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Generalized Gronwall fractional summation inequalities and their applications

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Abstract
Based on an iteration method, some generalized discrete fractional Gronwall inequalities are developed, which can be used in the qualitative analysis of the solutions to fractional difference equations and summation equations.

MSC: 39A12; 26A33

Keywords: Gronwall's inequality; fractional difference; fractional sum; fractional difference equation; fractional sum equation

1 Introduction

In recent years, the fractional differential and fractional integrals have been adopted in various fields of science and engineering, which can be used to describe certain phenomena, reflect some physiochemical properties, and provide accurate models for the systems under consideration. Some applications of fractional calculus include fluid flow, rheology, dynamical processes in self-similar and porous structures, electrical networks, probability and statistics, control theory of dynamical systems, chemical physics, optics, and signal processing, economics, and so on. Therefore, they are receiving extensive attention from a variety of domains. References [1, 2] introduced the definitions of fractional calculus, theorems and basic analytic solutions of the fractional equation in detail. At the same time, the fractional difference equations, fractional sum equations, and fractional inequalities also play important roles in many areas. In 1989, Miller and Ross [3] defined a fractional sum of order $\alpha > 0$ via the solution of a linear difference equation and proved some basic properties of this operator. After that, many authors followed up in various directions [4–17].

In 2000, Hirota [18] defined the fractional order difference operator ∇^{α} where α was a real number, using Taylor's series. In 2003, Nagai [19] adopted another definition for the fractional difference by modifying Hirota's definition. The definition in [19] contained the ∇ operator and the term $(-1)^j$ inside the summation index. Therefore this definition was difficult use in studying the properties of solutions of fractional difference equations. To avoid this, Deekshitulu and Mohan [20] modified the definition and rearranged the terms in the definition of Nagai [19]. Then they defined the fractional sum operator $\nabla^{-\alpha}$ and the fractional difference operator ∇^{α}. Recently, they discussed some basic inequalities, comparison theorems, and qualitative properties of the solutions of fractional difference equations in [21–24]. Under the definition in [20], they also considered an initial value problem of fractional order and obtained some useful fractional difference inequalities of Gronwall-Bellman type [25].

In 2009, Atici and Eloe [26] introduced the definition of the fractional sum and the fractional difference: $\nabla^{-\alpha}$, ∇^{α}, $\triangle^{-\alpha}$, $\triangle^{\alpha}$. In 2012, based on the definition, Atici and Eloe [27] and Ferreira [28] studied some discrete fractional Gronwall's inequalities as regards $\nabla^{-\alpha}$, $\triangle^{-\alpha}$ separately. They all adopted the same method: After establishing a comparison theorem, they gave an explicit solution to the linear discrete fractional sum equation of the initial value problem, which allowed them to state and prove an analog of Gronwall's inequality on discrete fractional calculus.

In 2011, Cheng [29] presented another form of definition. He provided some basic properties of fractional difference and summation, and established the theory symmetrical to the fractional differential equation.

In this paper, we will introduce some inequalities and their applications based on the definition in [29] to demonstrate the qualitative properties of solutions to some fractional summation equations. The proof is based on the iterative method.

Definition 1.1 [29] Let ν be nonnegative real number, define the ν-order summation of $x(n)$ as

$$\nabla^{-\nu}x(n) = \begin{bmatrix} \nu \\ n \end{bmatrix} * x(n) = \sum_{r=0}^{n} \begin{bmatrix} \nu \\ n-r \end{bmatrix} x(r),$$

where $\left[\begin{smallmatrix} \nu \\ n \end{smallmatrix}\right] = \frac{\nu(\nu+1)\cdots(\nu+n-1)}{n!}$, $*$ is the convolution operator.

Definition 1.2 [29] Let μ be positive real number and m be the minimum positive integer which is greater than μ ($m-1 \leq \mu \leq m$). Define the μ-order difference of $x(n)$ as

$$\nabla^{\mu}x(n) = \nabla^{m}\nabla^{-(m-\mu)}x(n),$$

where ∇^{m} is the mth-order backward difference operator.

Definition 1.3 [29] Define the discrete Mittag-Leffler function $F_{\alpha,\beta}(\lambda,n)$ as

$$F_{\alpha,\beta}(\lambda,n) = \sum_{k=0}^{\infty} \lambda^{k} \begin{bmatrix} \alpha k+\beta \\ n \end{bmatrix} \quad \bigl(|\lambda|<1\bigr),$$

$$F_{\alpha}(\lambda,n) = F_{\alpha,1}(\lambda,n) = \sum_{k=0}^{\infty} \lambda^{n} \begin{bmatrix} \alpha k+1 \\ n \end{bmatrix} \quad \bigl(|\lambda|<1\bigr).$$

In [11], Cheng gave the discrete fractional summation Gronwall inequality.

