Babenko and Kovalenko *Journal of Inequalities and Applications* (2015) 2015:295
DOI 10.1186/s13660-015-0822-1

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On modulus of continuity of differentiation operator on weighted Sobolev classes

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Abstract

In this paper we investigate the modulus of continuity of kth order differential operator, $k \in \mathbb{N}$, on the classes of functions defined on half-line that have positive non-increasing continuous majorants of functions and their higher derivatives.

Keywords: inequalities for derivatives; modulus of continuity; Kolmogorov type inequalities; perfect splines

1 Introduction

The following theorem was proved in [1] (1912) by Hardy and Littlewood (see also [2], Theorem 1.1.2).

Theorem A *Suppose that a function $x(t)$ is defined for $t > 0$ and its second derivative $x''(t)$ exists for $t > 0$. Let $f(t)$ and $g(t)$ be positive functions (both decreasing or both increasing). Then (as $t \to +\infty$) the following statements hold:*

1. *If $x(t) = O(f(t))$, $x''(t) = O(g(t))$, then*

$$x'(t) = O\big(\sqrt{f(t)g(t)}\big).$$

2. *If $x(t) = o(f(t))$, $x''(t) = O(g(t))$, then*

$$x'(t) = o\big(\sqrt{f(t)g(t)}\big).$$

3. *If $x(t) = O(f(t))$, $x''(t) = o(g(t))$, then*

$$x'(t) = o\big(\sqrt{f(t)g(t)}\big).$$

This theorem had a great impact on formation of the whole field of problems connected with inequalities between derivatives. To confirm this, it is sufficient to note that fundamental possibility of inequalities for upper bounds of derivatives of functions defined on the whole real line or half-line can be easily derived from this theorem (see [2], p.18).

In 1928 Mordell [3] (see also [2], Theorem 1.4.1) proved the following refinement of Theorem A for non-increasing functions.

Theorem B *Let $f(t)$ and $g(t)$ be positive non-increasing on the half-line $\mathbb{R}_+$ functions. If a function $x(t)$ is defined on the half-line $\mathbb{R}_+$ and for all $t > 0$ there exists $x''(t)$ such that*

$$\left|x(t)\right| \leq f(t), \qquad \left|x''(t)\right| \leq g(t),$$

then for all $t > 0$

$$\left|x'(t)\right| \leq 2\sqrt{f(t)g(t)}.$$

Let I be a finite interval, the whole real line $\mathbb{R}$, or a positive half-line $\mathbb{R}_+$. Denote by $C^m(I)$ ($m \in \mathbb{Z}_+$) the set of all m-times continuously differentiable (continuous in the case $m = 0$) functions $x : I \to \mathbb{R}$; by $L_\infty(I)$ we denote the space of all measurable functions $x : I \to \mathbb{R}$ with finite norms

$$\|x\|_\infty := \operatorname{ess\,sup}\left\{\left|x(t)\right| : t \in I\right\}.$$

Suppose that X is $C(I)$ or $L_\infty(I)$, $f \in C(I)$ is a positive non-increasing function. For $x \in X$, set

$$\|x\|_{X,f} := \left\|\frac{x(\cdot)}{f(\cdot)}\right\|_X.$$

For positive functions $f, g \in C(I)$ and natural r, set

$$L^r_{f,g}(I) := \left\{x \in C(I) : \|x\|_{C(I),f} < \infty, x^{(r-1)} \in \mathrm{AC}_{\mathrm{loc}}, \left\|x^{(r)}\right\|_{L_\infty(I),g} < \infty\right\},$$
$$W^r_{f,g}(I) := \left\{x \in L^r_{f,g}(I) : \left\|x^{(r)}\right\|_{L_\infty(I),g} \leq 1\right\}.$$

In the case when $f \equiv 1$ and $g \equiv 1$, we write $L^r_{\infty,\infty}(I)$ instead of $L^r_{f,g}(I)$.

Using above notations, the result of Theorem B can be rewritten in the following way:

$$\left\|x'\right\|_{C(\mathbb{R}_+),\sqrt{fg}} \leq 2\|x\|^{\frac{1}{2}}_{C(\mathbb{R}_+),f}\left\|x''\right\|^{\frac{1}{2}}_{L_\infty(\mathbb{R}_+),g}. \tag{1}$$

In the case when $f(t) \equiv 1$ and $g(t) \equiv 1$, from (1) one can obtain Landau's inequality [4] established in 1913:

$$\left\|x'\right\|_{C(\mathbb{R}_+)} \leq 2\|x\|^{\frac{1}{2}}_{C(\mathbb{R}_+)}\left\|x''\right\|^{\frac{1}{2}}_{L_\infty(\mathbb{R}_+)}. \tag{2}$$

Similar to (2), sharp inequalities for functions defined on the whole real line are also, in fact, contained in [4] (see [2], Section 1.2).

Later inequalities of type (2) for functions defined on $\mathbb{R}$ and $\mathbb{R}_+$ were generalized in many directions by many mathematicians. One of the brightest and the most important results in the whole field is Kolmogorov's inequality [5–7] for functions defined on the real line $\mathbb{R}$. After this result inequalities of type (2) are called Kolmogorov type inequalities. In articles [8–13] and monographs [2, 14], one can find a detailed overview of classical results about sharp inequalities for derivatives and further references. Articles [15] and [16] are devoted to inequalities between derivatives on classes with non-constant restrictions on the higher derivatives; in [17] and [18] inequalities for derivatives on classes of functions defined on a finite interval are considered; in [19] discrete analogues of inequalities are considered; in [20] and [21] one can find results connected to inequalities for fractional derivatives and further references.

We discuss some of the results for functions defined on the half-line in a more detailed way.

Let $T_r(t) := \cos r \arccos t$, $t \in [-1, 1]$, be Chebyshev polynomials of the first kind. Matorin in 1955 proved the following theorem (see [22]).

Theorem C *Let $k, r \in \mathbb{N}$, $k < r$. For arbitrary function $x \in L^r_{\infty,\infty}(\mathbb{R}_+)$, the following inequality holds*:

$$\left\|x^{(k)}\right\|_\infty \leq \frac{T_r^{(k)}(1)}{[T_r^{(r)}(1)]^{\frac{k}{r}}}\|x\|_\infty^{1-\frac{k}{r}}\left\|x^{(r)}\right\|_\infty^{\frac{k}{r}}. \tag{3}$$

In the cases $r = 2$ and $r = 3$, the inequality above is sharp.

For $r > 3$, inequality (3) is not sharp. Sharp inequality that estimates $\|x^{(k)}\|_{C(\mathbb{R}_+)}$ using $\|x\|_{C(\mathbb{R}_+)}$ and $\|x^{(r)}\|_{L_\infty(\mathbb{R}_+)}$ for functions $x \in L^r_{\infty,\infty}(\mathbb{R}_+)$ was received by Schoenberg and Cavaretta (see [23, 24]) in 1970 (see also [2], Section 3.3).

The function

$$\omega(\delta) = \omega\left(D^k, \delta\right) := \sup_{x \in W^r_{f,g}(I), \|x\|_{C(I),f} \leq \delta} \left\|x^{(k)}\right\|_{C(I)}, \quad \delta \geq 0, \tag{4}$$

is called modulus of continuity of kth order differentiation operator on the class $W^r_{f,g}(I)$ ($k = 1, 2, \dots, r-1$), where, as before, I denotes a finite interval, the whole real line $\mathbb{R}$, or a positive half-line $\mathbb{R}_+$.

