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Complete moment convergence for moving average process generated by ρ^--mixing random variables

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Abstract
Let $\{Y_i, -\infty < i < \infty\}$ be a sequence of ρ^--mixing random variables without the assumption of identical distributions, and $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers. In this paper, under some suitable conditions, we establish the complete moment convergence for the partial sum of moving average processes $\{X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1\}$. These results promote and improve the corresponding results obtained by Li and Zhang (Stat. Probab. Lett. 70:191-197, 2004) from NA to the case of a ρ^--mixing setting.

Keywords: complete moment convergence; moving average process; ρ^--mixing; Marcinkiewicz-Zygmund strong law of large numbers

1 Introduction

Let $\{Y_i, -\infty < i < \infty\}$ be a sequence of random variables and $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers, and for $n \geq 1$ set $X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}$. The limit behavior of the moving average process $\{X_n, n \geq 1\}$ has been extensively investigated by many authors. For example, Baek *et al.* [1] have obtained the convergence of moving average processes, Burton and Dehling [2] have obtained a large deviation principle, Ibragimov [3] has established the central limit theorem, Račkauskas and Suquet [4] have proved the functional central limit theorems for self-normalized partial sums of linear processes, and Chen *et al.* [5], Guo [6], Kim *et al.* [7, 8], Ko *et al.* [9], Li *et al.* [10], Li and Zhang [11], Qiu *et al.* [12], Wang and Hu [13], Yang and Hu [14], Zhang [15], Zhen *et al.* [16], Zhou *et al.* [17], Zhou and Lin [18], Shen *et al.* [19] have obtained the complete (moment) convergence of moving average process based on a sequence of dependent (or mixing) random variables, respectively. But very few results for moving average process based on a ρ^--mixing random variables are known. Firstly, we recall some definitions.

For two nonempty disjoint sets S, T of real numbers, we define $\operatorname{dist}(S, T) = \min\{|j-k|; j \in S, k \in T\}$. Let $\sigma(S)$ be the σ-field generated by $\{Y_k, k \in S\}$, and define $\sigma(T)$ similarly.

Definition 1.1 A sequence $\{Y_i, -\infty < i < \infty\}$ is called ρ^--mixing, if

$$\rho^-(s) = \sup\{\rho^-(S,T); S, T \subset Z, \operatorname{dist}(S,T) \geq s\} \to 0 \quad \text{as } s \to \infty,$$

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where

$$\rho^-(S,T) = 0 \vee \sup\{\operatorname{corr}\bigl(f(X_i, i \in S), g(X_j, j \in T)\bigr),$$

where the supremum is taken over all coordinatewise increasing real functions f on R^S and g on R^T.

Definition 1.2 A sequence $\{Y_i, -\infty < i < \infty\}$ is called ρ^*-mixing if

$$\rho^*(s) = \sup\bigl\{\rho(S,T); S, T \subset Z, \operatorname{dist}(S,T) \geq s\bigr\} \to 0 \quad \text{as } s \to \infty,$$

where

$$\rho(S,T) = \sup\bigl\{\bigl|\operatorname{corr}(f,g)\bigr|; f \in L_2\bigl(\sigma(S)\bigr), g \in L_2\bigl(\sigma(T)\bigr)\bigr\}.$$

Definition 1.3 A sequence $\{Y_i, i \in Z\}$ is called negatively associated (NA) if for every pair of disjoint subsets S, T of Z and any real coordinatewise increasing functions f on R^S and g on R^T

$$\operatorname{Cov}\bigl\{f(Y_i, i \in S), g(Y_j, j \in T)\bigr\} \leq 0.$$

Definition 1.4 A sequence $\{Y_i, -\infty < i < \infty\}$ of random variables is said to be stochastically dominated by a random variable Y if there exists a constant C such that

$$P\bigl\{|Y_i| > x\bigr\} \leq CP\bigl\{|Y| > x\bigr\}, \quad x \geq 0, -\infty < i < \infty.$$

Definition 1.5 A real valued function $l(x)$, positive and measurable on $[0,\infty)$, is said to be slowly varying at infinity if for each $\lambda > 0$, $\lim_{x\to\infty} \frac{l(\lambda x)}{l(x)} = 1$.

Li and Zhang [11] obtained the following complete moment convergence of moving average processes under NA assumptions.

Theorem A *Suppose that* $\{X_n = \sum_{i=-\infty}^{\infty} a_i \varepsilon_{i+n}, n \geq 1\}$, *where* $\{a_i, -\infty < i < \infty\}$ *is a sequence of real numbers with* $\sum_{i=-\infty}^{\infty} |a_i| < \infty$ *and* $\{\varepsilon_i, -\infty < i < \infty\}$ *is a sequence of identically distributed NA random variables with* $E\varepsilon_1 = 0$, $E\varepsilon_1^2 < \infty$. *Let h be a function slowly varying at infinity*, $1 \leq q < 2$, $r > 1 + q/2$. *Then* $E|\varepsilon_1|^r h(|\varepsilon_1|^q) < \infty$ *implies*

$$\sum_{n=1}^{\infty} n^{r/q-2-1/q} h(n) E\left\{\left|\sum_{j=1}^{n} X_j\right| - \varepsilon n^{1/q}\right\}^+ < \infty$$

for all $\varepsilon > 0$.

