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On approximating Mills ratio

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Abstract

In the article, we present several sharp bounds for the Mills ratio $R(x) = e^{x^2/2}\int_x^\infty e^{-t^2/2}\,dt$ ($x > 0$) in terms of the functions $I_a(x) = a/[\sqrt{x^2+2a} + (a-1)x]$ and $J(x) = a/[\sqrt{x^2+2a^2/\pi} + 2ax/\pi]$ with parameter $a > 0$.

MSC: 60E15; 26A48; 26D15

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1 Introduction

The Mills ratio [1] is the function

$$R(x) = \frac{1-\Phi(x)}{\phi(x)} = e^{\frac{x^2}{2}} \int_x^\infty e^{-\frac{t^2}{2}}\,dt, \quad x > 0, \tag{1.1}$$

where $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$ is the density function of a standard Gaussian law and $\Phi(x) = \int_{-\infty}^x \phi(t)\,dt$ its cumulative distribution function. The study of the Mills ratio is much older than the work of Mills [1], and through its relation with the function $F(x) = e^{x^2}\int_x^\infty e^{-t^2}\,dt$ given by $R(x) = \sqrt{2}F(x/\sqrt{2})$, its introduction can be traced back to Laplace [2], Livre X, Chapter 1, n°5, while he was analyzing different hypotheses related to the refraction of the light in the atmosphere. Laplace gave many of the essential results, like the continued fraction and the asymptotic expansion. Since the function F is related to the error function, and also to the upper incomplete Gamma function of parameter $1/2$, the properties of the Mills ratio are spread over papers and books of probability and statistics, mathematical analysis, numerical analysis, *etc.*, and many results have been discovered and rediscovered by different authors.

It is well known that the function Φ cannot be expressed as the composition of elementary functions, therefore, it is valuable to find sharp bounds for the Mills ratio by certain simple and elementary functions.

Gordon [3] proved that the double inequality

$$\frac{x}{x^2+1} \le R(x) \le \frac{1}{x}$$

holds for all $x > 0$.

Birnbaum [4] and Komatu [5] proved that the double inequality

$$\frac{2}{\sqrt{x^2+4}+x} < R(x) < \frac{2}{\sqrt{x^2+2}+x}$$

holds for all $x > 0$.

An improvement for the upper bound of Mills ratio,

$$R(x) < \frac{2}{\sqrt{x^2+8}+3x},$$

is due to Sampford [6] and Shenton [7].

Pollak [8] proved that $b = 4/\pi$ is the best possible parameter such that $2/[\sqrt{x^2+b}+x]$ is the upper bound of Mills ratio. Boyd [9] dealt with the bounds for the Mills ratio of the form

$$\psi(x) = \frac{a}{\sqrt{x^2+b}+c} \tag{1.2}$$

with $a, b, c > 0$ such that

$$\psi(0) = R(0), \qquad \lim_{x\to\infty}\bigl[x\bigl(\psi(x)-R(x)\bigr)\bigr] = 0$$

and proved that

$$\frac{\pi}{\sqrt{x^2+2\pi}+(\pi-1)x} < R(x) < \frac{\pi}{\sqrt{(\pi-2)^2x^2+2\pi x}+2x}.$$

Very recently, Gasull and Utzet [10] proved the double inequality

$$\max\bigl\{W_{3,0}(x), W_{1,2}(x)\bigr\} < R(x) < \max\bigl\{W_{0,3}(x), W_{2,1}(x)\bigr\} \tag{1.3}$$

for all $x > 0$, where

$$W_{3,0}(x) = \frac{\pi}{\sqrt{2(4-\pi)x^2+2\pi}+2x}, \qquad W_{1,2}(x) = \frac{\pi}{\sqrt{x^2+2\pi}+(\pi-1)x}, \tag{1.4}$$

$$W_{2,1}(x) = \frac{\pi}{\sqrt{(\pi-2)^2x^2+2\pi}+2x}, \qquad W_{0,3}(x) = \frac{4}{\sqrt{x^2+8}+3x}. \tag{1.5}$$

More inequalities involving the Mills ratio $R(x)$ can be found in the literature [11–21] and the references therein.

Let $\psi(x)$ be defined by (1.2). Then making use of the asymptotic expansion of the Mills ratio $R(x)$ at infinity (see [22], p. 44)

$$R(x) \sim \frac{1}{x} - \frac{1}{x^3} + \frac{1\cdot 3}{x^5} - \frac{1\cdot 3\cdot 5}{x^7} + \cdots \quad (x\to\infty)$$

we get

$$\psi(x) = \frac{a}{\sqrt{x^2+2a}+(a-1)x} := I_a(x) \tag{1.6}$$

if $\psi(x)$ satisfies

$$\lim_{x\to\infty}\bigl[x\bigl(\psi(x)-R(x)\bigr)\bigr] = 0, \qquad \lim_{x\to\infty}\bigl[x^3\bigl(\psi(x)-R(x)\bigr)\bigr] = 0,$$

and

$$\psi(x) = \frac{a}{\sqrt{x^2 + 2a^2/\pi} + 2ax/\pi} := J_a(x) \tag{1.7}$$

if $\psi(0) = R(0)$ and $\psi'(0) = R'(0)$.

The main purpose of this paper is to present the sharp bounds for the Mills ratio $R(x)$ in terms of $I_a(x)$ and $J_a(x)$.

2 Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

Lemma 2.1 (See [23], Proposition 1.1 or [24], Proposition 1.2) *Let* $-\infty \leq a < b \leq \infty, f, g : (a,b) \to \mathbb{R}$ *be differentiable on* (a,b) *with* $f(a^+) = g(a^+) = 0$ *or* $f(b^-) = g(b^-) = 0$, *and* $g'(x) \neq 0$ *on* (a,b). *If* $f'(x)/g'(x)$ *is increasing* (*decreasing*) *on* (a,b), *then so is* $f(x)/g(x)$.

