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A new half-discrete Mulholland-type inequality with multi-parameters

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Abstract

By means of weight functions and Hermite-Hadamard's inequality, a new half-discrete Mulholland-type inequality with a best constant factor is given. A best extension with multi-parameters, some equivalent forms, the operator expressions as well as some particular cases are considered.

MSC: 26D15; 47A07

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1 Introduction

Assuming that $f,g \in L^2(\mathbf{R}_+)$, $\|f\| = \{\int_0^\infty f^2(x)\,dx\}^{\frac{1}{2}} > 0$, $\|g\| > 0$, we have the following Hilbert integral inequality (*cf.* [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \pi \|f\|\|g\|, \tag{1.1}$$

where the constant factor π is best possible. If $a = \{a_m\}_{m=1}^\infty, b = \{b_n\}_{n=1}^\infty \in l^2$, $\|a\| = \{\sum_{m=1}^\infty a_m^2\}^{\frac{1}{2}} > 0$, $\|b\| > 0$, then we have the following discrete Hilbert inequality:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \pi \|a\|\|b\|, \tag{1.2}$$

with the same best constant factor π. Inequalities (1.1) and (1.2) are important in analysis and its applications (*cf.* [2, 3]). On the other hand, we have the following Mulholland inequality with the same best constant factor π (*cf.* [1, 4]):

$$\sum_{m=2}^\infty \sum_{n=2}^\infty \frac{a_m b_n}{\ln mn} < \pi \left\{ \sum_{m=2}^\infty m a_m^2 \sum_{n=2}^\infty n b_n^2 \right\}^{\frac{1}{2}}. \tag{1.3}$$

In 1998, by introducing an independent parameter $\lambda \in (0,1]$, Yang [5] gave an extension of (1.1). Generalizing the results from [5], Yang [3] gave some extensions of (1.1) and (1.2) as follows: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda_1 + \lambda_2 = \lambda \in \mathbf{R}$, $k_\lambda(x,y)$ is a non-negative homogeneous function of degree $-\lambda$ satisfying

$$k(\lambda_1) = \int_0^\infty k_\lambda(t,1) t^{\lambda_1 - 1}\,dt \in \mathbf{R}_+,$$

$\phi(x) = x^{p(1-\lambda_1)-1}$, $\psi(x) = x^{q(1-\lambda_2)-1}$, $f(x), g(y) \geq 0$,

$$f \in L_{p,\phi}(\mathbf{R}_+) = \left\{ f \,\middle|\, \|f\|_{p,\phi} := \left\{ \int_0^\infty \phi(x) \left| f(x) \right|^p dx \right\}^{\frac{1}{p}} < \infty \right\},$$

$g \in L_{q,\psi}(\mathbf{R}_+)$, and $\|f\|_{p,\phi}, \|g\|_{q,\psi} > 0$, then

$$\int_0^\infty \int_0^\infty k_\lambda(x,y) f(x) g(y)\, dx\, dy < k(\lambda_1) \|f\|_{p,\phi} \|g\|_{q,\psi}, \tag{1.4}$$

where the constant factor $k(\lambda_1)$ is best possible. Moreover, if $k_\lambda(x,y)$ is finite and $k_\lambda(x,y)x^{\lambda_1-1}$ ($k_\lambda(x,y)y^{\lambda_2-1}$) is decreasing for $x > 0$ ($y > 0$), then for $a_m, b_n \geq 0$,

$$a = \{a_m\}_{m=1}^\infty \in l_{p,\phi} = \left\{ a \,\middle|\, \|a\|_{p,\phi} := \left\{ \sum_{m=1}^\infty \phi(m) |a_m|^p \right\}^{\frac{1}{p}} < \infty \right\},$$

and $b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}$, $\|a\|_{p,\phi}, \|b\|_{q,\psi} > 0$, we have

$$\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda(m,n) a_m b_n < k(\lambda_1) \|a\|_{p,\phi} \|b\|_{q,\psi}, \tag{1.5}$$

where the constant factor $k(\lambda_1)$ is still the best possible. Clearly, for $p = q = 2$, $\lambda = 1$, $k_1(x,y) = \frac{1}{x+y}$ and $\lambda_1 = \lambda_2 = \frac{1}{2}$, (1.4) reduces to (1.1), while (1.5) reduces to (1.2).