Chen *et al.* [20] also established the following results for moving average processes under NA assumptions.

Theorem B *Let* $q > 0$, $1 \leq p < 2$, $r \geq 1$, $rp \neq 1$. *Suppose that* $\{X_n = \sum_{i=-\infty}^{\infty} a_i \varepsilon_{i+n}, n \geq 1\}$, *where* $\{a_i, -\infty < i < \infty\}$ *is a sequence of real numbers with* $\sum_{i=-\infty}^{\infty} |a_i| < \infty$ *and* $\{\varepsilon_i, -\infty < i <$

$\infty\}$ *is a sequence of identically distributed NA random variables. If* $E\varepsilon_1 = 0$ *and* $E|\varepsilon_1|^{rp} < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k} X_j\right| \ge \varepsilon n^{1/p}\right\} < \infty$$

for all $\varepsilon > 0$. *Furthermore if* $E\varepsilon_1 = 0$ *and* $E|\varepsilon_1|^{rp} < \infty$ *for* $q < rp$, $E|\varepsilon_1|^{rp}\log(1+|\varepsilon_1|) < \infty$ *for* $q = rp$, $E|\varepsilon_1|^{q} < \infty$ *for* $q > rp$, *then*

$$\sum_{n=1}^{\infty} n^{r-2-q/p} E\left(\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k} X_j\right| - \varepsilon n^{1/q}\right\}^{+}\right)^{q} < \infty$$

for all $\varepsilon > 0$.

Recently, Zhou and Lin [18] obtained the following complete moment convergence of moving average processes under ρ-mixing assumptions.

Theorem C *Let h be a function slowly varying at infinity,* $p \ge 1$, $p\alpha > 1$ *and* $\alpha > 1/2$. *Suppose that* $\{X_n, n \ge 1\}$ *is a moving average process based on a sequence* $\{Y_i, -\infty < i < \infty\}$ *of identically distributed* ρ*-mixing random variables. If* $EY_1 = 0$ *and* $E|Y_1|^{p+\delta}h(|Y_1|^{1/\alpha}) < \infty$ *for some* $\delta > 0$, *then for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} h(n) E\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k} X_j\right| - \varepsilon n^{\alpha}\right\}^{+} < \infty$$

and

$$\sum_{n=1}^{\infty} n^{p\alpha-2} h(n) E\left\{\sup_{k\ge n}\left|k^{-\alpha}\sum_{j=1}^{k} X_j\right| - \varepsilon\right\}^{+} < \infty.$$

Obviously, ρ^--mixing random variables include NA and ρ^*-mixing random variables, which have a lot of applications, their limit properties have aroused wide interest recently, and a lot of results have been obtained; we refer to Wang and Lu [21] for a Rosenthal-type moment inequality and weak convergence, Budsaba *et al.* [22, 23] for complete convergence for moving average process based on a ρ^--mixing sequence, Tan *et al.* [24] for the almost sure central limit theorem. But there are few results on the complete moment convergence of moving average process based on a ρ^--mixing sequence. Therefore, in this paper, we establish some results on the complete moment convergence for maximum partial sums with less restrictions. Throughout the sequel, C represents a positive constant although its value may change from one appearance to the next, $I\{A\}$ denotes the indicator function of the set A.

2 Preliminary lemmas

In this section, we list some lemmas which will be useful to prove our main results.

Lemma 2.1 (Zhou [17]) *If l is slowly varying at infinity, then*

(1) $\sum_{n=1}^{m} n^s l(n) \le Cm^{s+1} l(m)$ *for* $s > -1$ *and positive integer* m,
(2) $\sum_{n=m}^{\infty} n^s l(n) \le Cm^{s+1} l(m)$ *for* $s < -1$ *and positive integer* m.

Lemma 2.2 (Wang and Lu [21]) *For a positive real number* $q \ge 2$, *if* $\{X_n, n \ge 1\}$ *is a sequence of* ρ^-*-mixing random variables, with* $EX_i = 0$, $E|X_i|^q < \infty$ *for every* $i \ge 1$, *then for all* $n \ge 1$, *there is a positive constant* $C = C(q, \rho^-(\cdot))$ *such that*

$$E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_i\right|^q\right) \le C\left\{\sum_{i=1}^{n} E|X_i|^q + \left(\sum_{i=1}^{n} EX_i^2\right)^{\frac{q}{2}}\right\}.$$

Lemma 2.3 (Wang *et al.* [25]) *Let* $\{X_n, n \ge 1\}$ *be a sequence of random variables which is stochastically dominated by a random variable* X. *Then for any* $a > 0$ *and* $b > 0$,

$$E|X_n|^a I\{|X_n| \le b\} \le C\left[E|X|^a I\{|X| \le b\} + b^a P(|X| > b)\right],$$
$$E|X_n|^a I\{|X_n| > b\} \le CE|X|^a I\{|X| > b\}.$$