Note that in the case $r = 2$ Theorem B gives an estimate for the modulus of continuity $\omega(D^1, \delta) \leq 2\delta^{\frac{1}{2}}$, $\delta > 0$.

The result by Schoenberg and Cavaretta gives a sharp Kolmogorov type inequality for arbitrary orders of derivatives $k < r$. The exact constant $C(k, r)$ in this inequality is given implicitly in terms of a limit of a perfect splines sequence. In the case when $f = g \equiv 1$, the sharp Kolmogorov type inequality is equivalent to the equality $\omega(D^k, \delta) = C(k, r)\delta^{1-\frac{k}{r}}$, $\delta > 0$. So we can think that in the case of constant f and g this result gives the value of $\omega(D^k, \delta)$ for all $\delta \geq 0$ (although rather implicitly). Theorem C gives $\omega(D^k, \delta)$, $\delta \geq 0$ for $r \leq 3$ (with explicit constant).

Information about the connection between modulus of continuity of differentiation operator and Kolmogorov type inequalities and further references can be found in [2], Section 7 and Chapter 7.

The aim of this article is to study the function $\omega(D^k, \delta)$ for arbitrary $k, r \in \mathbb{N}$, $k < r$ and non-increasing continuous positive functions f and g.

The article is organized in the following way. In Section 2 some auxiliary statements and in Section 3 main statements are given. Sections 4 and 5 are devoted to proofs.

2 Auxiliary results

Let a positive function $g \in C[a, b]$ be given. A function $G \in C^{r-1}[a, b]$ is called a perfect g-spline of order r with knots $a < t_1 < \cdots < t_n < b$ if the following conditions hold:

(a) the derivative $G^{(r)}$ exists for all $t \in (t_i, t_{i+1})$, $i = 0, 1, \ldots, n$, where $t_0 := a$ and $t_{n+1} := b$;
(b) there exists $\epsilon \in \{1, -1\}$ such that $\frac{G^{(r)}(t)}{g(t)} \equiv \epsilon \cdot (-1)^i$ for $t \in (t_i, t_{i+1})$, $i = 0, 1, \ldots, n$.

Denote by $\Gamma^r_{n,g}[0, a]$ the set of all perfect g-splines G defined on $[0, a]$ of order r with not more than n knots.

Below f and g will denote continuous positive non-increasing on $[0, \infty)$ functions.

The next theorem proves existence and some properties of the perfect g-spline $G_{r,n,f,a} \in \Gamma^r_{n,g}[0, a]$ that least deviates from zero in $\|\cdot\|_{C[0,a],f}$ norm.

Theorem 1 *Let numbers $a > 0$, $r \in \mathbb{N}$, $n \in \mathbb{Z}_+$ be given. Then there exists a perfect g-spline $G_{r,n,f,a} \in \Gamma^r_{n,g}[0, a]$ that has $n + r + 1$ oscillation points, i.e., such that there exist $n + r + 1$ points $0 \le t_1 < t_2 < \cdots < t_{n+r+1} = a$ such that*

$$G_{r,n,f,a}(t_i) = (-1)^{i+r+1} \|G_{r,n,f,a}\|_{C[0,a],f} \cdot f(t_i), \quad i = 1, 2, \ldots, n + r + 1. \tag{5}$$

For $a > 0$, set $\varphi_{r,n,f}(a) := \|G_{r,n,f,a}\|_{C[0,a],f}$. Then $\varphi_{r,n,f}(a)$ is a continuous and non-decreasing function of $a \in (0, \infty)$.

Remark We do not prove the uniqueness of the spline $G_{r,n,f,a} \in \Gamma^r_{n,g}[0, a]$ satisfying (5). However, from arguments similar to the ones used in the proof of Theorem 2, it follows that if two splines $G_1, G_2 \in \Gamma^r_{n,g}[0, a]$ satisfy (5), then $G_1^{(k)}(0) = G_2^{(k)}(0)$ for all $k = 1, \ldots, r - 1$.

The role of perfect g-splines becomes clearer due to the following theorem.

Theorem 2 *Let $r \in \mathbb{N}$, $n \in \mathbb{Z}_+$ and $\delta > 0$ be such that $\varphi_{r,n,f}(a) = \delta$ for some $a > 0$. Then, for $k = 1, 2, \ldots, r - 1$,*

$$\omega\bigl(D^k, \delta\bigr) \le \bigl\|G^{(k)}_{r,n,f,a}\bigr\|_{C[0,a]}.$$

3 Main results

If $f(t) \equiv 1$ and $g(t) \equiv 1$, then $\varphi_{r,n,f}(\infty) := \lim_{a\to+\infty} \varphi_{r,n,f}(a) = \infty$ for all $r \in \mathbb{N}$, $n \in \mathbb{Z}_+$. In the case when f, g are arbitrary positive non-increasing continuous functions, this is not always true.

Set $g_k(t) := \int_0^t g_{k-1}(s)\, ds$, $k = 1, 2, \ldots, r$, where $g_0 := g$. The following theorem holds.

Theorem 3 *Let numbers $n \in \mathbb{Z}_+$ and $r \in \mathbb{N}$ be given. $\varphi_{r,n,f}(\infty) < \infty$ if and only if the following conditions hold*:

$$\begin{aligned} A_0 &:= \int_0^\infty g(t)\, dt < \infty, \\ A_k &:= \int_0^\infty \left[\sum_{s=0}^{k-1} \frac{(-1)^{k-s-1} A_s}{(k-s-1)!} t^{k-s-1} + (-1)^k g_k(t) \right] dt < \infty, \quad k = 1, \ldots, r - 1 \end{aligned} \tag{6}$$

and

$$\sup_{t\in[0,\infty)} \frac{|\sum_{s=0}^{r-1} \frac{(-1)^{r-s-1} A_s}{(r-s-1)!} t^{r-s-1} + (-1)^r g_r(t)|}{f(t)} < \infty. \tag{7}$$

Remark From Theorem 3 it follows that for all $r \in \mathbb{N}$, $n \in \mathbb{Z}_+$, $\varphi_{r,n,f}(\infty) < \infty$ if and only if $\varphi_{r,0,f}(\infty) < \infty$.

In the case when conditions (6) hold, set

$$P_k(t) := \sum_{s=0}^{k-1} \frac{(-1)^{s+1} A_s}{(k-s-1)!} t^{k-s-1} + g_k(t), \quad k = 1, 2, \dots, r. \tag{8}$$

The functions P_k have the following properties. P_k is kth primitive of the function g, which does not change sign on $[0,\infty)$ (positive for even k and negative for odd k), $k = 1, 2, \dots, r$; $P_k' = P_{k-1}$, $k = 2, \dots, r$; $P_k(0) = (-1)^k A_{k-1}$, $k = 1, \dots, r$.

If $\varphi_{r,0,f}(\infty) = \infty$, then, in virtue of Theorems 1 and 3, for all $r \in \mathbb{N}$, $n \in \mathbb{Z}_+$ and $\delta > 0$, there exists a number $\delta_{r,n} = \delta_{r,n}(\delta) > 0$ such that $\|G_{r,n,f,\delta_{r,n}}\|_{C[0,\delta_{r,n}],f} = \delta$ (if such number $\delta_{r,n}$ is not unique, we can take the minimal value). In this case, for all $\delta > 0$, the function $\omega(D^k, \delta)$ is characterized by the following theorem.