Theorem 1.1 [29] *Suppose that* $\beta > 0$. *Let* u_n, a_n, *and* g_n *be nonnegative functions, where* g_n *is also monotone and nondecreasing and satisfies* $0 \leq g_n \leq M$ $(0 \leq M < 1)$, $0 \leq n \leq N$. *If*

$$u_n \leq a_n + g_n \nabla^{-\beta} u_n,$$

then

$$u_n \le a_n + \sum_{k=1}^{\infty} (g_n)^k \nabla^{-k\beta} a_n.$$

In [30], Zhang *et al.* have extended the result and given the following conclusion.

Theorem 1.2 [30] *Let $\beta > 0$, $p \ge r > 0$, u_n, a_n, g_n be nonnegative functions, where g_n is also monotone and nondecreasing and satisfies $0 \le g_n \le M$ $(0 \le M < 1)$, $0 \le n \le N$. If*

$$(u_n)^p \le a_n + g_n \nabla^{-\beta} (u_n)^r,$$

then

$$u_n \le \left\{ a_n + D_n + \sum_{k=1}^{\infty} (Cg_n)^k \nabla^{-k\beta} D_n \right\}^{\frac{1}{p}},$$

where

$$D_n = g_n \nabla^{-\beta} \left\{ \frac{r}{p} k^{\frac{r-p}{p}} a_n + \frac{p-r}{p} k^{\frac{r}{p}} \right\},$$
$$C = \frac{r}{p} k^{\frac{r-p}{p}}, \quad k > 1.$$

In order to prove our results, we need the following basic information.

Lemma 1.1 [29] *Let $\mu > 0$, $\nu > 0$, then $\nabla^{-\mu} \nabla^{-\nu} x(n) = \nabla^{-(u+v)} x(n)$.*

Lemma 1.2 [29] $\nabla^{-\beta} \left[{k \atop n} \right] = \left[{k+\beta \atop n} \right]$, $k, n, \beta > 0$.

2 Main results

Theorem 2.1 *Suppose that $\beta > 0$, u_n, a_n, g_n, h_n are nonnegative functions, g_n and h_n are also monotone and nondecreasing, $u_n \le c$, $g_n \le M_1$, $h_n \le M_2$ $(M_1 M_2 < 1, 0 \le n \le N)$. If*

$$u_n \le a_n + g_n \nabla^{-\beta} h_n u_n, \tag{2.1}$$

then

$$u_n \le a_n + \sum_{k=1}^{\infty} g_n^k h_n^k \nabla^{-k\beta} a_n. \tag{2.2}$$

Proof Let $Bu_n = g_n \nabla^{-\beta} h_n u_n$. We find that B is linear and nondecreasing under the conditions that u_n, g_n, h_n are nonnegative functions and g_n, h_n are also nondecreasing. Therefore, (2.1) turns into

$$u_n \le a_n + Bu_n.$$

Then we have

$$\begin{aligned} u_n &\le a_n + B(a_n + Bu_n) = a_n + Ba_n + B^2 u_n \le \cdots \\ &\le a_n + \sum_{k=1}^{m-1} B^k a_n + B^m u_n \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &Bu_n = g_n \nabla^{-\beta} h_n u_n \le c M_1 M_2 \begin{bmatrix} 1+\beta \\ n \end{bmatrix}, \\ &B^2 u_n = g_n \nabla^{-\beta} h_n \big(g_n \nabla^{-\beta} h_n u_n \big) \le c (M_1 M_2)^2 \begin{bmatrix} 1+2\beta \\ n \end{bmatrix}, \\ &\vdots \\ &B^m u_n = g_n^m h_n^m \nabla^{-m\beta} u_n \le c (M_1 M_2)^m \begin{bmatrix} 1+m\beta \\ n \end{bmatrix}. \end{aligned} \tag{2.4}$$

Noticing that $|M_1 M_2| < 1$, then

$$\sum_{m=1}^{\infty} (M_1 M_2)^m \begin{bmatrix} 1+m\beta \\ n \end{bmatrix} = F_{\beta,1}(M_1 M_2, n) - 1.$$

Hence we have

$$\lim_{m\to\infty} (M_1 M_2)^m \begin{bmatrix} 1+m\beta \\ n \end{bmatrix} = 0.$$