Some other results about Hilbert-type inequalities can be found in [6–13]. On half-discrete Hilbert-type inequalities with the general non-homogeneous kernels, Hardy *et al.* provided a few results in Theorem 351 of [1]. But they did not prove that the constant factors are best possible. In 2005, Yang [14] gave a result with the kernel $\frac{1}{(1+nx)^\lambda}$ by introducing a variable and proved that the constant factor is best possible. Recently, Wang and Yang [15] gave a more accurate reverse half-discrete Hilbert-type inequality, and Yang [16] provided the following half-discrete Hilbert inequality with best constant factor:

$$\int_0^\infty f(x) \sum_{n=1}^\infty \frac{a_n}{x+n}\, dx < \pi \|f\| \|a\|. \tag{1.6}$$

In this paper, by means of weight functions and Hermite-Hadamard's inequality, a new half-discrete Mulholland-type inequality similar to (1.3) and (1.6) with a best possible constant factor is given as follows:

$$\int_1^\infty f(x) \sum_{n=2}^\infty \frac{a_n}{1 + \ln x \ln n}\, dx < \pi \left\{ \int_1^\infty x f^2(x)\, dx \sum_{n=2}^\infty n a_n^2 \right\}^{\frac{1}{2}}. \tag{1.7}$$

Moreover, a best extension of (1.7) with multi-parameters, some equivalent forms, the operator expressions as well as some particular cases are considered.

2 Some lemmas

Lemma 2.1 *If* $0 < \sigma < \lambda$ $(\sigma \leq 1)$, $\alpha > 0$, $\beta \geq \frac{2}{3}$, $\delta \in \{-1,1\}$, *the weight functions* $\omega(n)$ *and* $\varpi(x)$ *are defined by*

$$\omega(n) := (\ln \beta n)^{\sigma} \int_{\frac{1}{\alpha}}^{\infty} \frac{(\ln \alpha x)^{\delta\sigma-1}}{x(1 + \ln^{\delta} \alpha x \ln \beta n)^{\lambda}} \, dx, \quad n \in \mathbf{N}\backslash\{1\}, \tag{2.1}$$

$$\varpi(x) := (\ln \alpha x)^{\delta\sigma} \sum_{n=2}^{\infty} \frac{(\ln \beta n)^{\sigma-1}}{n(1 + \ln^{\delta} \alpha x \ln \beta n)^{\lambda}}, \quad x \in \left(\frac{1}{\alpha}, \infty\right), \tag{2.2}$$

then we have

$$\varpi(x) < \omega(n) = B(\sigma, \lambda - \sigma). \tag{2.3}$$

Proof Substituting $t = \ln^{\delta} \alpha x \ln \beta n$ in (2.1), and by a simple calculation, for $\delta \in \{-1,1\}$, we have

$$\omega(n) = \int_{0}^{\infty} \frac{1}{(1+t)^{\lambda}} t^{\sigma-1} \, dt = B(\sigma, \lambda - \sigma).$$

For fixed $x > \frac{1}{\alpha}$, in view of the conditions, it is easy to find that

$$h(x,y) := \frac{(\ln \beta y)^{\sigma-1}}{y(1 + \ln^{\delta} \alpha x \ln \beta y)^{\lambda}} = \frac{1}{y(1 + \ln^{\delta} \alpha x \ln \beta y)^{\lambda} (\ln \beta y)^{1-\sigma}}$$

is decreasing and strictly convex with $h'_{y}(x,y) < 0$ and $h''_{y^2}(x,y) > 0$, for $y \in (\frac{3}{2}, \infty)$. Hence by the Hermite-Hadamard inequality (*cf.* [17]), we find

$$\begin{aligned} \varpi(x) &< (\ln \alpha x)^{\delta\sigma} \int_{\frac{3}{2}}^{\infty} \frac{1}{y(1 + \ln^{\delta} \alpha x \ln \beta y)^{\lambda} (\ln \beta y)^{1-\sigma}} \, dy \\ &\overset{t=\ln^{\delta} \alpha x \ln \beta y}{=} \int_{\ln^{\delta} \alpha x \ln(\frac{3}{2}\beta)}^{\infty} \frac{t^{\sigma-1}}{(1+t)^{\lambda}} \, dt \leq B(\sigma, \lambda - \sigma), \end{aligned}$$

and then (2.3) follows. □

Lemma 2.2 *Let the assumptions of Lemma* 2.1 *be fulfilled and, additionally, let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geq 0$, $n \in \mathbf{N}\backslash\{1\}$, $f(x)$ *is a non-negative measurable function in* $(\frac{1}{\alpha}, \infty)$. *Then we have the following inequalities*:

