is a global minimum point by Lemma 2.1. Moreover, $f_\alpha(h_{11}^\alpha,\dots,h_{nn}^\alpha) = 0$. Therefore, we have

$$\mathcal{P} \geq 0, \tag{18}$$

and this implies

$$2\tau(p) \leq r\mathcal{C} + \frac{(n-1)(n+r)(n^2-n-r)}{rn}\mathcal{C}(L) + \frac{c}{4}\big[n(n-1) + 9n\cos^2\theta\big].$$

Therefore we derive

$$\rho \leq \frac{r}{n(n-1)}\mathcal{C} + \frac{(n+r)(n^2-n-r)}{rn^2}\mathcal{C}(L) + \frac{c}{4}\left(1 + \frac{9}{n-1}\cos^2\theta\right) \tag{19}$$

for every tangent hyperplane L of M and both inequalities (1) and (2) obviously follow from (19).

Moreover, we can easily see now that the equality sign holds in the inequalities (1) and (2) if and only if

$$h^\alpha_{ij} = 0, \quad \forall i,j \in \{1,\ldots,n\}, i \neq j, \tag{20}$$

and

$$h^\alpha_{nn} = \frac{n(n-1)}{r}h^\alpha_{11} = \frac{n(n-1)}{r}h^\alpha_{22} = \cdots = \frac{n(n-1)}{r}h^\alpha_{n-1,n-1} \tag{21}$$

for all $\alpha \in \{n+1,\ldots,4m\}$.

Finally, from (20) and (21) we deduce that the equality sign holds in (1) and (2) if and only if the submanifold M is invariantly quasi-umbilical with trivial normal connection in $\overline{M}$, such that the shape operators take the forms (3) with respect to suitable tangent and normal orthonormal frames.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Kyungpook National University, Daegu, 702-701, South Korea. [2]Department of Mathematics Education, Busan National University of Education, Busan, 611-736, South Korea. [3]Faculty of Economic Sciences, Petroleum-Gas University of Ploieşti, Bulevardul Bucureşti, Nr. 39, Ploieşti, 100680, Romania. [4]Faculty of Mathematics and Computer Science, Research Center in Geometry, Topology and Algebra, University of Bucharest, Str. Academiei, Nr. 14, Sector 1, Bucureşti, 70109, Romania.

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References

1. Chen, B-Y: Mean curvature and shape operator of isometric immersions in real-space-forms. Glasg. Math. J. **38**(1), 87-97 (1996)
2. Chen, B-Y: Some pinching and classification theorems for minimal submanifolds. Arch. Math. **60**, 568-578 (1993)
3. Casorati, F: Mesure de la courbure des surfaces suivant l'idée commune. Ses rapports avec les mesures de courbure gaussienne et moyenne. Acta Math. **14**(1), 95-110 (1890)
4. Kowalczyk, D: Casorati curvatures. Bull. Transilv. Univ. Braşov Ser. III **1**(50), 209-213 (2008)
5. Verstraelen, L: Geometry of submanifolds I. The first Casorati curvature indicatrices. Kragujev. J. Math. **37**(1), 5-23 (2013)
6. Decu, S, Haesen, S, Verstraelen, L: Optimal inequalities involving Casorati curvatures. Bull. Transilv. Univ. Brasov Ser. B (N.S.) **14**(49), suppl., 85-93 (2007)
7. Decu, S, Haesen, S, Verstraelen, L: Optimal inequalities characterising quasi-umbilical submanifolds. J. Inequal. Pure Appl. Math. **9**(3), 79 (2008)

8. Ghişoiu, V: Inequalities for the Casorati curvatures of slant submanifolds in complex space forms. In: Riemannian Geometry and Applications. Proceedings RIGA 2011, pp. 145-150. Ed. Univ. Bucureşti, Bucharest (2011)
9. Lee, CW, Yoon, DW, Lee, JW: Optimal inequalities for the Casorati curvatures of submanifolds of real space forms endowed with semi-symmetric metric connections. J. Inequal. Appl. **2014**, Article ID 327 (2014)
10. Slesar, V, Şahin, B, Vîlcu, GE: Inequalities for the Casorati curvatures of slant submanifolds in quaternionic space forms. J. Inequal. Appl. **2014**, Article ID 123 (2014)
11. Lee, JW, Vîlcu, GE: Inequalities for generalized normalized δ-Casorati curvatures of slant submanifolds in quaternionic space forms. Taiwan. J. Math. **19**(3), 691-702 (2015)
12. Oprea, T: Optimization methods on Riemannian submanifolds. An. Univ. Bucur., Mat. **54**(1), 127-136 (2005)
13. Malek, F, Nejadakbary, V: A lower bound for the Ricci curvature of submanifolds in generalized Sasakian space forms. Adv. Geom. **13**(4), 695-711 (2013)
14. Oprea, T: Ricci curvature of Lagrangian submanifolds in complex space forms. Math. Inequal. Appl. **13**(4), 851-858 (2010)
15. Oprea, T: Chen's inequality in the Lagrangian case. Colloq. Math. **108**(1), 163-169 (2007)
16. Rapcsák, T: Sectional curvatures in nonlinear optimization. J. Glob. Optim. **40**(1-3), 375-388 (2008)
17. Zhang, P: Inequalities for Casorati curvatures of submanifolds in real space forms. arXiv:1408.4996
18. Chen, B-Y: Pseudo-Riemannian Geometry, δ-Invariants and Applications. World Scientific, Hackensack (2011)
19. Ishihara, S: Quaternion Kählerian manifolds. J. Differ. Geom. **9**, 483-500 (1974)
20. Şahin, B: Slant submanifolds of quaternion Kaehler manifolds. Commun. Korean Math. Soc. **22**(1), 123-135 (2007)
21. Vîlcu, GE, Chen, B-Y: Inequalities for slant submanifolds in quaternionic space forms. Turk. J. Math. **34**(1), 115-128 (2010)