Lemma 2.2 (See [25], Theorem 9) *Let* $-\infty \leq a < b \leq \infty, f, g : (a,b) \to \mathbb{R}$ *be differentiable on* (a,b) *with* $f(b^-) = g(b^-) = 0$, $g'(x) \neq 0$ *on* (a,b), *and*

$$H_{f,g} = \frac{f'}{g'}g - f. \tag{2.1}$$

If there exists $c \in (a,b)$ *such that* f'/g' *is increasing* (*decreasing*) *on* (a,c) *and decreasing* (*increasing*) *on* (c,b). *Then the follows statements are true*:

(i) f/g *is decreasing* (*increasing*) *on* (a,b) *if* $g' > 0$ *on* (a,b) *and* $H_{f,g}(a^+) \leq (\geq) 0$ *or* $g' < 0$ *on* (a,b) *and* $H_{f,g}(a^+) \geq (\leq) 0$;

(ii) *there exists* $c_0 \in (a,b)$ *such that* f/g *is increasing* (*decreasing*) *on* (a,c_0) *and decreasing* (*increasing*) *on* (c_0,b) *if* $g' > 0$ *on* (a,b) *and* $H_{f,g}(a^+) > (<) 0$ *or* $g' < 0$ *on* (a,b) *and* $H_{f,g}(a^+) < (>) 0$.

Lemma 2.3 *Let* $a, b \in \mathbb{R}$ *with* $a < b$ *and* $f : [a,b] \to \mathbb{R}$ *be continuous and strictly convex* (*concave*) *with* $f(a)f(b) < 0$. *Then there exists* $c \in (a,b)$ *such that* $f(x)f(a) > 0$ *for* $x \in (a,c)$ *and* $f(x)f(b) > 0$ *for* $x \in (c,b)$.

Proof We only prove the case of f being convex with

$$f(a) < 0, \qquad f(b) > 0, \tag{2.2}$$

other cases can be proved by similar methods. Let $x \in (a,b)$ and

$$F(x) = \frac{f(x) - f(a)}{x - a}. \tag{2.3}$$

Then from the convexity of f on $[a,b]$ we know that F is increasing on (a,b).

We divide the proof into two cases.

Case 1 $F(a^+) \geq 0$. Then $F(x) > F(a^+) \geq 0$ for $x \in (a,b)$. It follows from the convexity of f on $[a,b]$ that

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} > \frac{f(x_2)-f(a)}{x_2-a} = F(x_2) > 0 \tag{2.4}$$

for all $x_1, x_2 \in (a,b)$ with $x_1 \neq x_2$.

Inequality (2.4) implies that f is increasing on (a,b), which together with (2.2) leads to the desired result.

Case 2 $F(a^+) < 0$. Then from the monotonicity of F given by (2.3) we clearly see that $F(b) > 0$ and there exists $c^* \in (a,b)$ such that $F(x) < 0$ for $x \in (a,c^*)$ and $F(x) > 0$ for $x \in (c^*,b)$.

From (2.2) and (2.3) together with Case 1 we know that

$$f(x) < f(a) < 0 \tag{2.5}$$

for $x \in (a,c^*)$ and f is increasing on (c^*,b).

Therefore, the desired assertion follows from (2.5) and the monotonicity of f on (c^*,b) together with (2.2). □

Lemma 2.4 *Let*

$$h(t) = -2\pi a t^3 - 3\pi^2 t^2 + 2a(2a^2 - 3\pi)t - \left[2(4-\pi)a^2 - \pi^2\right]. \tag{2.6}$$

Then the equation $h'(t) = 0$ *has the unique solution*

$$t_0 = t_0(a) = \frac{\sqrt{8a^4 - 12\pi a^2 + 3\pi^3} - \sqrt{3\pi}\pi}{2\sqrt{3\pi}a} \tag{2.7}$$

on $(0,1)$ *such that* $h(t_0) > 0$ *if* $a \in (\pi/\sqrt{2(4-\pi)}, \sqrt{\pi+1}+1)$.

Proof It follows from (2.6) that

$$h'(t) = -6\pi a t^2 - 6\pi^2 t + 2a(2a^2 - 3\pi), \tag{2.8}$$

$$h'(0) = 2a(2a^2 - 3\pi) > 0. \tag{2.9}$$

From (2.8) and (2.9) we know that $t_0 = t_0(a)$ given by (2.7) is the unique positive solution of the equation $h'(t) = 0$. Equation (2.7) gives

$$\frac{dt_0(a)}{da} = \frac{\sqrt{3\pi}\pi\sqrt{8a^4 - 12\pi a^2 + 3\pi^3} + (8a^4 - 3\pi^3)}{2\sqrt{3\pi}a^2\sqrt{8a^4 - 12\pi a^2 + 3\pi^3}} > 0 \tag{2.10}$$

for $a \in (\pi/\sqrt{2(4-\pi)}, \sqrt{\pi+1}+1)$.

Inequality (2.10) leads to

$$t_0(a) < t_0(\sqrt{\pi+1}+1) < t_0\left(\frac{10}{3}\right) = 0.7857\cdots < 1 \tag{2.11}$$

for $a \in (\pi/\sqrt{2(4-\pi)}, \sqrt{\pi+1}+1)$.