Theorem 4 *Let $r \in \mathbb{N}$ and $\varphi_{r,0,f}(\infty) = \infty$. Then, for all $\delta > 0$ and $k = 1, 2, \dots, r-1$,*

$$\omega\left(D^k, \delta\right) = \lim_{n\to\infty} \left|G^{(k)}_{r,n,f,\delta_{r,n}}(0)\right|.$$

Information about the function $\omega(D^k, \delta)$ in the case when $\varphi_{r,0,f}(\infty) < \infty$ is given by the following theorem.

Theorem 5 *Let $r \in \mathbb{N}$, $n \in \mathbb{Z}_+$ and $\varphi_{r,0,f}(\infty) < \infty$. Then, for all $k = 1, 2, \dots, r-1$,*

$$\omega\left(D^k, \varphi_{r,n,f}(\infty)\right) = \lim_{a\to\infty} \left|G^{(k)}_{r,n,f,a}(0)\right|.$$

In the case when $\varphi_{r,0,f}(\infty) < \infty$, information about asymptotic behavior of the function $\varphi_{r,n,f}(\infty)$ as $n \to \infty$ and fixed r is given by the following theorem.

Theorem 6 *Let $r \in \mathbb{N}$ and $\varphi_{r,0,f}(\infty) < \infty$. $\lim_{n\to\infty} \varphi_{r,n,f}(\infty) > 0$ if and only if $\underline{\lim}_{t\to\infty} \frac{f(t)}{|P_r(t)|} < \infty$, where the function $P_r(t)$ is defined in* (8).

4 Proofs of the auxiliary results

4.1 Proof of Theorem 1

Proof of existence and uniqueness of the perfect g-spline $G_{r,n,f,a}$ uses ideas that were used to prove Theorem 3.3.1 in monograph [2].

In the space $\mathbb{R}^{n+1}_1$ consider the sphere S^n with radius a, *i.e.*,

$$S^n = \left\{ \xi = (\xi_1, \xi_2, \dots, \xi_{n+1}) : \sum_{i=1}^{n+1} |\xi_i| = a \right\}.$$

For each $\xi \in S^n$, consider the partition of the segment $[0,a]$ by points

$$\xi_0 := 0, \qquad |\xi_1|, \qquad |\xi_1| + |\xi_2|, \qquad \dots, \qquad \sum_{i=1}^{n} |\xi_i|, \qquad \sum_{i=1}^{n+1} |\xi_i| = a.$$

Set $I_k := (\sum_{i=0}^{k-1} |\xi_i|, \sum_{i=0}^{k} |\xi_i|)$, $k = 1, 2, \dots, n+1$. For each of the partitions, consider the function

$$G_\xi^a(t) := \frac{1}{(r-1)!} \int_0^a (t-u)_+^{r-1} g_\xi^a(u)\,du,$$

where $g_\xi^a(t) = g(t) \operatorname{sgn} \xi_k$ on each segment I_k, $k = 1, 2, \dots, n+1$.

Then we have $(G_\xi^a)^{(r)} = g_\xi^a$ and hence G_ξ^a is a g-spline with knots at the points of partition. Let $Q_{n+r-1}^{\xi,a}(t) = \sum_{i=0}^{n+r-1} a_i(\xi) t^i$ be the polynomial on which $\inf_{Q_{n+r-1}} \|G_\xi^a - Q_{n+r-1}\|_{C[0,a],f}$ over all polynomials of degree less than or equal to $n+r-1$ is attained. Consider the mapping $\phi : S^n \to \mathbb{R}^n$, $\phi(\xi) := (a_r(\xi), \dots, a_{n+r-1}(\xi))$. From the definition of ϕ and properties of polynomials of the best approximation, it follows that ϕ is continuous and odd. Hence from Borsuk's theorem it follows that there exists $\xi_0 \in S^n$ such that $\phi(\xi_0) = 0$. This means that the polynomial $Q_{n+r-1}^{\xi_0,a}$ has order less than or equal to $r-1$. Therefore, for the function $G_{r,n,f,a} := G_{\xi_0} - Q_{n+r-1}^{\xi_0,a}$, we have $G_{r,n,f,a}^{(r)} = g_{\xi_0}$. Due to the generalization of Chebyshev's theorem about oscillation (see, for example, [25, 26], Chapter 9, Section 5) $G_{r,n,f,a}$ has $n+r+1$ oscillation points $0 \le t_1 < t_2 < \cdots < t_{n+r+1} \le a$ and hence at least $n+r$ sign changes. Thus, in view of Rolle's theorem, $G_{r,n,f,a}^{(r)}$ has at least n sign changes (and hence exactly n sign changes due to construction). This means that $G_{r,n,f,a}$ is a perfect g-spline with exactly n nodes, in particular, $G_{r,n,f,a} \in \Gamma_{n,g}^r[0,a]$.

Let us prove that $t_{n+r+1} = a$. Assume the converse. Since f is non-increasing, we get that $G'_{r,n,f,a}$ has $n+r$ sign changes and hence $G_{r,n,f,a}^{(r)}$ has $n+1$ sign changes. However, this is impossible. Multiplying, if needed, the function $G_{r,n,f,a}$ by -1, we get a perfect g-spline for which equalities (5) hold.

The fact that for fixed $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ the function $\varphi_{r,n,f}$ is non-decreasing follows from its definition. The continuity of the function $\varphi_{r,n,f}$ follows from the continuity of functions f and g. The theorem is proved.

4.2 Proof of Theorem 2

We need the following lemma.

Lemma 1 *Let $r \in \mathbb{N}$, $n \in \mathbb{Z}_+$, $a > 0$ and $x \in L_{\infty,\infty}^r[0,a]$ be given. Assume that a function x has at least $n+r$ sign changes, $x^{(r)}$ has not more than n sign changes and $x^{(r)}$ is non-zero almost everywhere. Then, for all $s = 0, 1, \dots, r-1$,*

$$\operatorname{sgn} x^{(s)}(0) = -\operatorname{sgn} x^{(s+1)}(0). \tag{9}$$

Remark Notation $\operatorname{sgn} x^{(r)}(0) = \pm 1$ means that there exists $\varepsilon > 0$ such that $\operatorname{sgn} x^{(r)}(t) = \pm 1$ almost everywhere in the interval $(0, \varepsilon)$.

From conditions of the lemma it follows that the function $x^{(s)}$ has exactly $n+r-s$ sign changes, $s = 0, 1, \dots, r$. Hence the function $x^{(s)}$ changes sign on each of its monotonicity intervals, $s = 0, 1, \dots, r-1$. This implies that $x^{(s)}(0) \ne 0$, $s = 0, 1, \dots, r-1$, and that equalities (9) hold. The lemma is proved.