3 Main results and proofs

Theorem 3.1 *Let* l *be a function slowly varying at infinity,* $p \ge 1$, $\alpha > 1/2$, $\alpha p > 1$. *Assume that* $\{a_i, -\infty < i < \infty\}$ *is an absolutely summable sequence of real numbers. Suppose that* $\{X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \ge 1\}$ *is a moving average process generated by a sequence* $\{Y_i, -\infty < i < \infty\}$ *of* ρ^-*-mixing random variables which is stochastically dominated by a random variable* Y. *If* $EY_i = 0$ *for* $1/2 < \alpha \le 1$, $E|Y|^p l(|Y|^{1/\alpha}) < \infty$ *for* $p > 1$ *and* $E|Y|^{1+\delta} < \infty$ *for* $p = 1$ *and some* $\delta > 0$, *then for any* $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) E\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k} X_j\right| - \varepsilon n^{\alpha}\right\}^{+} < \infty \tag{3.1}$$

and

$$\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) E\left\{\sup_{k\ge n}\left|k^{-\alpha}\sum_{j=1}^{k} X_j\right| - \varepsilon\right\}^{+} < \infty. \tag{3.2}$$

Proof Firstly to prove (3.1). Let $f(n) = n^{\alpha p-2-\alpha} l(n)$ and $Y_{xj}^{(1)} = -xI\{Y_j < -x\} + Y_j I\{|Y_j| \le x\} + xI\{Y_j > x\}$ and $Y_{xj}^{(2)} = Y_j - Y_{xj}^{(1)}$ be the monotone truncations of $\{Y_j, -\infty < j < \infty\}$ for $x > 0$. Then by the property of ρ^--mixing random variables (*cf.* Property P2 in Wang and Lu [21]), $\{Y_{xj}^{(1)} - EY_{xj}^{(1)}, -\infty < j < \infty\}$ and $\{Y_{xj}^{(2)}, -\infty < j < \infty\}$ are two sequences of ρ^--mixing random variables. Note that $\sum_{k=1}^{n} X_k = \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j$. Since $\sum_{i=-\infty}^{\infty} |a_i| < \infty$, by Lemma 2.3, we have for $x > n^{\alpha}$, if $\alpha > 1$

$$\begin{aligned} &x^{-1}\left|E\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_{xj}^{(1)}\right| \\ &\quad\le x^{-1}\sum_{i=-\infty}^{\infty} |a_i| \sum_{j=i+1}^{i+n}\left[E|Y_j| I\{|Y_j| \le x\} + xP(|Y| > x)\right] \\ &\quad\le Cx^{-1} n\left[E|Y| I\{|Y| \le x\} + xP(|Y| > x)\right] \le Cn^{1-\alpha} \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

If $1/2 < \alpha \leq 1$, note $\alpha p > 1$, this means $p > 1$. By $E|Y|^p l(|Y|^{1/\alpha}) < \infty$ and l is slowly varying at infinity, for any $0 < \epsilon < p - 1/\alpha$, we have $E|Y|^{p-\epsilon} < \infty$. Then noting $EY_i = 0$, by Lemma 2.3 we have

$$\begin{aligned} x^{-1}\left|E\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_{xj}^{(1)}\right| &= x^{-1}\left|E\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_{xj}^{(2)}\right| \\ &\leq Cx^{-1}\sum_{i=-\infty}^{\infty} |a_i| \sum_{j=i+1}^{i+n} E|Y_j| I\{|Y_j| > x\} \leq Cx^{-1} n E|Y| I\{|Y| > x\} \\ &\leq Cx^{1/\alpha - 1} E|Y| I\{|Y| > x\} \leq CE|Y|^{1/\alpha} I\{|Y| > x\} \\ &\leq E|Y|^{p-\epsilon} I\{|Y| > x\} \to 0, \quad \text{as } x \to \infty. \end{aligned}$$

Hence for $x > n^\alpha$ large enough, we get

$$x^{-1}\left|E\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_{xj}^{(1)}\right| < \varepsilon/4.$$

Therefore

$$\begin{aligned} &\sum_{n=1}^{\infty} f(n) E\left\{\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} X_j\right| - \varepsilon n^\alpha\right\}^+ \\ &\quad\leq \sum_{n=1}^{\infty} f(n) \int_{\varepsilon n^\alpha}^{\infty} P\left\{\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} X_j\right| \geq x\right\} dx \\ &\quad\leq C\sum_{n=1}^{\infty} f(n) \int_{n^\alpha}^{\infty} P\left\{\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} X_j\right| \geq \varepsilon x\right\} dx \\ &\quad\leq C\sum_{n=1}^{\infty} f(n) \int_{n^\alpha}^{\infty} P\left\{\max_{1\leq k\leq n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{xj}^{(2)}\right| \geq \varepsilon x/2\right\} dx \\ &\qquad + C\sum_{n=1}^{\infty} f(n) \int_{n^\alpha}^{\infty} P\left\{\max_{1\leq k\leq n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} \left(Y_{xj}^{(1)} - EY_{xj}^{(1)}\right)\right| \geq \varepsilon x/4\right\} dx \\ &\quad=: I_1 + I_2. \end{aligned} \tag{3.3}$$