Next, we prove that $h(t_0) > 0$ for $a \in (\pi/\sqrt{2(4-\pi)}, \sqrt{\pi+1}+1)$. From (2.6) and (2.7) we have

$$h(t_0) = \frac{(8a^4 - 12\pi a^2 + 3\pi^3)^{3/2} - 3\sqrt{3\pi}(\pi^2 - 4a^2)^2}{6\sqrt{3\pi}a^2}.$$

It is enough to prove that

$$\begin{aligned}
&\left(8u^2 - 12\pi u + 3\pi^3\right)^3 - \left[3\sqrt{3\pi}\left(4u - \pi^2\right)^2\right]^2 \\
&\quad = 4u\big[128u^5 - 576\pi u^4 + 144\pi\left(\pi^2 + 6\pi - 12\right)u^3 \\
&\qquad - 432\pi^3(\pi - 3)u^2 - 54\pi^5(6-\pi)u + 27\pi^7\big] \\
&\quad := 4uh_1(u) > 0
\end{aligned}$$

for $u \in (\pi^2/[2(4-\pi)], (\sqrt{\pi+1}+1)^2)$.

Let $v = u - \pi^2/[2(4-\pi)] > 0$. Then $h_1(u)$ can be rewritten as

$$\begin{aligned}
h_1(u) = {}& \frac{2}{(4-\pi)^5}\big[64(4-\pi)^5 v^5 + 64\pi(4-\pi)^4(7\pi - 18)v^4 \\
&+ 8\pi(4-\pi)^3\left(9\pi^4 + 74\pi^3 - 684\pi^2 + 1{,}728\pi - 1{,}728\right)v^3 \\
&+ 4\pi^3(4-\pi)^2\left(81\pi^4 - 736\pi^3 + 2{,}916\pi^2 - 6{,}048\pi + 5{,}184\right)v^2 \\
&+ \pi^5(4-\pi)\left(27\pi^5 - 324\pi^4 + 2{,}000\pi^3 - 7{,}272\pi^2 + 13{,}824\pi - 10{,}368\right)v \\
&+ 4\pi^7\left(9\pi^2 - 40\pi + 48\right)(\pi - 3)^2\big] > 0
\end{aligned}$$

due to all the coefficients of the quintic polynomial being positive. □

Lemma 2.5 *Let $h(t)$ be defined by* (2.6). *Then the following statements are true*:

(i) *there exists $t_1 \in (0,1)$ such that $h(t) > 0$ for $t \in (0,t_1)$ and $h(t) < 0$ for $t \in (t_1,1)$ if $a \in (0, \pi/\sqrt{2(4-\pi)}]$*;

(ii) *there exists $t_{11}, t_{12} \in (0,1)$ with $t_{11} < t_{12}$ such that $h(t) < 0$ for $t \in (0,t_{11}) \cup (t_{12},1)$ and $h(t) > 0$ for $t \in (t_{11},t_{12})$ if $a \in (\pi/\sqrt{2(4-\pi)}, \sqrt{\pi+1}+1)$*;

(iii) *there exists $t_1^* \in (0,1)$ such that $h(t) < 0$ for $t \in (0,t_1^*)$ and $h(t) > 0$ for $t \in (t_1^*,1)$ if $a \in [\sqrt{\pi+1}+1, \infty)$*.

Proof It follows from (2.6) that

$$h(0) = \pi^2 - 2(4-\pi)a^2 = -2(4-\pi)\left(a + \frac{\pi}{\sqrt{2(4-\pi)}}\right)\left(a - \frac{\pi}{\sqrt{2(4-\pi)}}\right), \tag{2.12}$$

$$\begin{aligned}
h(1) &= 4a^3 - 2(4-\pi)a^2 - 8\pi a - 2\pi^2 \\
&= 2(\pi + 2a)(a + \sqrt{\pi+1} - 1)\big[a - (\sqrt{\pi+1}+1)\big],
\end{aligned} \tag{2.13}$$

$$h'(t) = -6\pi a t^2 - 6\pi^2 t + 2a\left(2a^2 - 3\pi\right), \tag{2.14}$$

$$h'(0) = 2a\left(2a^2 - 3\pi\right) = 4a\left(a + \frac{\sqrt{6\pi}}{2}\right)\left(a - \frac{\sqrt{6\pi}}{2}\right), \tag{2.15}$$

$$h'(1) = 4a^3 - 12\pi a - 6\pi^2. \tag{2.16}$$

From (2.14) we clearly see that $h'(t)$ is decreasing and $h(t)$ is strictly concave on $(0,1)$. We divide the proof into five cases.

Case 1 $a \in (0, \pi/\sqrt{2(4-\pi)})$. Then (2.12) and (2.13) lead to

$$h(0) > 0, \qquad h(1) < 0. \tag{2.17}$$

Therefore, the desired assertion follows easily from (2.17) and the concavity of h on $(0,1)$ together with Lemma 2.3.

Case 2 $a = \pi/\sqrt{2(4-\pi)}$. Then (2.12), (2.13), (2.15), and (2.16) give

$$h(0) = 0, \qquad h(1) < 0, \tag{2.18}$$

$$h'(0) > 0, \qquad h'(1) = -\frac{2\pi^2}{\sqrt{2(4-\pi)}}\left[\frac{24-7\pi}{4-\pi} + 3\sqrt{2(4-\pi)}\right] < 0. \tag{2.19}$$

From (2.19) and the monotonicity of $h'(t)$ on $(0,1)$ we know that there exists $\lambda \in (0,1)$ such that $h(t)$ is increasing on $(0,\lambda]$ and decreasing on $[\lambda,1)$. Therefore, the desired result follows from (2.18) and the piecewise monotonicity of $h(t)$ on $(0,1)$.