From (2.4) we get

$$\lim_{m\to\infty} B^m u_n \le 0.$$

Due to the fact that $B^m u_n \ge 0$, we have

$$\lim_{m\to\infty} B^m u_n = 0.$$

Taking the limit as $m \to \infty$ on both sides of (2.3) we get

$$u_n \le a_n + \sum_{k=1}^{\infty} B^k a_n \le a_n + \sum_{k=0}^{\infty} g_n^k h_n^k \nabla^{-k\beta} a_n.$$

This completes the proof of Theorem 2.1. □

Remark 2.1 If $h_n \equiv 1$, (2.2) becomes Theorem 2.2 of Chapter 3 in [29]:

$$u_n \le a_n + \sum_{k=1}^{\infty} g_n^k \nabla^{-k\beta} a_n.$$

Corollary 2.1 *With the conditions of Theorem* 2.1, *let* $g_n = a$, $0 < aM_2 < 1$. *If*

$$u_n \le a_n + a\nabla^{-\beta} h_n u_n,$$

then

$$u_n \le a_n + \sum_{k=1}^{\infty} a^k h_n^k \nabla^{-k\beta} a_n.$$

Corollary 2.2 *Suppose that the conditions of Corollary* 2.1 *hold*, a_n *is monotone increasing and* $0 < aM_2 < 1$. *If*

$$u_n \le a_n + a\nabla^{-\beta} h_n u_n,$$

then

$$u_n \le a_n F_{\beta,1}(aM_2, n).$$

Proof Under the conditions of Corollary 2.1 and a_n is monotone increasing, and we get

$$\begin{aligned} u_n &\le a_n + \sum_{k=1}^{\infty} a^k h_n^k \nabla^{-k\beta} a_n \le a_n + a_n \sum_{k=1}^{\infty} a^k h_n^k \nabla^{-k\beta} \begin{bmatrix} 1 \\ n \end{bmatrix} \\ &\le a_n \sum_{k=0}^{\infty} (aM_2)^k \begin{bmatrix} k\beta + 1 \\ n \end{bmatrix} = a_n F_{\beta,1}(aM_2, n) \quad (0 < aM_2 < 1). \end{aligned}$$

□

Corollary 2.3 *Suppose that the conditions of Theorem* 2.1 *hold, if* a_n *is monotone increasing, then from*

$$u_n \le a_n + g_n \nabla^{-\beta} h_n u_n,$$

we get

$$u_n \le a_n \sum_{k=0}^{\infty} (M_1 M_2)^k \nabla^{-k\beta} \begin{bmatrix} 1 \\ n \end{bmatrix} = a_n F_{\beta,1}(M_1 M_2, n).$$

Theorem 2.2 *Suppose that* $\beta > 0$, u_n, *and* a_n *are nonnegative functions*, $\varphi(t)$ *is monotone and nondecreasing*, $\varphi(t+s) \le \varphi(t) + \varphi(s)$, $\varphi(t) \le Lt$, $0 < L < 1$. *If*

$$u_n \le a_n + \nabla^{-\beta} \varphi(u_n), \tag{2.5}$$

then

$$u_n \le \sum_{k=0}^{\infty} \nabla^{-k\beta} \varphi^k(a_n). \tag{2.6}$$

Proof Let $Bu_n = \nabla^{-\beta}\varphi(u_n)$. Under the conditions that $\varphi(a+b) \leq \varphi(a) + \varphi(b)$ and $\varphi(t)$ is monotone and nondecreasing, we have

$$\begin{aligned}
&\varphi\bigl(\nabla^{-\beta}\varphi(u_n)\bigr) = \varphi\left(\sum_{r=0}^{n}\begin{bmatrix} \nu \\ n-r \end{bmatrix}\varphi\bigl(u(r)\bigr)\right) \leq \sum_{r=0}^{n}\begin{bmatrix} \nu \\ n-r \end{bmatrix}\varphi^2\bigl(u(r)\bigr) = \nabla^{-\beta}\varphi^2(u_n), \\
&\varphi^2\bigl(\nabla^{-\beta}\varphi(u_n)\bigr) \leq \nabla^{-\beta}\varphi^3(u_n), \\
&\vdots
\end{aligned}$$