Firstly we show $I_1 < \infty$. Noting $|Y_{xj}^{(2)}| < |Y_j| I\{|Y_j| > x\}$, by Markov's inequality and Lemma 2.3, we have

$$\begin{aligned} I_1 &\leq C\sum_{n=1}^{\infty} f(n) \int_{n^\alpha}^{\infty} x^{-1} E\max_{1\leq k\leq n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{xj}^{(2)}\right| dx \\ &\leq C\sum_{n=1}^{\infty} f(n) \int_{n^\alpha}^{\infty} x^{-1} \sum_{i=-\infty}^{\infty} |a_i| \sum_{j=i+1}^{i+n} E\left|Y_{xj}^{(2)}\right| dx \\ &\leq C\sum_{n=1}^{\infty} n f(n) \int_{n^\alpha}^{\infty} x^{-1} E|Y| I\{|Y| > x\} dx \end{aligned}$$

$$
\begin{aligned}
&= C\sum_{n=1}^{\infty} nf(n)\sum_{m=n}^{\infty}\int_{m^\alpha}^{(m+1)^\alpha} x^{-1}E|Y|I\{|Y|>x\}\,dx \\
&\le C\sum_{n=1}^{\infty} nf(n)\sum_{m=n}^{\infty} m^{-1}E|Y|I\{|Y|>m^\alpha\} \\
&= C\sum_{m=1}^{\infty} m^{-1}E|Y|I\{|Y|>m^\alpha\}\sum_{n=1}^{m} n^{\alpha p-1-\alpha}l(n).
\end{aligned}
$$

If $p>1$, then $\alpha p-1-\alpha>-1$, and, by Lemma 2.1, we obtain

$$
\begin{aligned}
I_1 &\le C\sum_{m=1}^{\infty} m^{\alpha p-1-\alpha}l(m)E|Y|I\{|Y|>m^\alpha\} \\
&= C\sum_{m=1}^{\infty} m^{\alpha p-1-\alpha}l(m)\sum_{k=m}^{\infty}E|Y|I\{k^\alpha<|Y|\le(k+1)^\alpha\} \\
&= C\sum_{k=1}^{\infty}E|Y|I\{k^\alpha<|Y|\le(k+1)^\alpha\}\sum_{m=1}^{k} m^{\alpha p-1-\alpha}l(m) \\
&\le C\sum_{k=1}^{\infty} k^{\alpha p-\alpha}l(k)E|Y|I\{k^\alpha<|Y|\le(k+1)^\alpha\} \\
&\le CE|Y|^p l\big(|Y|^{1/\alpha}\big)<\infty.
\end{aligned}
$$

If $p=1$, notice that $E|Y|^{1+\delta}<\infty$ implies $E|Y|^{1+\delta'}l(|Y|^{1/\alpha})<\infty$ for any $0<\delta'<\delta$, then by Lemma 2.1, we obtain

$$
\begin{aligned}
I_1 &\le C\sum_{m=1}^{\infty} m^{-1}E|Y|I\{|Y|>m^\alpha\}\sum_{n=1}^{m} n^{-1}l(n) \\
&\le C\sum_{m=1}^{\infty} m^{-1}E|Y|I\{|Y|>m^\alpha\}\sum_{n=1}^{m} n^{-1+\alpha\delta'}l(n) \\
&\le C\sum_{m=1}^{\infty} m^{\alpha\delta'-1}l(m)E|Y|I\{|Y|>m^\alpha\} \\
&\le CE|Y|^{1+\delta'}l\big(|Y|^{1/\alpha}\big)\le CE|Y|^{1+\delta}<\infty.
\end{aligned}
$$

So, we get

$$
I_1<\infty. \tag{3.4}
$$

Next we show $I_2<\infty$. By Markov's inequality, the Hölder inequality, and Lemma 2.2, we conclude

$$
\begin{aligned}
I_2 &\le C\sum_{n=1}^{\infty} f(n)\int_{n^\alpha}^{\infty} x^{-r}E\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i\sum_{j=i+1}^{i+k}\big(Y_{xj}^{(1)}-EY_{xj}^{(1)}\big)\right|^r dx \\
&\le C\sum_{n=1}^{\infty} f(n)\int_{n^\alpha}^{\infty} x^{-r}E\left[\sum_{i=-\infty}^{\infty}\big(|a_i|^{\frac{r-1}{r}}\big)\left(|a_i|^{1/r}\max_{1\le k\le n}\left|\sum_{j=i+1}^{i+k}\big(Y_{xj}^{(1)}-EY_{xj}^{(1)}\big)\right|\right)\right]^r dx
\end{aligned}
$$

$$
\begin{aligned}
&\le C\sum_{n=1}^{\infty}f(n)\int_{n^\alpha}^{\infty}x^{-r}\left(\sum_{i=-\infty}^{\infty}|a_i|\right)^{r-1}\left(\sum_{i=-\infty}^{\infty}|a_i|E\max_{1\le k\le n}\left|\sum_{j=i+1}^{i+k}\left(Y_{xj}^{(1)}-EY_{xj}^{(1)}\right)\right|^r\right)dx\\
&\le C\sum_{n=1}^{\infty}f(n)\int_{n^\alpha}^{\infty}x^{-r}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}E\left|Y_{xj}^{(1)}-EY_{xj}^{(1)}\right|^r dx\\
&\quad+C\sum_{n=1}^{\infty}f(n)\int_{n^\alpha}^{\infty}x^{-r}\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n}E\left|Y_{xj}^{(1)}-EY_{xj}^{(1)}\right|^2\right)^{r/2}dx\\
&=: I_{21}+I_{22},
\end{aligned}
\tag{3.5}
$$

where $r \ge 2$ will be specialized later.