Case 3 $a \in (\pi/\sqrt{2(4-\pi)}, \sqrt{\pi+1}+1)$. Then (2.12) and (2.13) imply that

$$h(0) < 0, \qquad h(1) < 0. \tag{2.20}$$

Therefore, the desired assertion follows from (2.20) and Lemma 2.4 together with the concavity of h on $(0,1)$.

Case 4 $a = \sqrt{\pi+1}+1$. Then (2.12), (2.13), (2.15), and (2.16) lead to

$$h(0) < 0, \qquad h(1) = 0 \tag{2.21}$$

and (2.19) again holds. From (2.19) and the monotonicity of $h'(t)$ on $(0,1)$ we know that there exists $\mu \in (0,1)$ such that $h(t)$ is increasing on $(0,\mu]$ and decreasing on $[\mu,1)$. Therefore, the desired result follows from (2.21) and the piecewise monotonicity of $h(t)$ on $(0,1)$.

Case 5 $a \in (\sqrt{\pi+1}+1, \infty)$. Then (2.12) and (2.13) imply that

$$h(0) < 0, \qquad h(1) > 0. \tag{2.22}$$

Therefore, the desired assertion follows from Lemma 2.3 and (2.22) together with the concavity of f on $(0,1)$. □

3 Main results

Theorem 3.1 *The following statements are true for all* $x > 0$:

(1) *if* $a \in (0, (\sqrt{\pi+1}+1)^2/\pi]$, *then*

$$\frac{a}{\sqrt{x^2+2a}+(a-1)x} < R(x) < \sqrt{\frac{\pi}{a}}\frac{a}{\sqrt{x^2+2a}+(a-1)x}; \tag{3.1}$$

(2) *if* $a \in [4, \infty)$, *then*

$$\sqrt{\frac{\pi}{a}}\frac{a}{\sqrt{x^2+2a}+(a-1)x} < R(x) < \frac{a}{\sqrt{x^2+2a}+(a-1)x}; \tag{3.2}$$

(3) *if* $a \in ((\sqrt{\pi+1}+1)^2/\pi, 4)$, *then*

$$\min\left\{1, \sqrt{\frac{\pi}{a}}\right\} \frac{a}{\sqrt{x^2+2a}+(a-1)x} < R(x) < \lambda(a) \frac{a}{\sqrt{x^2+2a}+(a-1)x} \tag{3.3}$$

with $\lambda(a) = (\sqrt{x_0^2+2a}+(a-1)x_0)R(x_0)/a$, *where* x_0 *is the unique solution of the equation*

$$\frac{d[(\sqrt{x^2+2a}+(a-1)x)R(x)]}{dx} = 0$$

on the interval $(0, \infty)$. *In particular, if* $a = \pi$, *then* $x_0 = 0.590\cdots$, $\lambda(\pi) = 1.011\cdots$, *and*

$$\frac{\pi}{\sqrt{x^2+2\pi}+(\pi-1)x} < R(x) < \frac{\pi\lambda(\pi)}{\sqrt{x^2+2\pi}+(\pi-1)x}. \tag{3.4}$$

Proof Let

$$f_1(x) = \frac{ae^{-x^2/2}}{\sqrt{x^2+2a}+(a-1)x}, \qquad g(x) = \int_x^\infty e^{-t^2/2}\,dt$$

and $I_a(x)$ be defined by (1.6). Then simple computations lead to

$$\frac{f_1(x)}{g(x)} = \frac{I_a(x)}{R(x)}, \tag{3.5}$$

$$f_1(\infty) = g(\infty) = 0, \tag{3.6}$$

$$\lim_{x\to 0^+} \frac{f_1(x)}{g(x)} = \sqrt{\frac{a}{\pi}}, \qquad \lim_{x\to\infty} \frac{f_1(x)}{g(x)} = 1, \tag{3.7}$$

and

$$\frac{f_1'(x)}{g'(x)} = \frac{a(a-1+\frac{x}{\sqrt{x^2+2a}})}{[\sqrt{x^2+2a}+(a-1)x]^2} + \frac{ax}{\sqrt{x^2+2a}+(a-1)x}. \tag{3.8}$$

Let $t = x/\sqrt{x^2+2a} \in (0,1)$ or $x = \sqrt{2a}t/\sqrt{1-t^2}$. Then (3.8) can be rewritten as

$$\frac{f_1'(x)}{g'(x)} = \frac{-t^3+(2a^2-3a+1)t^2+(2a+1)t+(a-1)}{2(1-t+at)^2}. \tag{3.9}$$

Differentiating (3.9) gives

$$\begin{aligned}\left(\frac{f_1'(x)}{g'(x)}\right)' &= \frac{d}{dt}\left[\frac{-t^3+(2a^2-3a+1)t^2+(2a+1)t+(a-1)}{2(1-t+at)^2}\right] \times \frac{dt}{dx} \\ &= \frac{a(1-t)}{(1-t+at)^3(x^2+2a)^{3/2}} l(t),\end{aligned} \tag{3.10}$$

where

$$l(t) = (a-1)t^2+(a+2)t-2a^2+6a-1, \tag{3.11}$$

$$l(0) = -2\left(a - \frac{3-\sqrt{7}}{2}\right)\left(a - \frac{3+\sqrt{7}}{2}\right), \qquad l(1) = -2a(a-4). \tag{3.12}$$

Next, we divide our analysis into three cases to determine the sign of $l(t)$ on the interval $(0,1)$.

Case 1 $a = 1$. We clearly see that $l(t) = 3(t+1) > 0$ for $t \in (0,1)$.

Case 2 $a > 1$. It follows from (3.11) that $l(t)$ is strictly convex on $(0,1)$.

We divide the discussions into three subcases.

Subcase 2.1 $a \in (1, (3+\sqrt{7})/2]$. Then (3.11) and (3.12) lead to $l(t) > l(0) \geq 0$ for $t \in (0,1)$.