then for all $n \in N_0^+$,

$$\begin{aligned}
u_n &\leq a_n + Bu_n = a_n + \nabla^{-\beta}\varphi(u_n) \\
&\leq a_n + \nabla^{-\beta}\varphi(a_n + Bu_n) = a_n + \nabla^{-\beta}\varphi\bigl(a_n + \nabla^{-\beta}\varphi(u_n)\bigr) \\
&\leq a_n + \nabla^{-\beta}\varphi(a_n) + \nabla^{-\beta}\varphi\bigl(\nabla^{-\beta}\varphi(u_n)\bigr) \\
&\leq a_n + \nabla^{-\beta}\varphi(a_n) + \nabla^{-2\beta}\varphi^2(u_n) \\
&\leq a_n + \nabla^{-\beta}\varphi(a_n) + \nabla^{-2\beta}\varphi^2\bigl(a_n + \nabla^{-\beta}\varphi(u_n)\bigr) \\
&\leq a_n + \nabla^{-\beta}\varphi(a_n) + \nabla^{-2\beta}\varphi^2(a_n) + \nabla^{-3\beta}\varphi^3(u_n) \\
&\leq \cdots \\
&\leq a_n + \nabla^{-\beta}\varphi(a_n) + \nabla^{-2\beta}\varphi^2(a_n) + \nabla^{-3\beta}\varphi^3(a_n) + \cdots \\
&\quad + \nabla^{-(m-1)\beta}\varphi^{m-1}(a_n) + \nabla^{-m\beta}\varphi^m(u_n),
\end{aligned}$$

that is,

$$u_n \leq \sum_{k=0}^{m-1}\nabla^{-k\beta}\varphi^k(a_n) + \nabla^{-m\beta}\varphi^m(u_n). \tag{2.7}$$

Noticing that $\varphi(u_n) \leq Lu_n$, $|L| < 1$, and u_n is bounded for $0 \leq n \leq N$, suppose that $0 < u_n \leq C$, then we get

$$\begin{aligned}
&\nabla^{-\beta}\varphi(u_n) \leq L\nabla^{-\beta}u_n \leq L\nabla^{-\beta}C = CL\begin{bmatrix} 1+\beta \\ n \end{bmatrix}, \\
&\nabla^{-2\beta}\varphi^2(u_n) \leq CL^2\begin{bmatrix} 1+2\beta \\ n \end{bmatrix}, \\
&\nabla^{-3\beta}\varphi^3(u_n) \leq CL^3\begin{bmatrix} 1+3\beta \\ n \end{bmatrix}, \\
&\vdots \\
&\nabla^{-m\beta}\varphi^m(u_n) \leq CL^m\begin{bmatrix} 1+m\beta \\ n \end{bmatrix}.
\end{aligned} \tag{2.8}$$

From the fact that the series

$$\sum_{m=0}^{\infty} L^m \begin{bmatrix} 1+m\beta \\ n \end{bmatrix}$$

converges to $F_{\beta,1}(L,n)$, $|L|<1$, we have

$$\lim_{m\to\infty} L^m \begin{bmatrix} 1+m\beta \\ n \end{bmatrix} = 0. \tag{2.9}$$

Due to the fact that $\nabla^{-m\beta}\varphi^m(u_n) \geq 0$, from (2.8) and (2.9) we have

$$\lim_{m\to\infty} \nabla^{-m\beta}\varphi^m(u_n) = 0,$$

taking the limit as $m \to \infty$ on both sides of (2.7), we have

$$u_n \leq \sum_{k=0}^{\infty} \nabla^{-k\beta}\varphi^k(a_n).$$

This completes the proof of Theorem 2.2. □

By choosing some particular functions $\varphi(t)$, we can get the corresponding results. For example, suppose that a_n, u_n, β are the same as in Theorem 2.2. Let $\varphi(t) = \frac{1}{2}\sin t$ $(0 \leq t \leq \frac{\pi}{2})$, $0 \leq u_n \leq \frac{\pi}{2}$, then $\varphi(t)$ satisfies $\varphi(t+s) \leq \varphi(t) + \varphi(s)$, $\varphi(t) \leq \frac{1}{2}t$. If

$$u_n \leq a_n + \frac{1}{2}\nabla^{-\beta}\sin u_n,$$

then

$$u_n \leq \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \nabla^{-k\beta}\sin^k a_n.$$

Furthermore, if $a_n \equiv 1$, then from $\sin t \leq t$, we have

$$u_n \leq \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \nabla^{-k\beta}1 \leq \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \begin{bmatrix} 1+k\beta \\ n \end{bmatrix}.$$

3 Applications

Consider the fractional difference equation

$$\begin{cases} \nabla^{\alpha} y(n) = f\bigl(n, y(n)\bigr), \\ \nabla^{\alpha-1} y(n)|_{n=-1} = \eta, \end{cases} \tag{3.1}$$

where $0<\alpha<1$, $0 \leq n \leq N < \infty$.