For I_{21}, if $p > 1$, take $r > \max\{2, p\}$, then by C_r inequality, Lemma 2.3, and Lemma 2.1, we get

$$
\begin{aligned}
I_{21}&\le C\sum_{n=1}^{\infty}f(n)\int_{n^\alpha}^{\infty}x^{-r}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}\left[E|Y_j|^rI\{|Y_j|\le x\}+x^rP\left(|Y_j|>x\right)\right]dx\\
&\le C\sum_{n=1}^{\infty}nf(n)\int_{n^\alpha}^{\infty}x^{-r}\left[E|Y|^rI\{|Y|\le x\}+x^rP\left(|Y|>x\right)\right]dx\\
&\le C\sum_{n=1}^{\infty}nf(n)\sum_{m=n}^{\infty}\int_{m^\alpha}^{(m+1)^\alpha}\left[x^{-r}E|Y|^rI\{|Y|\le x\}+P\left(|Y|>x\right)\right]dx\\
&\le C\sum_{n=1}^{\infty}nf(n)\sum_{m=n}^{\infty}\left[m^{\alpha(1-r)-1}E|Y|^rI\{|Y|\le (m+1)^\alpha\}+m^{\alpha-1}P\left(|Y|>m^\alpha\right)\right]\\
&= C\sum_{m=1}^{\infty}\left[m^{\alpha(1-r)-1}E|Y|^rI\{|Y|\le (m+1)^\alpha\}+m^{\alpha-1}P\left(|Y|>m^\alpha\right)\right]\sum_{n=1}^{m}nf(n)\\
&\le C\sum_{m=1}^{\infty}m^{\alpha(p-r)-1}l(m)\sum_{k=1}^{m}E|Y|^rI\{k^\alpha<|Y|\le (k+1)^\alpha\}\\
&\quad+C\sum_{m=1}^{\infty}m^{\alpha p-1}l(m)\sum_{k=m}^{\infty}EI\{k^\alpha<|Y|\le (k+1)^\alpha\}\\
&= C\sum_{k=1}^{\infty}E|Y|^rI\{k^\alpha<|Y|\le (k+1)^\alpha\}\sum_{m=k}^{\infty}m^{\alpha(p-r)-1}l(m)\\
&\quad+C\sum_{k=1}^{\infty}EI\{k^\alpha<|Y|\le (k+1)^\alpha\}\sum_{m=1}^{k}m^{\alpha p-1}l(m)\\
&\le C\sum_{k=1}^{\infty}k^{\alpha(p-r)}l(k)E|Y|^p|Y|^{r-p}I\{k^\alpha<|Y|\le (k+1)^\alpha\}\\
&\quad+C\sum_{k=1}^{\infty}k^{\alpha p}l(k)E|Y|^p|Y|^{-p}I\{k^\alpha<|Y|\le (k+1)^\alpha\}\\
&\le CE|Y|^pl\left(|Y|^{1/\alpha}\right)<\infty.
\end{aligned}
\tag{3.6}
$$

For I_{21}, if $p = 1$, take $r > \max\{1 + \delta', 2\}$, where $0 < \delta' < \delta$, then by the same argument as above we have

$$\begin{aligned}
I_{21} &\le C\sum_{m=1}^{\infty}\bigl[m^{\alpha(1-r)-1}E|Y|^r I\bigl\{|Y| \le (m+1)^\alpha\bigr\} + m^{\alpha-1}P\bigl(|Y| > m^\alpha\bigr)\bigr]\sum_{n=1}^{m} nf(n) \\
&\le C\sum_{m=1}^{\infty}\bigl[m^{\alpha(1-r)-1}E|Y|^r I\bigl\{|Y| \le (m+1)^\alpha\bigr\} + m^{\alpha-1}P\bigl(|Y| > m^\alpha\bigr)\bigr]\sum_{n=1}^{m} n^{-1+\alpha\delta'}l(n) \\
&\le C\sum_{m=1}^{\infty}\bigl[m^{\alpha(1-r+\delta')-1}l(m)E|Y|^r I\bigl\{|Y| \le (m+1)^\alpha\bigr\} \\
&\quad + m^{\alpha(1+\delta')-1}l(m)EI\bigl\{|Y| > m^\alpha\bigr\}\bigr] \\
&\le CE|Y|^{1+\delta'}l\bigl(|Y|^{1/\alpha}\bigr) \le CE|Y|^{1+\delta} < \infty. \qquad (3.7)
\end{aligned}$$