Let us return to the proof of the theorem. Assume the converse, let a function $x \in W_{f,g}^r(\mathbb{R}_+)$ be such that $\|x\|_{C[0,\infty),f} \le \delta$ and $\|x^{(k)}\|_{C[0,\infty)} > \|G_{r,n,f,a}^{(k)}\|_{C[0,a)}$. We can assume that

$$\left|x^{(k)}(0)\right| > \left|G_{r,n,f,a}^{(k)}(0)\right| \tag{10}$$

(if this is not true, then there exists a point $t_0 > 0$ such that $|x^{(k)}(t_0)| > \|G^{(k)}_{r,n,f,a}\|_{C[0,a)}$ and instead of $x(t)$ we can consider the function $y(t) := x(t + t_0)$); then $y \in W^r_{f,g}(\mathbb{R}_+)$ and $\|y\|_{C[0,\infty),f} \le \delta$; moreover, we can assume that $\|x\|_{C(0,\infty),f} < \delta$ and $\|x^{(r)}\|_{L_\infty(0,\infty),g} < 1$; otherwise we can consider the function $(1-\varepsilon)x$ instead of the function x with $\varepsilon > 0$ so small that inequality (10) remains true. Further, multiplying, if needed, functions x and $G_{r,n,f,a}$ by -1, we can suppose that

$$x^{(k)}(0) > G^{(k)}_{r,n,f,a}(0) > 0. \tag{11}$$

Set $\Delta(t) := x(t) - G_{r,n,f,a}(t)$. Note that in view of construction for g-splines $G_{r,n,f,a}(t)$, functions $\Delta(t)$ and $G_{r,n,f,a}(t)$ have not less than $n+r$ sign changes ($G_{r,n,f,a}(t)$ has exactly $n+r$ sign changes); functions $\Delta^{(r)}(t)$ and $G^{(r)}_{r,n,f,a}(t)$ can have sign changes only in the knots of the g-spline $G_{r,n,f,a}(t)$, and hence not more than n sign changes. From assumptions above it follows that the function $\Delta^{(r)}(t)$ is non-zero almost everywhere. From Lemma 1 and (11) we get

$$(-1)^k G_{r,n,f,a}(0) > 0. \tag{12}$$

Due to (12), $(-1)^k G_{r,n,f,a}(t_1) > 0$, where t_1 is the first oscillation point of $G_{r,n,f,a}$. This means that $(-1)^k \Delta(t_1) < 0$. Since all sign changes of the function Δ are located inside the interval (t_1, a), we get $(-1)^k \Delta(0) < 0$, and hence, in virtue of Lemma 1, we get $\Delta^{(k)}(0) < 0$. But this contradicts (11). The theorem is proved.

5 Proofs of the main results

5.1 Proof of Theorem 3

We prove first that the statement of the theorem is true in the case $n = 0$. In the case $n = 0$, we write $\varphi_{r,f}$ instead of $\varphi_{r,0,f}$ and $G_{r,f,M}$ instead of $G_{r,0,f,M}$. To prove the theorem, we need the following lemma.

Lemma 2 *Let conditions* (6) *hold. Suppose $M > 0$ and $h_m(t)$ is the mth primitive of the function $g(t)$ on the interval $[0,M]$ that has m zeroes $(1 \le m \le r)$. Denote by α_m the first zero of the function $h_m(t)$. Then the following inequalities hold*:

$$\bigl|h_m(t)\bigr| < \bigl|P_m(t)\bigr|, \quad t \in [0, \alpha_m]$$

and

$$\bigl|P_m(t) - P_m(0) + h_m(0)\bigr| \le \bigl|h_m(t)\bigr|, \quad t \in [0, \gamma_m],$$

where γ_m is the unique zero of the function $P_m(t) - P_m(0) + h_m(0)$.

We will proceed using induction on m in order to prove the first inequality.

Let $m = 1$. $P_1(t) = -A_0 + g_1(t)$. Since conditions (6) hold and $P_1(0) = -A_0$, we have $P_1(\infty) = 0$. Since the function $h_1(t) = g_1(t) + C$ has one zero, $C \in (-A_0, 0]$. This means that $|h_1(t)| < |P_1(t)|$ for all $t \in [0, \alpha_1]$.

Let the statement of the lemma hold in the case $m = k \le r-1$. We will show that it holds in the case $m = k+1$ too.

To be definite, assume that the number $k+1$ is even. Then $P_{k+1}(0) > 0$, and in view of Lemma 1,

$$h_{k+1}(0) > 0. \tag{13}$$

Assume the converse, let a point $t_0 \in [0, \alpha_{k+1})$ such that $h_{k+1}(t_0) \geq P_{k+1}(t_0)$ exist. Since the function $h_{k+1}(t)$ has $k+1$ zeroes, the function $h_{k+1}^{(k+1)}(t) = g(t)$ does not have zeroes, we have that the function $h'_{k+1}(t)$ has k zeroes, and hence, in virtue of induction assumption and $P'_{k+1}(t) = P_k(t)$, we get that $|h'_{k+1}(t)| < |P_k(t)|$ $\forall t \in [0, \beta]$, where β is the first zero of the function $h'_{k+1}(t)$. Hence, due to (13) for all $t \in [0, \beta]$, the following inequality holds:

$$0 \geq h'_{k+1}(t) > P_k(t). \tag{14}$$

In view of Rolle's theorem, $\beta > \alpha_k$. This means that inequality (14) holds for all $t \in [0, \alpha_k]$. Since $h_{k+1}(\alpha_{k+1}) = 0 < P_{k+1}(\alpha_{k+1})$, we have $P_{k+1}(t_0) - P_{k+1}(\alpha_{k+1}) < h_{k+1}(t_0) - h_{k+1}(\alpha_{k+1})$; on the other hand, inequality (14) holds, and hence

$$\begin{aligned} P_{k+1}(t_0) - P_{k+1}(\alpha_{k+1}) &= -\int_{t_0}^{\alpha_{k+1}} P_k(t)\,dt > -\int_{t_0}^{\alpha_{k+1}} h'_{k+1}(t)\,dt \\ &= h_{k+1}(t_0) - h_{k+1}(\alpha_{k+1}). \end{aligned}$$

Contradiction. The first inequality is proved.

From Lemma 1 it follows that $\operatorname{sgn} h_m(0) = \operatorname{sgn} P_m(0)$. From the proved part of the lemma it now follows that the function $P_m(t) - P_m(0) + h_m(0)$ has exactly one zero. The second inequality in the statement of the lemma can be proved using arguments similar to the ones used in the proof of the first inequality. The lemma is proved.

Let us return to the proof of the theorem in the case when $n = 0$.

Let conditions (6) and (7) hold. Set $K_r := \sup_{t \in [0,\infty)} \frac{|P_r(t)|}{f(t)}$. Assume the converse, let $\varphi_{r,f}(\infty) = \infty$. This means that there exists $M > 0$ such that $\varphi_{r,f}(M) > K_r$. Due to Lemma 2 the inequality $|G_{r,f,M}(t)| < |P_r(t)|$ holds on the interval $[0, \alpha_r^M]$. Moreover, $|P_r(t)| \leq K_r f(t) < \varphi_{r,f}(M) f(t)$. Thus, on the interval $[0, \alpha_r^M]$, the inequality $|G_{r,f,M}(t)| < \varphi_{r,f}(M) f(t)$ holds. However, in this case $G_{r,f,M}(t)$ has not more than r oscillating points. Contradiction. Sufficiency is proved.