$$\begin{aligned} J &:= \left\{ \sum_{n=2}^{\infty} \frac{1}{n} (\ln \beta n)^{p\sigma-1} \left[\int_{\frac{1}{\alpha}}^{\infty} \frac{f(x)}{(1 + \ln^{\delta} \alpha x \ln \beta n)^{\lambda}} \, dx \right]^{p} \right\}^{\frac{1}{p}} \\ &\leq \left[B(\sigma, \lambda - \sigma) \right]^{\frac{1}{q}} \left\{ \int_{\frac{1}{\alpha}}^{\infty} \varpi(x) x^{p-1} (\ln \alpha x)^{p(1-\delta\sigma)-1} f^{p}(x) \, dx \right\}^{\frac{1}{p}}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} L_1 &:= \left\{ \int_{\frac{1}{\alpha}}^{\infty} \frac{(\ln \alpha x)^{q\delta\alpha-1}}{x[\varpi(x)]^{q-1}} \left[\sum_{n=2}^{\infty} \frac{a_n}{(1 + \ln^{\delta} \alpha x \ln \beta n)^{\lambda}} \right]^{q} dx \right\}^{\frac{1}{q}} \\ &\leq \left\{ B(\sigma, \lambda - \sigma) \sum_{n=2}^{\infty} n^{q-1} (\ln \beta n)^{q(1-\sigma)-1} a_n^{q} \right\}^{\frac{1}{q}}. \end{aligned} \tag{2.5}$$

Proof By Hölder's inequality (*cf.* [17]) and (2.3), it follows that

$$\begin{aligned}
&\left[\int_{\frac{1}{\alpha}}^{\infty} \frac{f(x)\,dx}{(1+\ln^{\delta}\alpha x \ln\beta n)^{\lambda}}\right]^{p} \\
&\quad = \left\{\int_{\frac{1}{\alpha}}^{\infty} \frac{1}{(1+\ln^{\delta}\alpha x \ln\beta n)^{\lambda}}\left[\frac{(\ln\alpha x)^{(1-\delta\sigma)/q}}{(\ln\beta n)^{(1-\sigma)/p}}\frac{x^{\frac{1}{q}}f(x)}{n^{\frac{1}{p}}}\right]\right. \\
&\qquad \times \left.\left[\frac{(\ln\beta n)^{(1-\sigma)/p}}{(\ln\alpha x)^{(1-\delta\sigma)/q}}\frac{n^{\frac{1}{p}}}{x^{\frac{1}{q}}}\right]dx\right\}^{p} \leq \int_{\frac{1}{\alpha}}^{\infty} \frac{x^{p-1}(\ln\alpha x)^{(1-\delta\sigma)(p-1)}}{(1+\ln^{\delta}\alpha x \ln\beta n)^{\lambda}}\frac{f^{p}(x)\,dx}{n(\ln\beta n)^{1-\sigma}} \\
&\qquad \times \left\{\int_{\frac{1}{\alpha}}^{\infty} \frac{n^{q-1}}{(1+\ln^{\delta}\alpha x \ln\beta n)^{\lambda}}\frac{(\ln\beta n)^{(1-\sigma)(q-1)}}{x(\ln\alpha x)^{1-\delta\sigma}}\,dx\right\}^{p-1} \\
&\quad = \left\{\omega(n)n^{q-1}(\ln\beta n)^{q(1-\sigma)-1}\right\}^{p-1}\int_{\frac{1}{\alpha}}^{\infty} \frac{x^{p-1}(\ln\alpha x)^{(1-\delta\sigma)(p-1)}}{(1+\ln^{\delta}\alpha x \ln\beta n)^{\lambda}}\frac{f^{p}(x)\,dx}{n(\ln\beta n)^{1-\sigma}} \\
&\quad = \left[B(\sigma,\lambda-\sigma)\right]^{p-1}n(\ln\beta n)^{1-p\sigma}\int_{\frac{1}{\alpha}}^{\infty} \frac{x^{p-1}(\ln\alpha x)^{(1-\delta\sigma)(p-1)}}{(1+\ln^{\delta}\alpha x \ln\beta n)^{\lambda}}\frac{f^{p}(x)\,dx}{n(\ln\beta n)^{1-\sigma}}.
\end{aligned}$$