Subcase 2.2 $a \in ((3+\sqrt{7})/2, 4)$. Then (3.12) gives

$$l(0) < 0, \qquad l(1) > 0. \tag{3.13}$$

It follows from Lemma 2.3 and the convexity of $l(t)$ on $(0,1)$ together with (3.13) that there exists $t_1 \in (0,1)$ such that $l(t) < 0$ for $t \in (0,t_1)$ and $l(t) > 0$ for $t \in (t_1,1)$.

Subcase 2.3 $a \in [4,\infty)$. Then (3.12) gives

$$l(0) < 0, \qquad l(1) \leq 0. \tag{3.14}$$

Making use of the convexity of $l(t)$ on $(0,1)$ and (3.14) we get

$$l(t) \leq (1-t)l(0) + tl(1) < 0$$

for $t \in (0,1)$.

Case 3 $0 < a < 1$. Then from (3.11) we clearly see that $l(t)$ is strictly concave on $(0,1)$.

We divide the discussions into two subcases.

Subcase 3.1 $a \in (0, (3-\sqrt{7})/2)$. Then (3.12) and Lemma 2.3 lead to the conclusion that (3.13) again holds and there exists $t_2 \in (0,1)$ such that $l(t) < 0$ for $t \in (0,t_2)$ and $l(t) > 0$ for $t \in (t_2,1)$.

Subcase 3.2 $a \in [(3-\sqrt{7})/2, 1)$. Then (3.12) leads to

$$l(0) \geq 0, \qquad l(1) > 0. \tag{3.15}$$

Making use of the concavity of $l(t)$ on $(0,1)$ and (3.15) we have

$$l(t) \geq (1-t)l(0) + tl(1) > 0$$

for $t \in (t,1)$.

Now, we divide the discussion into three cases to prove the desired results.

Case A $a \in [(3-\sqrt{7})/2, (3+\sqrt{7})/2]$. Then Subcases 2.1 and 3.2 together with (3.10) and (3.11) lead to the conclusion that $f_1'(x)/g'(x)$ is increasing on $(0,\infty)$. From the monotonicity of $f_1'(x)/g'(x)$ on $(0,\infty)$ and (3.6) together with Lemma 2.1 and the fact that $g'(x) = -e^{-x^2/2} \neq 0$ we know that $f_1(x)/g(x)$ is also increasing on $(0,\infty)$. Therefore, (3.1) follows easily from (3.5), (3.7), and the monotonicity of $f_1(x)/g(x)$ on $(0,\infty)$.

Case B $a \in [4,\infty)$. Then (3.10) and (3.11) together with Subcase 2.3 lead to the conclusion that $f_1'(x)/g'(x)$ is decreasing on $(0,\infty)$. Making use of (3.6) and Lemma 2.1 together with

$g'(x) \neq 0$ we know that $f_1(x)/g(x)$ is also decreasing on $(0,\infty)$. Therefore, (3.2) follows easily from (3.5), (3.7), and the monotonicity of $f_1(x)/g(x)$ on $(0,\infty)$.

Case C $a \in (0,(3-\sqrt{7})/2) \cup ((3+\sqrt{7})/2,4)$. Then from Subcases 2.2 and 3.1 we know that there exists $t^* \in (0,1)$ such that $l(t) < 0$ for $t \in (0,t^*)$ and $l(t) > 0$ for $t \in (t^*,1)$, and (3.10) and (3.11) lead to the conclusion that there exists $x^* = \sqrt{2a}t^*/\sqrt{1-t^{*2}} \in (0,\infty)$ such that $f_1'(x)/g'(x)$ is decreasing on $(0,x^*)$ and increasing on (x^*,∞).

Note that

$$\begin{aligned} H_{f_1,g}(x) &= \frac{f_1'(x)}{g'(x)}g(x) - f_1(x) \\ &= \left[\frac{a(a-1+\frac{x}{\sqrt{x^2+2a}})}{(\sqrt{x^2+2a}+(a-1)x)^2} + \frac{ax}{\sqrt{x^2+2a}+(a-1)x}\right]\int_x^\infty e^{-t^2/2}\,dt \\ &\quad - \frac{ae^{-x^2/2}}{\sqrt{x^2+2a}+(a-1)x}, \\ H_{f_1,g}(0) &= \frac{\sqrt{2\pi}}{4}\left(\sqrt{a}+\frac{\sqrt{\pi+1}-1}{\sqrt{\pi}}\right)\left(\sqrt{a}-\frac{\sqrt{\pi+1}+1}{\sqrt{\pi}}\right). \end{aligned} \tag{3.16}$$

We divide the discussion into two subcases.

Subcase C(1) $a \in (0,(3-\sqrt{7})/2) \cup ((3+\sqrt{7})/2,(\pi+2\sqrt{\pi+1}+2)/\pi]$. Then (3.16) leads to

$$H_{f_1,g}(0) \leq 0. \tag{3.17}$$

It follows from (3.6), (3.17), and $g'(x) = -e^{-x^2/2} < 0$ together with the piecewise monotonicity of f_1'/g' and Lemma 2.2(i) that f_1/g is increasing on $(0,\infty)$. Therefore, (3.1) follows easily from (3.5), (3.7), and the monotonicity of $f_1(x)/g(x)$ on $(0,\infty)$.