Let now $\varphi_{r,f}(\infty) < \infty$. This means that for all $a \geq 0$, $t \in [0, a]$ we have $|G_{r,f,a}(t)| \leq \varphi_{r,f}(\infty) f(t) \leq \varphi_{r,f}(\infty) f(0)$. Passing to the limit as $a \to \infty$, we get existence of bounded on $[0, \infty)$ primitive Q_r of order r of the function $g(t)$. Really, for each $a > 0$, we have $G_{r,f,a}(t) = g_r(t) + \sum_{k=0}^{r-1} c_k(a) t^k$, where c_k are real functions, $k = 0, \dots, r-1$, and g_r is some (fixed) primitive of order r of the function g. For $a, t \geq 0$, set $R(a; t) := \sum_{k=0}^{r-1} c_k(a) t^k$. From Markov's inequality for polynomials we get that for $k = 0, \dots, r-1$ there exists a constant N_k independent of a such that

$$\begin{aligned} \bigl|c_k(a)\bigr| &= \bigl|R^{(k)}(a; 0)\bigr| \leq \max_{t \in [0,1]} \bigl|R^{(k)}(a; t)\bigr| \leq N_k \max_{t \in [0,1]} \bigl|R(a; t)\bigr| \\ &\leq N_k \Bigl[\max_{t \in [0,1]} \bigl|g_r(t)\bigr| + \varphi_{r,f}(\infty) f(0)\Bigr]. \end{aligned}$$

Hence the functions c_k are bounded, $k = 0, \dots, r-1$. This means that there exists a sequence $\{a_n\}_{n=1}^{\infty}$, $\lim_{n \to \infty} a_n = \infty$ such that all the sequences $c_k(a_n)$ have finite limits $c_k :=$

$\lim_{n\to\infty} c_k(a_n)$, $k = 0, \ldots, r-1$. For the function $Q_r(t) := g_r(t) + \sum_{k=0}^{r-1} c_k t^k$, fixed $t \geq 0$ and all natural n such that $a_n > t$, we have

$$\begin{aligned}
\left|Q_r(t)\right| &= \left|g_r(t) + \sum_{k=0}^{r-1} c_k(a_n)t^k + \sum_{k=0}^{r-1} c_k t^k - \sum_{k=0}^{r-1} c_k(a_n)t^k\right| \\
&\leq \left|g_r(t) + \sum_{k=0}^{r-1} c_k(a_n)t^k\right| + \left|\sum_{k=0}^{r-1}\bigl(c_k - c_k(a_n)\bigr)t^k\right| \\
&= \left|G_{r,f,a_n}(t)\right| + \left|\sum_{k=0}^{r-1}\bigl(c_k - c_k(a_n)\bigr)t^k\right|.
\end{aligned}$$

Since n can be arbitrarily large and all the functions $G_{r,n,f,a}$, $a > 0$ are bounded on $[0, a]$, by an absolute constant $\varphi_{r,f}(\infty)f(0)$, we get that Q_r is a bounded on $[0, \infty)$ primitive of order r of the function g.

Since functions $f(t)$ and $g(t)$ are bounded, we get that all functions $Q_r^{(k)}(t)$, $k = 1, \ldots, r-1$, are also bounded on $[0, \infty)$. Note that the only bounded on $[0, \infty)$ primitives of the function $g(t)$ of order $k \in \mathbb{N}$ are functions $P_k(t) + C_k$, where $C_k \in \mathbb{R}$, and only in the case when corresponding conditions (6) hold. This means that conditions (6) hold. Necessity of conditions (6) are proved.

Note that from arguments above it follows that the following lemma holds.

Lemma 3 *Let $r \in \mathbb{N}$, $r \geq 2$ and $\varphi_{r,f}(\infty) < \infty$. Then $|G_{r,f,M}^{(r-k-1)}(0)| \to A_k$ and $\alpha_{k+1}^M \to \infty$ when $M \to \infty$, where α_{k+1}^M is the first zero of the function $G_{r,f,M}^{(r-k-1)}$, $k = 0, 1, \ldots, r-2$.*

We will prove that condition (7) also holds. If $f(\infty) > 0$, then condition (7) holds always when conditions (6) hold. Below we will assume that

$$f(\infty) = 0. \tag{15}$$

From Lemma 2 we get

$$\begin{aligned}
\left|\int_0^{\alpha_{r-1}^M} P_{r-1}(t)\,dt\right| &\geq \left|G_{r,f,M}(0) - G_{r,f,M}\bigl(\alpha_{r-1}^M\bigr)\right| \\
&\geq \left|\int_0^{\gamma_M} \bigl[P_{r-1}(t) - P_{r-1}(0) + G'_{r,f,M}(0)\bigr]\,dt\right|,
\end{aligned}$$

where γ_M is zero of the function $P_{r-1}(t) - P_{r-1}(0) + G'_{r,f,M}(0)$.

In view of Lemma 3 (with $k = r-2$, $M \to \infty$) we get

$$\left|\int_0^{\alpha_{r-1}^M} P_{r-1}(t)\,dt\right| \to \left|\int_0^{\infty} P_{r-1}(t)\,dt\right| = A_{r-1}; \qquad -P_{r-1}(0) + G'_{r,f,M}(0) \to 0,$$

hence

$$\gamma_M \to \infty \quad \text{and} \quad \left|\int_0^{\gamma_M} \bigl[P_{r-1}(t) - P_{r-1}(0) + G'_{r,f,M}(0)\bigr]\,dt\right| \to A_{r-1}.$$

From Lemma 3 and equality (15) we get $G_{r,f,M}(\alpha_{r-1}^M) \to 0$ and hence

$$\left|G_{r,f,M}(0)\right| \to A_{r-1}, \quad M \to \infty. \tag{16}$$

We will show that $\varphi_{r,f}(\infty) \geq \sup_{t\in[0,\infty)} \frac{P_r(t)}{f(t)}$. Assume the converse. Suppose that a point t_0 such that $|P_r(t_0)| > \varphi_{r,f}(\infty)f(t_0)$ exists. We can choose $\varepsilon > 0$ in such a way that

$$\left|P_r(t_0) - \varepsilon \operatorname{sgn}\left[P_r(0)\right]\right| > \varphi_{r,f}(\infty)f(t_0). \tag{17}$$

In virtue of (16) and Lemma 2, we can choose $M > 0$ big enough, so that

$$\left|P_r(t) - \varepsilon \operatorname{sgn}\left[P_r(0)\right]\right| < \left|G_{r,f,M}(t)\right| < \left|P_r(t)\right| \quad \forall t \in [0, \gamma],$$

where γ is zero of the function $P_r(t) - \varepsilon \operatorname{sgn}[P_r(0)]$. Since inequality (17) holds, we have $\gamma > t_0$, and hence $|G_{r,f,M}(t_0)| > \varphi_{r,f}(\infty)f(t_0)$. Contradiction. Thus condition (7) is proved. The theorem is proved in the case when $n = 0$.

Let n be an arbitrary natural number now.

We will prove that for all $r, n \in \mathbb{N}$, $\varphi_{r,n,f}(\infty) < \infty$ if and only if $\varphi_{r,f}(\infty) < \infty$.

It is clear that $\varphi_{r,f}(M) \geq \varphi_{r,n,f}(M)$ for all $M > 0$, and hence $\varphi_{r,f}(\infty) < \infty$ implies $\varphi_{r,n,f}(\infty) < \infty$.

Assume that $\varphi_{r,n,f}(\infty) < \infty$. Denote by $t_{n,k}^M$ the kth knot of the g-spline $G_{r,n,f,M}(t)$, $k = 1, 2, \ldots, n$. Set $t_{n,0}^M := 0$, $t_{n,n+1}^M := M$. Let $1 \leq k \leq n+1$ be the smallest number of the knots of the g-spline $G_{r,n,f,M}(t)$ for which the set $\{t_{n,k}^M : M > 0\}$ is unbounded. We can choose an increasing sequence $\{M_l\}_{l=1}^{\infty}$, $M_l \to \infty$ as $l \to \infty$ such that $t_{n,s}^{M_l} \to t_{n,s} < \infty$, $s \leq k-1$ and $t_{n,k}^{M_l} \to \infty$ as $l \to \infty$.