Then by Lebesgue term-by-term integration theorem (*cf.* [18]), we have

$$\begin{aligned}
J &\leq \left[B(\sigma,\lambda-\sigma)\right]^{\frac{1}{q}}\left\{\sum_{n=2}^{\infty}\int_{\frac{1}{\alpha}}^{\infty} \frac{x^{p-1}(\ln\alpha x)^{(1-\delta\sigma)(p-1)}}{(1+\ln^{\delta}\alpha x \ln\beta n)^{\lambda}}\frac{f^{p}(x)\,dx}{n(\ln\beta n)^{1-\sigma}}\right\}^{\frac{1}{p}} \\
&= \left[B(\sigma,\lambda-\sigma)\right]^{\frac{1}{q}}\left\{\int_{\frac{1}{\alpha}}^{\infty}\sum_{n=2}^{\infty} \frac{x^{p-1}(\ln\alpha x)^{(1-\delta\sigma)(p-1)}}{(1+\ln^{\delta}\alpha x \ln\beta n)^{\lambda}}\frac{f^{p}(x)\,dx}{n(\ln\beta n)^{1-\sigma}}\right\}^{\frac{1}{p}} \\
&= \left[B(\sigma,\lambda-\sigma)\right]^{\frac{1}{q}}\left\{\int_{\frac{1}{\alpha}}^{\infty}\varpi(x)x^{p-1}(\ln\alpha x)^{p(1-\delta\sigma)-1}f^{p}(x)\,dx\right\}^{\frac{1}{p}},
\end{aligned}$$

hence, (2.4) follows.

By Hölder's inequality again, we have

$$\begin{aligned}
&\left[\sum_{n=2}^{\infty} \frac{a_n}{(1+\ln^{\delta}\alpha x \ln\beta n)^{\lambda}}\right]^{q} \\
&\quad = \left\{\sum_{n=2}^{\infty} \frac{1}{(1+\ln^{\delta}\alpha x \ln\beta n)^{\lambda}}\left[\frac{(\ln\alpha x)^{(1-\delta\sigma)/q}}{(\ln\beta n)^{(1-\sigma)/p}}\frac{x^{\frac{1}{q}}}{n^{\frac{1}{p}}}\right]\right. \\
&\qquad \times \left.\left[\frac{(\ln\beta n)^{(1-\sigma)/p}}{(\ln\alpha x)^{(1-\delta\sigma)/q}}\frac{n^{\frac{1}{p}}a_n}{x^{\frac{1}{q}}}\right]\right\}^{q} \leq \left\{\sum_{n=2}^{\infty} \frac{x^{p-1}(\ln\alpha x)^{(1-\delta\sigma)(p-1)}}{n(1+\ln^{\delta}\alpha x \ln\beta n)^{\lambda}(\ln\beta n)^{1-\sigma}}\right\}^{q-1} \\
&\qquad \times \sum_{n=2}^{\infty} \frac{n^{q-1}}{(1+\ln^{\delta}\alpha x \ln\beta n)^{\lambda}}\frac{(\ln\beta n)^{(1-\sigma)(q-1)}}{x(\ln\alpha x)^{1-\delta\sigma}}a_n^{q} \\
&\quad = \frac{x[\varpi(x)]^{q-1}}{(\ln\alpha x)^{q\delta\sigma-1}}\sum_{n=2}^{\infty} \frac{n^{q-1}}{x(1+\ln^{\delta}\alpha x \ln\beta n)^{\lambda}}(\ln\alpha x)^{\delta\sigma-1}(\ln\beta n)^{(1-\sigma)(q-1)}a_n^{q}.
\end{aligned}$$

By the Lebesgue term-by-term integration theorem, we have

$$\begin{aligned} L_1 &\leq \left\{ \int_{\frac{1}{\alpha}}^{\infty} \sum_{n=2}^{\infty} \frac{n^{q-1}}{x(1+\ln^{\delta}\alpha x \ln\beta n)^{\lambda}} (\ln\alpha x)^{\delta\sigma-1} (\ln\beta n)^{(1-\sigma)(q-1)} a_n^q \, dx \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_{n=2}^{\infty} \left[(\ln\beta n)^{\sigma} \int_{\frac{1}{\alpha}}^{\infty} \frac{(\ln\alpha x)^{\delta\sigma-1}\, dx}{x(1+\ln^{\delta}\alpha x \ln\beta n)^{\lambda}} \right] n^{q-1} (\ln\beta n)^{q(1-\sigma)-1} a_n^q \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_{n=2}^{\infty} \omega(n) n^{q-1} (\ln\beta n)^{q(1-\sigma)-1} a_n^q \right\}^{\frac{1}{q}}, \end{aligned}$$

and in view of (2.3), inequality (2.5) follows. □

3 Main results

We introduce the functions

$$\begin{aligned} &\Phi_{\delta}(x) := x^{p-1} (\ln\alpha x)^{p(1-\delta\sigma)-1} \quad \left(x > \frac{1}{\alpha} \right), \\ &\Psi(n) := n^{q-1} (\ln\beta n)^{q(1-\sigma)-1} \quad \bigl(n \in \mathbf{N}\backslash\{1\} \bigr), \end{aligned}$$

wherefrom $[\Phi_{\delta}(x)]^{1-q} = \frac{1}{x}(\ln\alpha x)^{q\delta\sigma-1}$, and $[\Psi(n)]^{1-p} = \frac{1}{n}(\ln\beta n)^{p\sigma-1}$.