For I_{22}, if $1 \le p < 2$, take $r > 2$, note $\alpha p + r/2 - \alpha pr/2 - 1 = (\alpha p - 1)(1 - r/2) < 0$, by the C_r inequality, Lemma 2.3, and Lemma 2.1, we obtain

$$\begin{aligned}
I_{22} &\le C\sum_{n=1}^{\infty} n^{r/2}f(n)\int_{n^\alpha}^{\infty} x^{-r}\bigl[\bigl(E|Y|^2 I\bigl\{|Y| \le x\bigr\}\bigr)^{r/2} + x^r P^{r/2}\bigl(|Y| > x\bigr)\bigr]\,dx \\
&\le C\sum_{n=1}^{\infty} n^{r/2}f(n)\sum_{m=n}^{\infty}\int_{m^\alpha}^{(m+1)^\alpha}\bigl[x^{-r}\bigl(E|Y|^2 I\bigl\{|Y| \le x\bigr\}\bigr)^{r/2} + P^{r/2}\bigl(|Y| > x\bigr)\bigr]\,dx \\
&\le C\sum_{n=1}^{\infty} n^{r/2}f(n)\sum_{m=n}^{\infty}\bigl[m^{\alpha(1-r)-1}\bigl(E|Y|^2 I\bigl\{|Y| \le (m+1)^\alpha\bigr\}\bigr)^{r/2} + m^{\alpha-1}P^{r/2}\bigl(|Y| > m^\alpha\bigr)\bigr] \\
&= C\sum_{m=1}^{\infty}\bigl[m^{\alpha(1-r)-1}\bigl(E|Y|^2 I\bigl\{|Y| \le (m+1)^\alpha\bigr\}\bigr)^{r/2} + m^{\alpha-1}P^{r/2}\bigl(|Y| > m^\alpha\bigr)\bigr]\sum_{n=1}^{m} n^{r/2}f(n) \\
&\le C\sum_{m=1}^{\infty} m^{\alpha(p-r)+r/2-2}l(m)\bigl(E|Y|^p|Y|^{2-p}I\bigl\{|Y| \le (m+1)^\alpha\bigr\}\bigr)^{r/2} \\
&\quad + C\sum_{m=1}^{\infty} m^{\alpha p+r/2-2}l(m)\bigl(E|Y|^p|Y|^{-p}I\bigl\{|Y| > m^\alpha\bigr\}\bigr)^{r/2} \\
&\le C\sum_{m=1}^{\infty} m^{\alpha p+r/2-\alpha pr/2-2}l(m)\bigl(E|Y|^p\bigr)^{r/2} < \infty. \qquad (3.8)
\end{aligned}$$

For I_{22}, if $p \ge 2$, take $r > (\alpha p - 1)/(\alpha - 1/2) > 2$; we have $\alpha(p - r) + r/2 - 2 < -1$, and therefore one gets

$$\begin{aligned}
I_{22} &\le C\sum_{m=1}^{\infty}\bigl[m^{\alpha(1-r)-1}\bigl(E|Y|^2 I\bigl\{|Y| \le (m+1)^\alpha\bigr\}\bigr)^{r/2} + m^{\alpha-1}P^{r/2}\bigl(|Y| > m^\alpha\bigr)\bigr]\sum_{n=1}^{m} n^{r/2}f(n) \\
&\le C\sum_{m=1}^{\infty} m^{\alpha(p-r)+r/2-2}l(m)\bigl(E|Y|^2 I\bigl\{|Y| \le (m+1)^\alpha\bigr\}\bigr)^{r/2} \\
&\quad + C\sum_{m=1}^{\infty} m^{\alpha p+r/2-2}l(m)\bigl(E|Y|^2|Y|^{-2}I\bigl\{|Y| > m^\alpha\bigr\}\bigr)^{r/2}
\end{aligned}$$

$$\leq C\sum_{m=1}^{\infty} m^{\alpha(p-r)+r/2-2} l(m)\bigl(E|Y|^2\bigr)^{r/2} < \infty. \tag{3.9}$$

Thus, (3.1) can be deduced by combining (3.3)-(3.9).