Denote by $G_{r,f,M}^K(t)$ the primitive of order r of the function g that least deviates from zero in norm $\|\cdot\|_{C[K,K+M],f}$. Set

$$\varphi_{r,f}^K(M) := \left\|G_{r,f,M}^K\right\|_{C[K,K+M],f}.$$

Then, for all l, $\|G_{r,n,f,M_l}\|_{C[t_{n,k-1}^{M_l},t_{n,k}^{M_l}],f} \geq \varphi_{r,f}^{t_{n,k-1}}(t_{n,k}^{M_l} - t_{n,k-1}^{M_l})$. Passing to the limit as $l \to \infty$, we get $\varphi_{r,f}^{t_{n,k-1}}(\infty) \leq \varphi_{r,n,f}(\infty) < \infty$. Note that from the case when $n = 0$ proved above, it follows that for all $K > 0$, $\varphi_{r,f}^K(\infty) < \infty$ if and only if $\varphi_{r,f}(\infty) < \infty$. The theorem is proved.

Remark In the case when $f \equiv 1$, for all natural r, $\varphi_{r,f}(\infty) < \infty$ implies $\varphi_{r-1,f}(\infty) < \infty$. At the same time not for all functions f, $\varphi_{r,f}(\infty) < \infty$ implies $\varphi_{r-1,f}(\infty) < \infty$. In particular this implies that for non-constant functions f,

$$\sup_{x\in W_{f,g}^r(\mathbb{R}_+), \|x\|_{C(\mathbb{R}_+),f}\leq\delta} \left\|x^{(k)}\right\|_{C(\mathbb{R}_+),f}$$

may be infinite for all $\delta > 0$.

Really, in the case when $f \equiv 1$, condition (7) holds always when conditions (6) hold. If

$$g(t) = \left[-1 + (t + \sqrt{3})^2\right]e^{-\frac{(t+\sqrt{3})^2}{2}}$$

and

$$f(t) = e^{-\frac{(t+\sqrt{3})^2}{2}},$$

then

$$P_1(t) = -\big[(t+\sqrt{3})\big]e^{-\frac{(t+\sqrt{3})^2}{2}},$$

$$P_2(t) = e^{-\frac{(t+\sqrt{3})^2}{2}},$$

and condition (7) holds when $r = 2$ and does not hold when $r = 1$.

5.2 Proof of Theorem 4

To prove Theorem 4, it is sufficient to prove that for all $\delta > 0$ there exists a g-spline $G_{r,\delta} = G_{r,\delta}(\cdot, f, g)$ of order r defined on $[0,\infty)$ with infinite number of knots y_k ($k = 1, 2, \ldots$), $0 = y_0 < y_1 < \cdots < y_k < \cdots$, $y_k \to \infty$ ($k \to \infty$), with the following properties:

1. $\|G_{r,\delta}\|_{C[0,\infty),f} = \delta$ and either $G^{(r)}_{r,\delta} \equiv g$ or $G^{(r)}_{r,\delta} \equiv -g$ on the intervals (y_k, y_{k+1}) ($k = 0, 1, 2, \ldots$).
2. For all $c > 0$, the sequences $\{G^{(k)}_{r,n,f,\delta_{r,n}}\}_{n=0}^{\infty}$ ($k = 0, 1, \ldots, r-1$) (whose elements are defined on $[0,c]$ for big enough n) converge to $G^{(k)}_{r,\delta}$ uniformly on $[0,c]$.

Really, from Theorem 2 it will follow that $\omega(D^k, \delta) = |G^{(k)}_{r,\delta}(0)|$, and from condition 2 it will follow that

$$\lim_{n\to\infty} \big|G^{(k)}_{r,n,f,\delta_{r,n}}(0)\big| = \big|G^{(k)}_{r,\delta}(0)\big|.$$

$\{\delta_{r,n}\}_{n=0}^{\infty}$ is a non-decreasing sequence. Moreover, this sequence is unbounded because otherwise we would get a perfect g-spline G with arbitrarily close oscillating points; this is impossible because the functions G and $G^{(r)}$ (and hence G') are bounded.

Denote by $t_{n,k}$ ($k = 1, \ldots, n$, $n = 1, 2, \ldots$) the knots of the g-spline $G_{r,n,f,\delta_{r,n}}$. We can choose a sequence n_s ($n_s \to \infty$ as $s \to \infty$) such that every sequence $\{t_{n_s,k}\}_{n_s \ge k}^{\infty}$ ($k = 1, 2, \ldots$) has a (finite or infinite) limit.

Let $0 \le y_1 < y_2 < \cdots$ be all distinct finite limits of these sequences, ordered in an ascending way. The number of the nodes y_k is infinite since from the statement of the theorem, we have $\varphi_{r,n,f}(\infty) = \infty$ for all $n \in \mathbb{N}$.

For all $i \in \mathbb{N}$ and for all small enough $\varepsilon > 0$, there exists $N = N(i,\varepsilon)$ such that for every $n > N(i,\varepsilon)$, $G^{(r)}_{r,n,f,\delta_{r,n}} \equiv g$ or $G^{(r)}_{r,n,f,\delta_{r,n}} \equiv -g$ on $I_i(\varepsilon) := (y_{i-1}+\varepsilon, y_i - \varepsilon)$. In other words, for each $i \in \mathbb{N}$ starting with some $n = N(i,\varepsilon)$, the restriction of the g-spline $G_{r,n,f,\delta_{r,n}}$ to the interval $I_i(\varepsilon)$ is a primitive of order r of g or $-g$. Since $\varepsilon > 0$ is arbitrary, on each interval (y_{i-1}, y_i) we get existence of point-wise limit $\lim_{n\to\infty} G_{r,n,f,\delta_{r,n}} =: G_{r,\delta}$; moreover, on the intervals (y_i, y_{i+1}), $G^{(r)}_{r,\delta} \equiv g$ or $G^{(r)}_{r,\delta} \equiv -g$ ($i = 1, 2, \ldots$). It is clear that $\|G_{r,\delta}\|_{C[0,\infty),f} = \delta$. Using arguments similar to the ones used to prove that $\lim_{n\to\infty} \delta_{r,n} = +\infty$, we can prove that $y_k \to \infty$ ($k \to \infty$).

Let us fix some $c > 0$. Starting with some n, all g-splines $G_{r,n,f,\delta_{r,n}}$ are defined on $[0,c]$. From $\|G_{r,n,f,\delta_{r,n}}\|_{C[0,\delta_{r,n}],f} = \delta$ and the fact that f is non-increasing (and hence is bounded) it follows that the sequence $\{G_{r,n,f,\delta_{r,n}}\}_{n=0}^{\infty}$ is uniformly bounded on $[0,c]$; from $|G^{(r)}_{r,n,f,\delta_{r,n}}(t)| \le g(t)$ almost everywhere on $[0,\infty)$ and the fact that g is non-increasing (and hence is bounded) it follows that sequences $\{G^{(k)}_{r,n,f,\delta_{r,n}}\}_{n=0}^{\infty}$, $k = 0, \dots, r-1$, are uniformly bounded on $[0, c]$ and equicontinuous. The later implies uniform convergence on $[0, c]$ of the sequence $G_{r,n,f,\delta_{r,n}}$ to $G_{r,f}$. The theorem is proved.