Theorem 3.1 *If* $0 < \sigma < \lambda$ $(\sigma \leq 1)$, $\alpha > 0$, $\beta \geq \frac{2}{3}$, $\delta \in \{-1,1\}$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x)$, $a_n \geq 0$, $f \in L_{p,\Phi}(\frac{1}{\alpha}, \infty)$, $a = \{a_n\}_{n=2}^{\infty} \in l_{q,\Psi}$, $\|f\|_{p,\Phi_{\delta}} > 0$, *and* $\|a\|_{q,\Psi} > 0$, *then we have the following equivalent inequalities*:

$$\begin{aligned} I &:= \sum_{n=2}^{\infty} a_n \int_{\frac{1}{\alpha}}^{\infty} \frac{f(x)\, dx}{(1+\ln^{\delta}\alpha x \ln\beta n)^{\lambda}} \\ &= \int_{\frac{1}{\alpha}}^{\infty} f(x) \sum_{n=2}^{\infty} \frac{a_n\, dx}{(1+\ln^{\delta}\alpha x \ln\beta n)^{\lambda}} < B(\sigma, \lambda-\sigma) \|f\|_{p,\Phi_{\delta}} \|a\|_{q,\Psi}, \end{aligned} \tag{3.1}$$

$$J = \left\{ \sum_{n=2}^{\infty} \bigl[\Psi(n)\bigr]^{1-p} \left[\int_{\frac{1}{\alpha}}^{\infty} \frac{f(x)\, dx}{(1+\ln^{\delta}\alpha x \ln\beta n)^{\lambda}} \right]^p \right\}^{\frac{1}{p}} < B(\sigma, \lambda-\sigma) \|f\|_{p,\Phi_{\delta}}, \tag{3.2}$$

$$L := \left\{ \int_{\frac{1}{\alpha}}^{\infty} \bigl[\Phi_{\delta}(x)\bigr]^{1-q} \left[\sum_{n=2}^{\infty} \frac{a_n}{(1+\ln^{\delta}\alpha x \ln\beta n)^{\lambda}} \right]^q dx \right\}^{\frac{1}{q}} < B(\sigma, \lambda-\sigma) \|a\|_{q,\Psi}, \tag{3.3}$$

where the constant $B(\sigma, \lambda-\sigma)$ *is the best possible in the above inequalities.*

Proof The two expressions for I in (3.1) follow from Lebesgue's term-by-term integration theorem. By (2.4) and (2.3), we have (3.2). By Hölder's inequality, we have

$$I = \sum_{n=2}^{\infty} \left[\Psi^{\frac{-1}{q}}(n) \int_{\frac{1}{\alpha}}^{\infty} \frac{f(x)\, dx}{(1+\ln^{\delta}\alpha x \ln\beta n)^{\lambda}} \right] \bigl[\Psi^{\frac{1}{q}}(n) a_n \bigr] \leq J \|a\|_{q,\Psi}. \tag{3.4}$$

Then by (3.2), we have (3.1).

On the other hand, assuming that (3.1) is valid, we set

$$a_n := \left[\Psi(n)\right]^{1-p}\left[\int_{\frac{1}{\alpha}}^{\infty} \frac{f(x)\,dx}{(1+\ln^{\delta}\alpha x\ln\beta n)^{\lambda}}\right]^{p-1}, \quad n \in \mathbf{N}\backslash\{1\}.$$

It follows that $J^{p-1} = \|a\|_{q,\Psi}$. By (2.4), we find $J < \infty$. If $J = 0$, then (3.2) is trivially valid; if $J > 0$, then by (3.1), we have

$$\|a\|_{q,\Psi}^{q} = J^{q(p-1)} = J^{p} = I < B(\sigma,\lambda-\sigma)\|f\|_{p,\Phi_\delta}\|a\|_{q,\Psi},$$

namely, $\|a\|_{q,\Psi}^{q-1} = J < B(\sigma,\lambda-\sigma)\|f\|_{p,\Phi_\delta}$. That is, (3.2) is equivalent to (3.1).