Now, we show (3.2). By Lemma 2.1 and (3.1) we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) E\left\{\sup_{k\geq n}\left|k^{-\alpha}\sum_{j=1}^{k} X_j\right| - \varepsilon\right\}^{+} \\
&\quad= \sum_{n=1}^{\infty} n^{\alpha p-2} l(n)\int_0^{\infty} P\left\{\sup_{k\geq n}\left|k^{-\alpha}\sum_{j=1}^{k} X_j\right| > \varepsilon + t\right\} dt \\
&\quad= \sum_{i=1}^{\infty}\sum_{n=2^{i-1}}^{2^i-1} n^{\alpha p-2} l(n)\int_0^{\infty} P\left\{\sup_{k\geq n}\left|k^{-\alpha}\sum_{j=1}^{k} X_j\right| > \varepsilon + t\right\} dt \\
&\quad\leq C\sum_{i=1}^{\infty}\int_0^{\infty} P\left\{\sup_{k\geq 2^{i-1}}\left|k^{-\alpha}\sum_{j=1}^{k} X_j\right| > \varepsilon + t\right\} dt \sum_{n=2^{i-1}}^{2^i-1} n^{\alpha p-2} l(n) \\
&\quad\leq C\sum_{i=1}^{\infty} 2^{i(\alpha p-1)} l\bigl(2^i\bigr)\int_0^{\infty} P\left\{\sup_{k\geq 2^{i-1}}\left|k^{-\alpha}\sum_{j=1}^{k} X_j\right| > \varepsilon + t\right\} dt \\
&\quad\leq C\sum_{i=1}^{\infty} 2^{i(\alpha p-1)} l\bigl(2^i\bigr)\sum_{l=i}^{\infty}\int_0^{\infty} P\left\{\max_{2^{l-1}\leq k<2^l}\left|k^{-\alpha}\sum_{j=1}^{k} X_j\right| > \varepsilon + t\right\} dt \\
&\quad\leq C\sum_{l=1}^{\infty}\int_0^{\infty} P\left\{\max_{2^{l-1}\leq k<2^l}\left|k^{-\alpha}\sum_{j=1}^{k} X_j\right| > \varepsilon + t\right\} dt \sum_{i=1}^{l} 2^{i(\alpha p-1)} l\bigl(2^i\bigr) \\
&\quad\leq C\sum_{l=1}^{\infty} 2^{l(\alpha p-1)} l\bigl(2^l\bigr)\int_0^{\infty} P\left\{\max_{2^{l-1}\leq k<2^l}\left|\sum_{j=1}^{k} X_j\right| > (\varepsilon + t)2^{(l-1)\alpha}\right\} dt \\
&\quad\leq C\sum_{l=1}^{\infty} 2^{l(\alpha p-1-\alpha)} l\bigl(2^l\bigr)\int_0^{\infty} P\left\{\max_{1\leq k<2^l}\left|\sum_{j=1}^{k} X_j\right| > \varepsilon 2^{(l-1)\alpha} + y\right\} dy \\
&\quad\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n)\int_0^{\infty} P\left\{\max_{1\leq k<n}\left|\sum_{j=1}^{k} X_j\right| > \varepsilon n^{\alpha} 2^{-\alpha} + y\right\} dy \\
&\quad= C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) E\left\{\max_{1\leq k<n}\left|\sum_{j=1}^{k} X_j\right| - \varepsilon_0 n^{\alpha}\right\}^{+} < \infty.
\end{aligned}$$

Hence the proof of Theorem 3.1 is completed. □

The next theorem treats the case $\alpha p = 1$.

Theorem 3.2 *Let l be a function slowly varying at infinity, $1 \leq p < 2$. Assume that $\sum_{i=-\infty}^{\infty} |a_i|^{\theta} < \infty$, where θ belong to $(0,1)$ if $p = 1$ and $\theta = 1$ if $1 < p < 2$. Suppose that $\{X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1\}$ is a moving average process generated by a sequence $\{Y_i, -\infty < i < \infty\}$ of ρ^--mixing random variables which is stochastically dominated by a random variable Y.*

If $EY_i = 0$ *and* $E|Y|^p l(|Y|^p) < \infty$, *then for any* $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{-1-1/p} l(n) E\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k} X_j\right| - \varepsilon n^{1/p}\right\}^{+} < \infty. \tag{3.10}$$

Proof Let $g(n) = n^{-1-1/p} l(n)$. Similarly to the proof of (3.3), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} g(n) E\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k} X_j\right| - \varepsilon n^{1/p}\right\}^{+} \\
&\quad\le C\sum_{n=1}^{\infty} g(n)\int_{n^{1/p}}^{\infty} P\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{xj}^{(2)}\right| \ge \varepsilon x/2\right\} dx \\
&\qquad + C\sum_{n=1}^{\infty} g(n)\int_{n^{1/p}}^{\infty} P\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k}\left(Y_{xj}^{(1)} - EY_{xj}^{(1)}\right)\right| \ge \varepsilon x/4\right\} dx \\
&\quad=: J_1 + J_2.
\end{aligned} \tag{3.11}$$

For J_1, by Markov's inequality, the C_r inequality, Lemma 2.3, and Lemma 2.1, one gets

$$\begin{aligned}
J_1 &\le C\sum_{n=1}^{\infty} g(n)\int_{n^{1/p}}^{\infty} x^{-\theta} E\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{xj}^{(2)}\right|^{\theta} dx \\
&\le C\sum_{n=1}^{\infty} ng(n)\int_{n^{1/p}}^{\infty} x^{-\theta} E|Y|^{\theta} I\{|Y| > x\} dx \\
&= C\sum_{n=1}^{\infty} ng(n)\sum_{m=n}^{\infty}\int_{m^{1/p}}^{(m+1)^{1/p}} x^{-\theta} E|Y|^{\theta} I\{|Y| > x\} dx \\
&\le C\sum_{n=1}^{\infty} ng(n)\sum_{m=n}^{\infty} m^{(1-\theta)/p-1} E|Y|^{\theta} I\{|Y| > m^{1/p}\} \\
&= C\sum_{m=1}^{\infty} m^{(1-\theta)/p-1} E|Y|^{\theta} I\{|Y| > m^{1/p}\}\sum_{n=1}^{m} ng(n) \\
&\le C\sum_{m=1}^{\infty} m^{-\theta/p} l(m) E|Y|^{\theta} I\{|Y| > m^{1/p}\} \\
&= C\sum_{m=1}^{\infty} m^{-\theta/p} l(m)\sum_{k=m}^{\infty} E|Y|^{\theta} I\{k^{1/p} < |Y| < (k+1)^{1/p}\} \\
&= C\sum_{k=1}^{\infty} E|Y|^{\theta} I\{k^{1/p} < |Y| < (k+1)^{1/p}\}\sum_{m=1}^{k} m^{-\theta/p} l(m) \\
&\le C\sum_{k=1}^{\infty} k^{1-\theta/p} l(k) E|Y|^{\theta} I\{k^{1/p} < |Y| < (k+1)^{1/p}\} \\
&\le CE|Y|^p l\left(|Y|^p\right) < \infty.
\end{aligned} \tag{3.12}$$