5.3 Proof of Theorem 5

Let $n \geq 0$. We can choose an increasing sequence $\{M_k\}_{k=1}^{\infty}$, $M_k \to \infty$ as $k \to \infty$ in such a way that all sequences $t^{M_k}_{n,s}$, $1 \leq s \leq n$ (as above, $t^{M_k}_{n,s}$ is the sth knot of the g-spline G_{r,n,f,M_k}) have limits (finite or infinite). Let $t_{n,1} < \cdots < t_{n,m}$ be all distinct finite limits of these sequences in ascending order. Analogously to the proof of Theorem 4, we get uniform on each segment $[0, c]$, $c > 0$ convergence of the sequence G_{r,n,f,M_k} to the g-spline $P_{r,n,f,\{M_k\}}$ with m knots (defined on the whole half-line) together with all derivatives up to the order $r - 1$ inclusively. For brevity we will write $P_{r,n,f}$ instead of $P_{r,n,f,\{M_k\}}$.

Let the function $x(t)$ be such that

$$\begin{aligned} &\|x\|_{C[0,\infty),f} \leq \varphi_{r,n,f}(\infty), \\ &\|x^{(r)}\|_{L_\infty(0,\infty),g} \leq 1. \end{aligned} \tag{18}$$

We will show that for all $s = 1, 2, \dots, r-1$,

$$\|x^{(s)}\|_{C[0,\infty)} \leq \|P^{(s)}_{r,n,f}\|_{C[0,\infty)}. \tag{19}$$

Assume the converse, let for some s, $\|x^{(s)}\|_{C[0,\infty)} > \|P^{(s)}_{r,n,f}\|_{C[0,\infty)}$. Then there exists $\varepsilon > 0$ such that $\|x^{(s)}\|_{C[0,\infty)} > (1+\varepsilon)\|P^{(s)}_{r,n,f}\|_{C[0,\infty)}$. We can suppose that $|x^{(s)}(0)| > (1+\varepsilon)|P^{(s)}_{r,n,f}(0)|$ (if this is not true, then there exists a point $t_0 > 0$ such that $|x^{(s)}(t_0)| > \|(1+\varepsilon)P^{(s)}_{r,n,f}\|_{C[0,\infty)}$ and instead of the function $x(t)$ we can consider $y(t) := x(t + t_0)$. Moreover, since the functions f and g are non-increasing, conditions (18) and (19) are not broken and uniform norms of the function x and its derivatives do not increase). Moreover, we can assume that

$$x^{(s)}(0) > (1+\varepsilon)P^{(s)}_{r,n,f}(0) > 0 \tag{20}$$

(if this is not true, we can multiply x and (or) $P_{r,n,f}$ by -1). Set $\Delta_k(t) := x(t) - (1+\varepsilon)P_{r,n,f,M_k}(t)$, $t \in [0, M_k]$. We can choose k so big that

$$x^{(s)}(0) > (1+\varepsilon)P^{(s)}_{r,n,f,M_k}(0) \tag{21}$$

and

$$(1+\varepsilon)\varphi_{r,n,f}(M_k) > \varphi_{r,n,f}(\infty). \tag{22}$$

From Lemma 1 we get

$$(-1)^s P_{r,n,f,M_k}(0) > 0. \tag{23}$$

In view of (22) and (23) $(-1)^s\Delta_k(0) < 0$, and hence due to Lemma 1 we get $\Delta^{(s)}_k(0) < 0$. However, this contradicts (21).

In virtue of property (19) proved above, the limit $\lim_{M_k\to\infty} |G^{(k)}_{r,n,f,M_k}(0)|$ does not depend on the choice of the sequence $\{M_k\}_{k=1}^{\infty}$. This finishes the proof of the theorem.

5.4 Proof of Theorem 6

We will need the following lemmas.

Lemma 4 *Suppose that $\varphi_{r,n,f}(\infty) < \infty$ and $Q_{r,n} \in L^r_{\infty,\infty}(\mathbb{R}_+)$ is a g-spline of rth order defined on half-line $[0,\infty)$ with $n \in \mathbb{Z}_+$ knots $0 =: t_0 < t_1 < \cdots < t_n$. Then there exists $\epsilon \in \{-1,1\}$ such that for $s = 1,2,\ldots,r$,*

$$Q^{(s)}_{r,n}(t) = \epsilon P^{(s)}_r(t), \quad t \geq t_n. \tag{24}$$

We will prove the lemma using induction. In the case $s = r$, equality (24) holds. Let it be true for $s = k \geq 2$. We will prove that it is true for $s = k-1$ too. In view of the induction assumption $Q^{(k)}_{r,n}(t) = \epsilon P^{(k)}_r(t)$, $t \geq t_n$. Moreover, $Q^{(k-1)}_{r,n}(\infty) = P^{(k-1)}_r(\infty) = 0$. Then, for $t \geq t_n$,

$$-Q^{(k-1)}_{r,n}(t) = \int_t^\infty Q^{(k)}_{r,n}(s)\,ds = \epsilon \int_t^\infty P^{(k)}_r(s)\,ds = -\epsilon P^{(k-1)}_r(t).$$

The lemma is proved.

Lemma 5 *Let $\varphi_{r,n,f}(\infty) < \infty$ and $\lim_{t\to\infty} \frac{f(t)}{P_r(t)} = \infty$. Then the number of oscillation points of the g-spline $P_{r,n,f}$ tends to infinity as $n \to \infty$.*

Let some $n \in \mathbb{N}$ be fixed. Suppose that $M > 0$ is such that for all $t > M$, $\frac{f(t)}{P_r(t)} > \frac{2}{\varphi_{r,n,f}(\infty)}$. Let the g-spline $P_{r,n,f}$ have k oscillating points $0 \leq a_1 < a_2 < \cdots < a_k$. Denote by $0 \leq b_1 < b_2 < \cdots < b_{n+r+1}$ all oscillation points of the g-spline $G_{r,n,f,K}$, where K is chosen so big that $\operatorname{sgn} G_{r,n,f,K}(b_s) = \operatorname{sgn} P_{r,n,f}(b_s)$, $s = 1,2,\ldots,k$, $b_{k+1} > \max\{a_k, M\}$ and $\varphi_{r,n,f}(K) > \frac{1}{2}\varphi_{r,n,f}(\infty)$. Choose $\varepsilon > 0$ so small that $(1-\varepsilon)\varphi_{r,n,f}(K) > \frac{1}{2}\varphi_{r,n,f}(\infty)$.

Set $\Delta(t) := P_{r,n,f}(t) - (1-\varepsilon)G_{r,n,f,K}(t)$. Then $\operatorname{sgn}\Delta(a_s) = \operatorname{sgn} P_{r,n,f}(a_s)$, $s = 1,2,\ldots,k$, since

$$\bigl|P_{r,n,f}(a_s)\bigr| = \varphi_{r,n,f}(\infty) f(a_s) > (1-\varepsilon)\varphi_{r,n,f}(K) f(a_s) \geq (1-\varepsilon)\bigl|G_{r,n,f,K}(a_s)\bigr|.$$

Hence the function $\Delta(t)$ has $k-1$ sign changes on the interval $[0,a_k]$. Moreover, for $s = k+1,\ldots,n+r+1$,

$$\bigl|P_{r,n,f}(b_s)\bigr| < \frac{\varphi_{r,n,f}(\infty)}{2} f(b_s) < (1-\varepsilon)\varphi_{r,n,f}(K) f(b_s) = (1-\varepsilon)\bigl|G_{r,n,f,K}(b_s)\bigr|.$$

This means that $\operatorname{sgn}\Delta(b_s) = \operatorname{sgn} G_{r,n,f,K}(b_s)$, $s = k+1,\ldots,n+r+1$. Hence the function $\Delta(t)$ has $n+r-k$ sign changes on the interval $[b_{k+1},a]$, and hence at least $n+r-1$ sign changes on the whole interval $[0,K]$. This means that the function $\Delta^{(r)}(t)$ has at least $n-1$ sign changes, and hence the g-spline $P_{r,n,f}$ has at least $n-1$ knots.