By (2.3) we have $[\varpi(x)]^{1-q} > [B(\sigma,\lambda-\sigma)]^{1-q}$. Then in view of (2.5), we have (3.3). By Hölder's inequality, we find

$$I = \int_{\frac{1}{\alpha}}^{\infty}\left[\Phi_\delta^{\frac{1}{p}}(x)f(x)\right]\left[\Phi_\delta^{\frac{-1}{p}}(x)\sum_{n=2}^{\infty}\frac{a_n}{(1+\ln^{\delta}\alpha x\ln\beta n)^{\lambda}}\right]dx \le \|f\|_{p,\Phi_\delta}L. \tag{3.5}$$

Then by (3.3), we have (3.1).

On the other hand, assume that (3.1) is valid. Setting

$$f(x) := \left[\Phi_\delta(x)\right]^{1-q}\left[\sum_{n=2}^{\infty}\frac{a_n}{(1+\ln^{\delta}\alpha x\ln\beta n)^{\lambda}}\right]^{q-1}, \quad x \in \left(\frac{1}{\alpha},\infty\right),$$

then $L^{q-1} = \|f\|_{p,\Phi_\delta}$. By (2.5), we find $L < \infty$. If $L = 0$, then (3.3) is trivially valid; if $L > 0$, then by (3.1), we have

$$\|f\|_{p,\Phi_\delta}^{p} = L^{p(q-1)} = L^{q} = I < B(\sigma,\lambda-\sigma)\|f\|_{p,\Phi_\delta}\|a\|_{q,\Psi},$$

therefore $\|f\|_{p,\Phi_\delta}^{p-1} = L < B(\sigma,\lambda-\sigma)\|a\|_{q,\Psi}$, that is, (3.3) is equivalent to (3.1). Hence, inequalities (3.1), (3.2), and (3.3) are equivalent.

For $0 < \varepsilon < p(\lambda-\sigma)$, setting $E_\delta := \{x; x > \frac{1}{\alpha}, \ln^{\delta}\alpha x \in (0,1)\}$,

$$\widetilde{f}(x) = \frac{1}{x}(\ln\alpha x)^{\delta(\sigma+\frac{\varepsilon}{p})-1}, \quad x \in E_\delta; \qquad \widetilde{f}(x) = 0, \quad \left\{x; x > \frac{1}{\alpha}\right\}\backslash E_\delta,$$

and $\widetilde{a}_n = \frac{1}{n}(\ln\beta n)^{\sigma-\frac{\varepsilon}{q}-1}$, $n \in \mathbf{N}\backslash\{1\}$, if there exists a positive number k ($\le B(\sigma,\lambda-\sigma)$), such that (3.1) is valid when replacing $B(\sigma,\lambda-\sigma)$ with k, then in particular, for $\delta = \pm 1$, setting $u = \ln^{\delta}\alpha x$, it follows that

$$\int_{E_\delta}\frac{dx}{x(\ln\alpha x)^{-\delta\varepsilon+1}} = \int_0^1 \frac{|\delta|u^{\delta-1}}{u^{-\varepsilon+\delta}}\,du = \frac{1}{\varepsilon},$$

$$\widetilde{I} := \sum_{n=2}^{\infty}\int_{\frac{1}{\alpha}}^{\infty}\frac{1}{(1+\ln^{\delta}\alpha x\ln\beta n)^{\lambda}}\widetilde{a}_n\widetilde{f}(x)\,dx < k\|\widetilde{f}\|_{p,\Phi_\delta}\|\widetilde{a}\|_{q,\Psi}$$

$$= k\left\{\int_{E_\delta}\frac{dx}{x(\ln\alpha x)^{-\delta\varepsilon+1}}\right\}^{\frac{1}{p}}\left\{\frac{1}{2(\ln 2\beta)^{\varepsilon+1}} + \sum_{n=3}^{\infty}\frac{1}{n(\ln\beta n)^{\varepsilon+1}}\right\}^{\frac{1}{q}}$$