For J_2, similar to the proof of I_2, take $r = 2$, by Lemma 2.2, Lemma 2.3, and Lemma 2.1, we conclude

$$\begin{aligned} J_2 &\le C\sum_{n=1}^{\infty} g(n)\int_{n^{1/p}}^{\infty} x^{-2}E\Big|\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k}\big(Y_{xj}^{(1)} - EY_{xj}^{(1)}\big)\right|^2 dx \\ &\le C\sum_{n=1}^{\infty} ng(n)\int_{n^{1/p}}^{\infty} x^{-2}\big[E|Y|^2 I\{|Y|\le x\} + x^2 P(|Y|>x)\big]\,dx \\ &= C\sum_{n=1}^{\infty} ng(n)\sum_{m=n}^{\infty}\int_{m^{1/p}}^{(m+1)^{1/p}} x^{-2}\big[E|Y|^2 I\{|Y|\le x\} + x^2 P(|Y|>x)\big]\,dx \\ &\le C\sum_{n=1}^{\infty} ng(n)\sum_{m=n}^{\infty}\big[m^{-1-1/p}E|Y|^2 I\{|Y|\le (m+1)^{1/p}\} + m^{1/p-1}P(|Y|>m^{1/p})\big] \\ &= C\sum_{m=1}^{\infty}\big[m^{-1-1/p}E|Y|^2 I\{|Y|\le (m+1)^{1/p}\} + m^{1/p-1}P(|Y|>m^{1/p})\big]\sum_{n=1}^{m} ng(n) \\ &\le C\sum_{m=1}^{\infty}\big[m^{-2/p}l(m)E|Y|^2 I\{|Y|\le (m+1)^{1/p}\} + l(m)P(|Y|>m^{1/p})\big] \\ &\le CE|Y|^p l(|Y|^p) < \infty. \end{aligned} \tag{3.13}$$

Hence from (3.11)-(3.13), (3.10) holds. □

For the complete convergence and strong law of large numbers, we have the following corollary from the above theorems immediately.

Corollary 3.3 *Under the assumptions of Theorem* 3.1, *for any* $\varepsilon > 0$ *we have*

$$\sum_{n=1}^{\infty} n^{\alpha p-2}l(n)P\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k} X_j\right| > \varepsilon n^{\alpha}\right\} < \infty. \tag{3.14}$$

Under the assumptions of Theorem 3.2, *for any* $\varepsilon > 0$ *we have*

$$\sum_{n=1}^{\infty} n^{-1}l(n)P\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k} X_j\right| > \varepsilon n^{1/p}\right\} < \infty; \tag{3.15}$$

in particular, the assumptions $EY_i = 0$ *and* $E|Y|^p < \infty$ *imply the following Marcinkiewicz-Zygmund strong law of large numbers*:

$$\lim_{n\to\infty}\frac{1}{n^{1/p}}\sum_{j=1}^{n} X_j = 0 \quad a.s. \tag{3.16}$$

Remark 3.4 Corollary 3.3 provides complete convergence for the maximum of partial sums, which extends the corresponding results of Budsaba *et al.* [22, 23] and Theorem 1 of Baek *et al.* [1] with less restrictions. Since ρ^--mixing random variables include NA and ρ^*-mixing random variables, our results also hold for NA and ρ^*-mixing, and therefore

Theorem 3.1 improves upon the above Theorem A from Li and Zhang [11] with less restrictions, and our results also extend and generalize the above Theorem B from Chen *et al.* [20] with $q = 1$ partly.

Remark 3.5 Obviously, the assumption that $\{Y_i, -\infty < i < \infty\}$ is stochastically dominated by a random variable Y is weaker than the assumption of identical distribution of the random variables $\{Y_i, -\infty < i < \infty\}$, therefore the above results also hold for identically distributed random variables.

Remark 3.6 Let $a_0 = 1, a_i = 0, i \neq 0$, then $S_n = \sum_{k=1}^{n} X_k = \sum_{k=1}^{n} Y_k$. Hence the above results hold when $\{X_k, k \geq 1\}$ is a sequence of ρ^--mixing random variables which is stochastically dominated by a random variable Y.

Competing interests
The author declares to have no competing interests.

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