Let $t^K_{n,s}$, $s = 1,\ldots,n$, be the knots of the g-spline $G_{r,n,f,K}$. Note that for all $s = 1,2,\ldots,n$, the g-spline $G_{r,n,f,K}$ has at least s oscillating points on the interval $[0,t^k_{n,s}]$. Really, assume the converse, suppose for some $1 \leq s \leq n$ that the g-spline $G_{r,n,f,K}$ has less than s oscillation points on the interval $[0,t^K_{n,s}]$. Then the g-spline $G_{r,n,f,K}$ has more than $n+r+1-s$ oscillation points on the interval $(t^K_{n,s},K]$, and hence more than $n+r-s$ sign changes. This means that the function $G^{(r)}_{r,n,f,K}$ has more than $n-s$ sign changes on the interval $(t^K_{n,s},K]$. However, this is impossible.

This means that the limiting g-spline $P_{r,n,f}$ has at least $n-1$ oscillation points. The lemma is proved.

Lemma 6 *For all* $s = 1, 2, \dots, r$ *and all* $t \geq 0$ (*almost everywhere is the case when* $s = r$), *the following inequality holds*:

$$\bigl|P^{(s)}_{r,n,f}(t)\bigr| \leq \bigl|P^{(s)}_r(t)\bigr|. \tag{25}$$

We will prove the statement of the lemma using induction. In the case $s = r$, inequality (25) holds with equality sign. Let inequality (25) hold with $s = k \geq 2$. We will prove that it is true for $s = k - 1$.

Assume the converse. Let

$$T := \bigl\{t \in [0,\infty) : \bigl|P^{(k-1)}_{r,n,f}(t)\bigr| > \bigl|P^{(k-1)}_r(t)\bigr|\bigr\} \neq \emptyset.$$

Denote by $0 < t_1 < \cdots < t_l$ all knots of the g-spline $P_{r,n,f}$. Then, due to Lemma 4, $T \subset [0, t_l)$ and

$$\bigl|P^{(k-1)}_{r,n,f}(t_l)\bigr| = \bigl|P^{(k-1)}_r(t_l)\bigr|. \tag{26}$$

Let $a \in T$. Then

$$\begin{aligned}
\bigl|P^{(k-1)}_r(t_l) - P^{(k-1)}_r(a)\bigr| &= \left|\int_a^{t_l} P^{(k)}_r(t)\,dt\right| = \int_a^{t_l} \bigl|P^{(k)}_r(t)\bigr|\,dt \\
&\geq \int_a^{t_l} \bigl|P^{(k)}_{r,n,f}(t)\bigr|\,dt \geq \left|\int_a^{t_l} P^{(k)}_{r,n,f}(t)\,dt\right| \\
&= \bigl|P^{(k-1)}_{r,n,f}(t_l) - P^{(k-1)}_{r,n,f}(a)\bigr|;
\end{aligned}$$

this is impossible in virtue of (26), and the facts that $\operatorname{sgn} P^{(k-1)}_r(t_l) = \operatorname{sgn} P^{(k-1)}_r(a)$, function $|P^{(k-1)}_r|$ is non-increasing and $a \in T$. Contradiction. Hence $T = \emptyset$ and the lemma is proved.

Let us return to the proof of the theorem.

Let

$$\varliminf_{t\to\infty} \frac{f(t)}{|P_r(t)|} =: c < \infty. \tag{27}$$

Then, in view of Theorem 3, $c > 0$. We will show that $\varphi_{r,n,f}(\infty) \geq \frac{1}{2c}$ $\forall n$. Assume the converse, let a number n_0 such that

$$\varphi_{r,n_0,f}(\infty) < \frac{1}{2c} \tag{28}$$

exist. Denote by $0 < t_1 < t_2 < \cdots < t_k$ all knots of $P_{r,n_0,f}$. Then, due to Lemma 4, $P'_{r,n_0,f}(t) = \pm P'_r(t)$, $t \geq t_k$. In virtue of (28) we have

$$\bigl|P_{r,n_0,f}(t)\bigr| < \frac{f(t)}{2c}$$

for all $t \geq 0$. $f(\infty) = 0$ since (27) holds. Then $P_{r,n_0,f}(\infty) = 0$, and hence

$$P_{r,n_0,f}(t) = \pm P_r(t),$$

$t \geq t_k$. But then $|P_r(t)| < \frac{f(t)}{2c}$, *i.e.*,

$$\frac{f(t)}{|P_r(t)|} > 2c, \quad t \geq t_n,$$

which contradicts (27). Sufficiency is proved.

We will prove the necessity now. $\lim_{n\to\infty} \varphi_{r,n,f}(\infty) = \delta > 0$. Assume the converse, let $\underline{\lim}_{t\to\infty} \frac{f(t)}{|P_r(t)|} = \infty$. Then there exists a number $M > 0$ such that for all $t > M$ the inequality

$$\frac{f(t)}{|P_r(t)|} > \frac{1}{\delta}$$

holds; this is equivalent to

$$\left|P_r(t)\right| < \delta f(t).$$

In view of Lemma 6, for all n, the following inequality holds:

$$\int_0^M \left|P'_{r,n,f}(t)\right| dt \leq \int_0^M \left|P_{r-1}(t)\right| dt. \tag{29}$$

Choose n so big that

$$n \cdot f(M) > \int_0^M \left|P_{r-1}(t)\right| dt. \tag{30}$$

Choose m such that the g-spline $P_{r,m,f}(t)$ has at least $n+1$ oscillation points (this is possible due to Lemma 5). Denote by $0 \leq a_1 < a_2 < \cdots < a_{n+1}$ the first oscillation points of the g-spline $P_{r,m,f}(t)$. Then, in virtue of (30), and by the facts that f is non-increasing function, $\bigvee_0^M P_{r,m,f} = \int_0^M |P'_{r,m,f}(t)|\, dt$ and (29), we get that $a_n > M$. Thus $a_{n+1} > a_n > M$ are oscillation points of the g-spline $P_{r,m,f}(t)$, and we get

$$\begin{aligned}
\left|P_{r,m,f}(a_{n+1}) - P_{r,m,f}(a_n)\right| &= \varphi_{r,m,f}(\infty) \cdot \bigl(f(a_{n+1}) + f(a_n)\bigr) \\
&\geq \delta \cdot \bigl(f(a_{n+1}) + f(a_n)\bigr) > \delta f(a_n) \\
&> \left|P_r(a_n)\right| = \int_{a_n}^{\infty} \left|P_{r-1}(t)\right| dt.
\end{aligned}$$

On the other hand, in view of Lemma 6,

$$\begin{aligned}
\left|P_{r,m,f}(a_{n+1}) - P_{r,m,f}(a_n)\right| &\leq \int_{a_n}^{a_{n+1}} \left|P'_{r,m,f}(t)\right| dt \\
&\leq \int_{a_n}^{a_{n+1}} \left|P_{r-1}(t)\right| dt \\
&< \int_{a_n}^{\infty} \left|P_{r-1}(t)\right| dt.
\end{aligned}$$

Contradiction. The theorem is proved.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have made an equal contribution to the paper, have read and approved the final manuscript.

Acknowledgements
The authors would like to thank the referees for their valuable remarks and suggestions.

Received: 19 December 2014 Accepted: 16 September 2015 Published online: 23 September 2015

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