$$\begin{aligned}
&< k\left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}\left\{\frac{1}{2(\ln 2\beta)^{\varepsilon+1}} + \int_2^\infty \frac{1}{x(\ln \beta x)^{\varepsilon+1}}\,dx\right\}^{\frac{1}{q}} \\
&= \frac{k}{\varepsilon}\left\{\frac{\varepsilon}{2(\ln 2\beta)^{\varepsilon+1}} + \frac{1}{(\ln 2\beta)^{\varepsilon}}\right\}^{\frac{1}{q}},
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
\widetilde{I} &= \sum_{n=2}^\infty \frac{1}{n}(\ln \beta n)^{\sigma-\frac{\varepsilon}{q}-1}\int_{E_\delta} \frac{(\ln \alpha x)^{\delta(\sigma+\frac{\varepsilon}{p})-1}}{x(1+\ln^\delta \alpha x \ln \beta n)^\lambda}\,dx \\
&\overset{t=\ln^\delta \alpha x \ln \beta n}{=} \sum_{n=2}^\infty \frac{1}{n(\ln \beta n)^{\varepsilon+1}}\int_0^{\ln \beta n} \frac{1}{(1+t)^\lambda} t^{\sigma+\frac{\varepsilon}{p}-1}\,dt \\
&= B\left(\sigma+\frac{\varepsilon}{p}, \lambda-\sigma-\frac{\varepsilon}{p}\right)\sum_{n=2}^\infty \frac{1}{n(\ln \beta n)^{\varepsilon+1}} - A(\varepsilon) \\
&> B\left(\sigma+\frac{\varepsilon}{p}, \lambda-\sigma-\frac{\varepsilon}{p}\right)\int_2^\infty \frac{1}{y(\ln \beta y)^{\varepsilon+1}}\,dy - A(\varepsilon) \\
&= \frac{1}{\varepsilon(\ln 2\beta)^\varepsilon} B\left(\sigma+\frac{\varepsilon}{p}, \lambda-\sigma-\frac{\varepsilon}{p}\right) - A(\varepsilon), \\
A(\varepsilon) &:= \sum_{n=2}^\infty \frac{1}{n(\ln \beta n)^{\varepsilon+1}}\int_{\ln \beta n}^\infty \frac{1}{(t+1)^\lambda} t^{\sigma+\frac{\varepsilon}{p}-1}\,dt.
\end{aligned} \tag{3.7}$$

We find

$$\begin{aligned}
0 < A(\varepsilon) &\le \sum_{n=2}^\infty \frac{1}{n(\ln \beta n)^{\varepsilon+1}}\int_{\ln \beta n}^\infty \frac{1}{t^\lambda} t^{\sigma+\frac{\varepsilon}{p}-1}\,dt \\
&= \frac{1}{\frac{\lambda}{2}-\frac{\varepsilon}{p}}\sum_{n=2}^\infty \frac{1}{n(\ln \beta n)^{\sigma+\frac{\varepsilon}{q}+1}} < \infty,
\end{aligned}$$

and so $A(\varepsilon) = O(1)(\varepsilon \to 0^+)$. Hence by (3.6) and (3.7), it follows that

$$\frac{1}{(\ln 2\beta)^\varepsilon} B\left(\sigma+\frac{\varepsilon}{p}, \lambda-\sigma-\frac{\varepsilon}{p}\right) - \varepsilon O(1) < k\left\{\frac{\varepsilon}{2(\ln 2\beta)^{\varepsilon+1}} + \frac{1}{(\ln 2\beta)^\varepsilon}\right\}^{\frac{1}{q}},$$

and $B(\sigma, \lambda-\sigma) \le k(\varepsilon \to 0^+)$. Hence $k = B(\sigma, \lambda-\sigma)$ is the best value of (3.1).

By the equivalence of the inequalities, the constant factor $B(\sigma, \lambda-\sigma)$ in (3.2) ((3.3)) is the best possible. Otherwise, we would reach the contradiction by (3.4) ((3.5)) that the constant factor in (3.1) is not the best possible. □

Remark 3.2 (i) Define the first type half-discrete Hilbert-type operator $T_1 : L_{p,\Phi_\delta}(\frac{1}{\alpha}, \infty) \to l_{p,\Psi^{1-p}}$ as follows: For $f \in L_{p,\Phi_\delta}(\frac{1}{\alpha}, \infty)$, we define $T_1 f \in l_{p,\Psi^{1-p}}$ by

$$T_1 f(n) = \int_{\frac{1}{\alpha}}^\infty \frac{1}{(1+\ln^\delta \alpha x \ln \beta n)^\lambda} f(x)\,dx, \quad n \in \mathbf{N}\backslash\{1\}.$$

Then by (3.2), $\|T_1 f\|_{p,\Psi^{1-p}} \le B(\sigma, \lambda-\sigma)\|f\|_{p,\Phi_\delta}$ and so T_1 is a bounded operator with $\|T_1\| \le B(\sigma, \lambda-\sigma)$. Since by Theorem 3.1, the constant factor in (3.2) is best possible, we have $\|T_1\| = B(\sigma, \lambda-\sigma)$.