Subcase C(2) $a \in ((\pi+2\sqrt{\pi+1}+2)/\pi,4)$. Then (3.16) gives

$$H_{f_1,g}(0) > 0. \tag{3.18}$$

From (3.6), (3.18), and $g'(x) < 0$ together with the piecewise monotonicity of f_1'/g' and Lemma 2.2(ii) we know that there exists $x_0 \in (0,\infty)$ such that f_1/g is decreasing on $(0,x_0)$ and increasing on (x_0,∞). Consequently, we get

$$\min\left\{\lim_{x\to 0}\frac{g(x)}{f_1(x)}, \lim_{x\to\infty}\frac{g(x)}{f_1(x)}\right\} < \frac{g(x)}{f_1(x)} = \frac{R(x)}{I_a(x)} < \frac{R(x_0)}{I_a(x_0)} = \lambda(a). \tag{3.19}$$

Therefore, (3.3) follows from (3.7) and (3.19). We clearly see that x_0 satisfies the equation $(f_1(x)/g(x))' = 0$ or $d[(\sqrt{x^2+2a}+(a-1)x)R(x)]/dx = 0$, and $\lambda(a) = (\sqrt{x_0^2+2a}+(a-1)x_0)R(x_0)/a$.

If $a = \pi$, then numerical computations show that $x_0 = 0.590\cdots$ and $\lambda(\pi) = 1.011\cdots$. □

Theorem 3.2 *The following statements are true for all* $x > 0$:

(1) *if* $a \in (0,\pi/\sqrt{2(4-\pi)}]$, *then*

$$\frac{a}{\sqrt{x^2+\frac{2a^2}{\pi}+\frac{2ax}{\pi}}} < R(x) < \frac{\pi+2a}{\pi a}\frac{a}{\sqrt{x^2+\frac{2a^2}{\pi}+\frac{2ax}{\pi}}}; \tag{3.20}$$

(2) *if* $a \in [\sqrt{\pi+1}+1, \infty)$, *then*

$$\frac{\pi+2a}{\pi a}\frac{a}{\sqrt{x^2+\frac{2a^2}{\pi}}+\frac{2ax}{\pi}} < R(x) < \frac{a}{\sqrt{x^2+\frac{2a^2}{\pi}}+\frac{2ax}{\pi}}; \tag{3.21}$$

(3) *if* $a \in (\pi/\sqrt{2(4-\pi)}, \sqrt{\pi+1}+1)$, *then*

$$\frac{a\theta(a)}{\sqrt{x^2+\frac{2a^2}{\pi}}+\frac{2ax}{\pi}} < R(x) < \max\left\{\frac{\pi+2a}{\pi a}, 1\right\}\frac{a}{\sqrt{x^2+\frac{2a^2}{\pi}}+\frac{2ax}{\pi}}, \tag{3.22}$$

where

$$\theta(a) = \frac{1}{a}\left(\sqrt{x_0^2+\frac{2a^2}{\pi}}+\frac{2ax_0}{\pi}\right)R(x_0)$$

and x_0 *is the unique solution of the equation*

$$\frac{d[(\sqrt{x^2+\frac{2a^2}{\pi}}+\frac{2ax}{\pi})R(x)]}{dx} = 0$$

on $(0,\infty)$. *In particular, if* $a = a_0 = \pi/(\pi-2) = 2.7519\cdots$, *then* $x_0 = 1.6108\cdots$, $\theta(a_0) = 0.9909\cdots$, *and*

$$\frac{\pi\theta(a_0)}{\sqrt{(\pi-2)^2x^2+2\pi}+2x} < R(x) < \frac{\pi}{\sqrt{(\pi-2)^2x^2+2\pi}+2x}.$$

Proof Let

$$f_2(x) = \frac{ae^{-x^2/2}}{\sqrt{x^2+\frac{2a^2}{\pi}}+\frac{2ax}{\pi}}, \qquad g(x) = \int_x^\infty e^{-t^2/2}\,dt$$

and $J_a(x)$ be defined by (1.7). Then simple computations lead to

$$\frac{f_2(x)}{g(x)} = \frac{J_a(x)}{R(x)}, \tag{3.23}$$

$$f_2(\infty) = g(\infty) = 0, \tag{3.24}$$

$$\lim_{x\to 0^+}\frac{f_2(x)}{g(x)} = 1, \qquad \lim_{x\to\infty}\frac{f_2(x)}{g(x)} = \frac{\pi a}{\pi+2a}, \tag{3.25}$$

$$\frac{f_2'(x)}{g'(x)} = \frac{ax}{\sqrt{x^2+\frac{2a^2}{\pi}}+\frac{2ax}{\pi}} + a\frac{\frac{2a}{\pi}+\frac{x}{\sqrt{x^2+\frac{2a^2}{\pi}}}}{(\sqrt{x^2+\frac{2a^2}{\pi}}+\frac{2ax}{\pi})^2}, \tag{3.26}$$

$$\begin{aligned} H_{f_2,g}(x) &= \frac{f_2'(x)}{g'(x)}g(x) - f_2(x) \\ &= \left[\frac{ax}{\sqrt{x^2+\frac{2a^2}{\pi}}+\frac{2ax}{\pi}} + a\frac{\frac{2a}{\pi}+\frac{x}{\sqrt{x^2+\frac{2a^2}{\pi}}}}{(\sqrt{x^2+\frac{2a^2}{\pi}}+\frac{2ax}{\pi})^2}\right]\int_x^\infty e^{-t^2}\,dt - \frac{ae^{-x^2/2}}{\sqrt{x^2+\frac{2a^2}{\pi}}+\frac{2ax}{\pi}}, \end{aligned}$$

$$H_{f_2,g}(0) = H_{f_2,g}(\infty) = 0. \tag{3.27}$$

Let $t = x/\sqrt{x^2 + 2a^2/\pi} \in (0,1)$ or $x = \sqrt{2}at/\sqrt{\pi(1-t^2)}$. Then (3.26) becomes

$$\frac{f_2'(x)}{g'(x)} = -\frac{\pi}{2a}\frac{\pi^2 t^3 + (2\pi a - 4a^3)t^2 - (2\pi a^2 + \pi^2)t - 2\pi a}{(\pi + 2at)^2}. \tag{3.28}$$