From Proposition 3.1 in Chapter 3 of [29], we know that the problem (3.1) is equivalent to the summation equation

$$y(n) = \eta \begin{bmatrix} \alpha \\ n \end{bmatrix} + \nabla^{-\alpha} f\bigl(n, y(n)\bigr). \tag{3.2}$$

Suppose that z_n is the solution of the initial value problem

$$\begin{cases} \nabla^{\alpha} z(n) = f\bigl(n, z(n)\bigr), \\ \nabla^{\alpha-1} z(n)|_{n=-1} = \widetilde{\eta}, \end{cases} \tag{3.3}$$

$0 < \alpha < 1$, $0 \le n \le N < \infty$, then (3.3) is equivalent to the summation equation

$$z(n) = \widetilde{\eta} \begin{bmatrix} \alpha \\ n \end{bmatrix} + \nabla^{-\alpha} f\bigl(n, z(n)\bigr). \tag{3.4}$$

Theorem 3.1 *Suppose that $f(n,t)$ satisfies the condition*

$$\bigl|f(n,z) - f(n,y)\bigr| \le h_n |z - y|, \tag{3.5}$$

where h_n is a monotone nondecreasing positive function, $|h_n| < 1$. Then the solutions of (3.1) *rely on the initial value continuously.*

Proof From (3.2), (3.4), and (3.5) we have

$$\begin{aligned} \bigl|z(n) - y(n)\bigr| &\le |\widetilde{\eta} - \eta| \begin{bmatrix} \alpha \\ n \end{bmatrix} + \nabla^{-\alpha} \bigl|f\bigl(n, z(n)\bigr) - f\bigl(n, y(n)\bigr)\bigr| \\ &\le |\widetilde{\eta} - \eta| \begin{bmatrix} \alpha \\ n \end{bmatrix} + \nabla^{-\alpha} h_n \bigl|z(n) - y(n)\bigr|. \end{aligned}$$

From Corollary 2.1, we have

$$\begin{aligned} \bigl|z(n) - y(n)\bigr| &\le |\widetilde{\eta} - \eta| \begin{bmatrix} \alpha \\ n \end{bmatrix} + \sum_{k=1}^{\infty} h_n^k \nabla^{-k\alpha} |\widetilde{\eta} - \eta| \begin{bmatrix} \alpha \\ n \end{bmatrix} \\ &= |\widetilde{\eta} - \eta| \sum_{k=0}^{\infty} h_n^k \begin{bmatrix} k\alpha + \alpha \\ n \end{bmatrix} = |\widetilde{\eta} - \eta| F_{\alpha,\alpha}(h_n, n), \quad |h_n| < 1. \end{aligned}$$

Then the solutions of (3.1) rely on the initial value continuously. □

Remark 3.1 The condition (3.5) generalizes the Lipschitz condition, then Theorem 3.1 is a promotion of Theorem 3.2 in Chapter 3 of [29].

Theorem 3.2 *Suppose that $f(n,t)$ satisfies the condition*

$$\bigl|f(n,z) - f(n,y)\bigr| \le \varphi\bigl(|z - y|\bigr), \tag{3.6}$$

where $\varphi(t)$ is the same as in Theorem 2.2. *Then the solutions of* (3.1) *rely on the initial value continuously.*

Proof From (3.2), (3.4), and (3.6) we have

$$\begin{aligned}\left|z(n)-y(n)\right| &\le |\widetilde{\eta}-\eta| \begin{bmatrix}\alpha\\ n\end{bmatrix} + \nabla^{-\alpha}\left|f\left(n,z(n)\right)-f\left(n,y(n)\right)\right| \\ &\le |\widetilde{\eta}-\eta| \begin{bmatrix}\alpha\\ n\end{bmatrix} + \nabla^{-\alpha}\varphi\left(\left|z(n)-y(n)\right|\right).\end{aligned}$$

By Theorem 2.2 we get

$$\begin{aligned}\left|z(n)-y(n)\right| &\le \sum_{k=0}^{\infty}\nabla^{-k\alpha}\varphi^{k}\left(|\widetilde{\eta}-\eta| \begin{bmatrix}\alpha\\ n\end{bmatrix}\right) \\ &\le |\widetilde{\eta}-\eta| \sum_{k=0}^{\infty} L^{k} \begin{bmatrix}k\alpha+\alpha\\ n\end{bmatrix} = |\widetilde{\eta}-\eta| F_{\alpha,\alpha}(L,n).\end{aligned}$$

Then the solutions of (3.1) rely on the initial value continuously. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
RX studied the generalized discrete fractional Gronwall inequalities and completed the corresponding proof. YZ obtained the results of Section 3: Applications. All authors read and approved the final manuscript.

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