(ii) Define the second type half-discrete Hilbert-type operator $T_2 : l_{q,\Psi} \to L_{q,\Phi_\delta^{1-q}}(\frac{1}{\alpha},\infty)$ as follows: For $a \in l_{q,\Psi}$, we define $T_2 a \in L_{q,\Phi_\delta^{1-q}}(\frac{1}{\alpha},\infty)$ by

$$T_2 a(x) = \sum_{n=2}^{\infty} \frac{1}{(1+\ln^\delta \alpha x \ln \beta n)^\lambda} a_n, \quad x \in \left(\frac{1}{\alpha},\infty\right).$$

Then by (3.3), $\|T_2 a\|_{q,\Phi_\delta^{1-q}} \leq B(\sigma,\lambda-\sigma)\|a\|_{q,\Psi}$ and so T_2 is a bounded operator with $\|T_2\| \leq B(\sigma,\lambda-\sigma)$. Since by Theorem 3.1, the constant factor in (3.3) is best possible, we have $\|T_2\| = B(\sigma,\lambda-\sigma)$.

Remark 3.3 For $p = q = 2$, $\lambda = 1$, $\sigma = \frac{1}{2}$, $\delta = 1$ in (3.1), (3.2), and (3.3), (i) if $\alpha = \beta = 1$, then we have (1.7) and the following equivalent inequalities:

$$\sum_{n=2}^{\infty} \frac{1}{n}\left(\int_1^\infty \frac{f(x)}{1+\ln x \ln n}\,dx\right)^2 < \pi^2 \int_1^\infty x f^2(x)\,dx, \tag{3.8}$$

$$\int_1^\infty \frac{1}{x}\left(\sum_{n=2}^{\infty} \frac{a_n}{1+\ln x \ln n}\right)^2 dx < \pi^2 \sum_{n=2}^{\infty} n a_n^2; \tag{3.9}$$

(ii) if $\alpha = \beta = \frac{2}{3}$, then we have the following equivalent inequalities:

$$\int_{\frac{3}{2}}^\infty \sum_{n=2}^{\infty} \frac{a_n f(x)\,dx}{1+\ln \frac{2}{3}x \ln \frac{2}{3}n} < \pi \left\{\int_{\frac{3}{2}}^\infty x f^2(x)\,dx \sum_{n=2}^{\infty} n a_n^2\right\}^{\frac{1}{2}}, \tag{3.10}$$

$$\sum_{n=2}^{\infty} \frac{1}{n}\left(\int_{\frac{3}{2}}^\infty \frac{f(x)}{1+\ln \frac{2}{3}x \ln \frac{2}{3}n}\,dx\right)^2 < \pi^2 \int_{\frac{3}{2}}^\infty x f^2(x)\,dx, \tag{3.11}$$

$$\int_{\frac{3}{2}}^\infty \frac{1}{x}\left(\sum_{n=2}^{\infty} \frac{a_n}{1+\ln \frac{2}{3}x \ln \frac{2}{3}n}\right)^2 dx < \pi^2 \sum_{n=2}^{\infty} n a_n^2. \tag{3.12}$$

Remark 3.4 For $\delta = -1$ in (3.1), (3.2), and (3.3), setting $F(x) = \ln^\lambda(\alpha x) f(x)$, $\mu = \lambda - \sigma$ (> 0), and $\Phi(x) := x^{p-1}(\ln \alpha x)^{p(1-\mu)-1}$, we have the following new equivalent inequalities with the same best possible constant factor $B(\sigma,\mu)$:

$$\begin{aligned}\sum_{n=2}^{\infty} a_n \int_{\frac{1}{\alpha}}^\infty \frac{F(x)\,dx}{\ln^\lambda(\alpha\beta n x)} &= \int_{\frac{1}{\alpha}}^\infty F(x) \sum_{n=2}^{\infty} \frac{a_n}{\ln^\lambda(\alpha\beta n x)}\,dx \\ &< B(\sigma,\mu)\|F\|_{p,\Phi}\|a\|_{q,\Psi},\end{aligned} \tag{3.13}$$

$$\left\{\sum_{n=2}^{\infty} \left[\Psi(n)\right]^{1-p} \left[\int_{\frac{1}{\alpha}}^\infty \frac{F(x)\,dx}{\ln^\lambda(\alpha\beta n x)}\right]^p\right\}^{\frac{1}{p}} < B(\sigma,\mu)\|F\|_{p,\Phi}, \tag{3.14}$$

$$\left\{\int_{\frac{1}{\alpha}}^\infty \left[\Phi(x)\right]^{1-q} \left[\sum_{n=2}^{\infty} \frac{a_n}{\ln^\lambda(\alpha\beta n x)}\right]^q dx\right\}^{\frac{1}{q}} < B(\sigma,\mu)\|a\|_{q,\Psi}. \tag{3.15}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. QH participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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