Differentiating (3.28) gives

$$\begin{aligned}\left(\frac{f_2'(x)}{g'(x)}\right)' &= -\frac{\pi}{2a}\frac{d}{dt}\left(\frac{\pi^2 t^3 + (2\pi a - 4a^3)t^2 - (2\pi a^2 + \pi^2)t - 2\pi a}{(\pi + 2at)^2}\right) \times \frac{dt}{dx} \\ &= -\frac{\pi}{2a}\left(\pi\frac{2\pi a t^3 + 3\pi^2 t^2 + (6\pi a - 4a^3)t + (8a^2 - \pi^2 - 2\pi a^2)}{(\pi + 2at)^3}\right) \\ &\quad \times \frac{2a^2\sqrt{\pi}}{(\pi x^2 + 2a^2)^{3/2}} \\ &= \frac{a\pi^{5/2}}{(\pi + 2at)^3(2a^2 + \pi x^2)^{3/2}}h(t),\end{aligned} \tag{3.29}$$

where $h(t)$ is defined by Lemma 2.4.

We divide the proof into three cases.

Case 1 $a \in (0, \pi/\sqrt{2(4-\pi)}]$. Then from Lemma 2.5(i) and (3.29) we know that there exists $x_1 = \sqrt{2}at_1/\sqrt{\pi(1-t_1)^2} \in (0,\infty)$ such that f_2'/g' is increasing on $(0, x_1)$ and decreasing on (x_1, ∞). It follows from the piecewise monotonicity of f_2'/g', (3.24), (3.27), $g'(x) = -e^{-x^2/2} < 0$, and Lemma 2.2(i) that f_2/g is decreasing on $(0,\infty)$. Therefore, (3.20) follows from (3.23) and (3.25) together with the monotonicity of f_2/g.

Case 2 $a \in [\sqrt{\pi + 1} + 1, \infty)$. Then from Lemma 2.5(iii) and (3.29) we know that there exists $x_1^* = \sqrt{2}at_1^*/\sqrt{\pi(1-t_1^*)^2} \in (0,\infty)$ such that f_2'/g' is decreasing on $(0, x_1^*)$ and increasing on (x_1^*, ∞). It follows from the piecewise monotonicity of f_2'/g', (3.24), (3.27), $g'(x) < 0$, and Lemma 2.2(i) that f_2/g is increasing on $(0,\infty)$. Therefore, (3.21) follows from (3.23) and (3.25) together with the monotonicity of f_2/g.

Case 3 $a \in (\pi/\sqrt{2(4-\pi)}, \sqrt{\pi+1} + 1)$. Then from Lemma 2.5(ii) and (3.29) together with $g > 0$ and $(H_{f_2,g})' = (f_2'/g')'g$ we know that there exists $x_{11} = \sqrt{2}at_{11}/\sqrt{\pi(1-t_{11})^2}$, $x_{12} = \sqrt{2}at_{12}/\sqrt{\pi(1-t_{12})^2} \in (0,\infty)$ with $x_{11} < x_{12}$ such that $H_{f_2,g}$ is decreasing on $(0, x_{11}) \cup (x_{12}, \infty)$ and increasing on (x_{11}, x_{12}).

Making use of the piecewise monotonicity of $H_{f_2,g}$ and (3.27) we conclude that there exists $x_0 \in (0,\infty)$ such that $H_{f_2,g}(x) < 0$ for $x \in (0, x_0)$ and $H_{f_2,g}(x) > 0$ for $x \in (x_0, \infty)$, then the identity $(f_2/g)' = g'H_{f_2,g}/g^2$ and $g' < 0$ lead to the conclusion that f_2/g is increasing on $(0, x_0)$ and decreasing on (x_0, ∞). Therefore, (3.22) follows easily from (3.23) and (3.25) together with the piecewise monotonicity of f_2/g, where

$$\theta(a) = \frac{R(x_0)}{J_a(x_0)} = \frac{1}{a}\left(\sqrt{x_0^2 + \frac{2a^2}{\pi}} + \frac{2ax_0}{\pi}\right)R(x_0).$$

We clearly see that x_0 satisfies the equation $(f_2/g)' = 0$, namely x_0 is the unique solution of the equation

$$\frac{d[(\sqrt{x^2 + \frac{2a^2}{\pi}} + \frac{2ax}{\pi})R(x)]}{dx} = 0$$

on $(0,\infty)$. In particular, if $a = a_0 = \pi/(\pi - 2) = 2.7519\cdots$, then numerical computations show that $x_0 = 1.6108\cdots$ and $\theta(a_0) = 0.9909\cdots$. □

Remark 3.1 Let $x > 0$, and $I_a(x)$ and $J_a(x)$ be defined by (1.6) and (1.7), respectively. Then the functions $a \to I_a(x)$ and $a \to J_a(x)$ are increasing on $(0,\infty)$, and the functions $a \to \sqrt{\pi/a}I_a(x)$ and $a \to (\pi + 2a)J_a(x)/(\pi a)$ are decreasing on $(0,\infty)$ due to

$$\frac{\partial I_a(x)}{\partial a} = \frac{x^2 + a - x\sqrt{x^2+2a}}{\sqrt{x^2+2a} + (\sqrt{x^2+2a} + (a-1)x)^2} > 0,$$

$$\frac{\partial}{\partial a}\left(\sqrt{\frac{\pi}{a}}I_a(x)\right) = -\frac{1}{2}\sqrt{\frac{\pi}{a}}x\frac{(1+a)\sqrt{x^2+2a} - x}{\sqrt{x^2+2a}(\sqrt{x^2+2a} + (a-1)x)^2} < 0,$$

$$\frac{\partial J_a(x)}{\partial a} = \frac{x^2}{\sqrt{x^2+2a^2/\pi}(\sqrt{x^2+2a^2/\pi} + 2ax/\pi)^2} > 0,$$

$$\frac{\partial}{\partial a}\left(\frac{\pi+2a}{\pi a}J_a(x)\right) = -\frac{2}{\pi}\frac{x(\sqrt{x^2+2a^2/\pi} - x) + a}{\sqrt{x^2+2a^2/\pi}(\sqrt{x^2+2a^2/\pi} + 2ax/\pi)^2} < 0.$$

From Theorems 3.1 and 3.2, and their proofs together with Remark 3.1 we get Corollary 3.1.

Corollary 3.1 *Let $a_1, b_1, a_2, b_2 > 0$. Then the double inequalities*

$$\frac{a_1}{\sqrt{x^2+2a_1} + (a_1-1)x} < R(x) < \frac{b_1}{\sqrt{x^2+2b_1} + (b_1-1)x} \tag{3.30}$$

and

$$\frac{a_2}{\sqrt{x^2+2a_2^2/\pi} + 2a_2x/\pi} < R(x) < \frac{b_2}{\sqrt{x^2+2b_2^2/\pi} + 2b_2x/\pi} \tag{3.31}$$

hold for all $x > 0$ if and only if $a_1 \le \pi$, $b_1 \ge 4$, $a_2 \le \pi/\sqrt{2(4-\pi)} = 2.3976\cdots$, and $b_2 \ge \pi/(\pi-2) = 2.7516\cdots$.

Remark 3.2 Letting $a_1 = \pi$, $b_1 = 4$, $a_2 = \pi/\sqrt{2(4-\pi)}$, and $b_2 = \pi/(\pi-2)$. Then (3.30) and (3.31) lead to

$$W_{1,2}(x) < R(x) < W_{0,3}(x), \qquad W_{3,0}(x) < R(x) < W_{2,1}(x)$$

for all $x > 0$, which implies inequality (1.3), where $W_{3,0}(x)$, $W_{1,2}(x)$, $W_{2,1}(x)$, and $W_{0,3}(x)$ are defined by (1.4) and (1.5).

Letting $a = 0^+, 1, 2, (\sqrt{\pi+1}+1)^2/\pi$ in Theorem 3.1(1), $a = 4, 5, \infty$ in Theorem 3.1(2), $a = 0^+, 1, 2, \pi/\sqrt{2(4-\pi)}$ in Theorem 3.2(1) and $a = \sqrt{\pi+1}+1, \pi, 4, \infty$ in Theorem 3.2(2), respectively. Then we get Corollary 3.2 immediately.

Corollary 3.2 *The following inequalities for the Mills ratio $R(x)$:*

$$\frac{x}{x^2+1} < \frac{1}{\sqrt{x^2+2}} < \frac{2}{\sqrt{x^2+4}+x} < \frac{(\sqrt{\pi+1}+1)^2}{\sqrt{\pi^2x^2+2\pi(\sqrt{\pi+1}+1)^2}+2(\sqrt{\pi+1}+1)x}$$
$$< R(x) < \frac{\pi(\sqrt{\pi+1}+1)}{\sqrt{\pi^2x^2+2\pi(\sqrt{\pi+1}+1)^2}+2(\sqrt{\pi+1}+1)x} < \frac{\sqrt{2\pi}}{\sqrt{x^2+4}+x} < \frac{\sqrt{\pi}}{\sqrt{x^2+2}},$$

$$\frac{\sqrt{5\pi}}{\sqrt{x^2+10}+4x} < \frac{2\sqrt{\pi}}{\sqrt{x^2+8}+3x} < R(x) < \frac{4}{\sqrt{x^2+8}+3x} < \frac{5}{\sqrt{x^2+10}+4x} < \frac{1}{x},$$

$$\frac{\pi}{\sqrt{\pi^2x^2+2\pi}+2x} < \frac{2\pi}{\sqrt{\pi^2x^2+8\pi}+4x} < \frac{\pi}{\sqrt{2(4-\pi)x^2+2\pi}+2x} < R(x)$$
$$< \frac{\sqrt{2(4-\pi)}+2}{\sqrt{2(4-\pi)x^2+2\pi}+2x} < \frac{\pi+4}{\sqrt{\pi^2x^2+8\pi}+4x}$$
$$< \frac{\pi+2}{\sqrt{\pi^2x^2+2\pi}+2x} < \frac{1}{x},$$

$$\frac{2}{2x+\sqrt{2\pi}} < \frac{\pi+8}{\sqrt{\pi^2x^2+32\pi}+8x} < \frac{3}{\sqrt{x^2+2\pi}+2x}$$
$$< \frac{(\sqrt{\pi+1}+1)^2}{\sqrt{\pi^2x^2+2\pi(\sqrt{\pi+1}+1)^2}+2(\sqrt{\pi+1}+1)x} < R(x)$$
$$< \frac{\pi(\sqrt{\pi+1}+1)}{\sqrt{\pi^2x^2+2\pi(\sqrt{\pi+1}+1)^2}+2(\sqrt{\pi+1}+1)x}$$
$$< \frac{\pi}{\sqrt{x^2+2\pi}+2x} < \frac{4\pi}{\sqrt{\pi^2x^2+32\pi}+8x} < \frac{\pi}{2x+\sqrt{2\pi}}$$

hold for all $x > 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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