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Multi-C^*-ternary algebras and applications

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Abstract

In this paper, we look at the concept of multi-C^*-ternary algebras and consider some properties. As an application we approximate multi-C^*-ternary algebra homomorphisms and derivations in these spaces.

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1 Introduction and preliminaries

Ternary algebraic structures arise naturally in theoretical and mathematical physics, for example, the quark model inspired a particular brand of ternary algebraic system. We also refer the reader to 'Nambu mechanics' [1] (see also [2, 3] and [4]).

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x,y,z)\mapsto[x,y,z]$ of A^3 into A, which is $\mathbf{C}$-linear in the outer variables, conjugate $\mathbf{C}$-linear in the middle variable, and associative in the sense that $[x,y,[z,w,v]]=[x,[w,z,y],v]=[[x,y,z],w,v]$, and satisfies $\|[x,y,z]\|\le\|x\|\cdot\|y\|\cdot\|z\|$ and $\|[x,x,x]\|=\|x\|^3$ (see [4]).

If a C^*-ternary algebra $(A,[\cdot,\cdot,\cdot])$ has the identity, *i.e.*, the element $e\in A$ such that $x=[x,e,e]=[e,e,x]$ for all $x\in A$, then it is routine to verify that A, endowed with $x\circ y:=[x,e,y]$ and $x^*:=[e,x,e]$, is a unital C^*-algebra. Conversely, if $(A,\circ)$ is a unital C^*-algebra, then $[x,y,z]:=x\circ y^*\circ z$ makes A into a C^*-ternary algebra.

A $\mathbf{C}$-linear mapping $H:A\to B$ is called a C^**-ternary algebra homomorphism* if

$$H([x,y,z])=[H(x),H(y),H(z)]$$

for all $x,y,z\in A$. A $\mathbf{C}$-linear mapping $\delta:A\to A$ is called a C^**-ternary derivation* if

$$\delta([x,y,z])=[\delta(x),y,z]+[x,\delta(y),z]+[x,y,\delta(z)]$$

for all $x,y,z\in A$ (see [5]).

Ternary structures and their generalization, the so-called n-ary structures, are important in view of their applications in physics (see [6]).

Let X be a set. A function $d:X\times X\to[0,\infty]$ is called a *generalized metric* on X if d satisfies the following conditions:

(1) $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(3) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Theorem 1.1 ([7]) *Let (X,d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for each $x \in X$, either*

$$d\big(J^n x, J^{n+1} x\big) = \infty$$

for all non-negative integers n or there exists a positive integer n_0 such that
(1) $d(J^n x, J^{n+1} x) < \infty$ *for all* $n \geq n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

2 Multi-normed spaces

The notion of a multi-normed space was introduced by Dales and Polyakov in [8] and many examples are given in [8–10].

Let $(\mathcal{E}, \|\cdot\|)$ be a complex normed space and let $k \in \mathbf{N}$. We denote by $\mathcal{E}^k$ the linear space $\mathcal{E} \oplus \cdots \oplus \mathcal{E}$ consisting of k-tuples $(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k \in \mathcal{E}$. The linear operations on $\mathcal{E}^k$ are defined coordinate-wise. The zero element of either $\mathcal{E}$ or $\mathcal{E}^k$ is denoted by 0. We denote by $\mathbf{N}_k$ the set $\{1, 2, \ldots, k\}$ and by Σ_k the group of permutations on k symbols.

Definition 2.1 A *multi-norm* on $\{\mathcal{E}^k : k \in \mathbf{N}\}$ is a sequence

$$\big(\|\cdot\|_k\big) = \big(\|\cdot\|_k : k \in \mathbf{N}\big)$$

such that $\|\cdot\|_k$ is a norm on $\mathcal{E}^k$ for each $k \in \mathbf{N}$ with $k \geq 2$:
(A1) $\|(x_{\sigma(1)}, \ldots, x_{\sigma(k)})\|_k = \|(x_1, \ldots, x_k)\|_k$ for any $\sigma \in \Sigma_k$ and $x_1, \ldots, x_k \in \mathcal{E}$;
(A2) $\|(\alpha_1 x_1, \ldots, \alpha_k x_k)\|_k \leq (\max_{i \in \mathbf{N}_k} |\alpha_i|)\|(x_1, \ldots, x_k)\|_k$ for any $\alpha_1, \ldots, \alpha_k \in \mathbf{C}$ and $x_1, \ldots, x_k \in \mathcal{E}$;
(A3) $\|(x_1, \ldots, x_{k-1}, 0)\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1}$ for any $x_1, \ldots, x_{k-1} \in \mathcal{E}$;
(A4) $\|(x_1, \ldots, x_{k-1}, x_{k-1})\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1}$ for any $x_1, \ldots, x_{k-1} \in \mathcal{E}$.
In this case, we say that $((\mathcal{E}^k, \|\cdot\|_k) : k \in \mathbf{N})$ is a *multi-normed space*.

Lemma 2.2 ([10]) *Suppose that $((\mathcal{E}^k, \|\cdot\|_k) : k \in \mathbf{N})$ is a multi-normed space and let $k \in \mathbf{N}$. Then*
(1) $\|(x, \ldots, x)\|_k = \|x\|$ *for any* $x \in \mathcal{E}$;
(2) $\max_{i \in \mathbf{N}_k} \|x_i\| \leq \|x_1, \ldots, x_k\|_k \leq \sum_{i=1}^k \|x_i\| \leq k \max_{i \in \mathbf{N}_k} \|x_i\|$ *for any* $x_1, \ldots, x_k \in \mathcal{E}$.

It follows from (2) that, if $(\mathcal{E}, \|\cdot\|)$ is a Banach space, then $(\mathcal{E}^k, \|\cdot\|_k)$ is a Banach space for each $k \in \mathbf{N}$. In this case, $((\mathcal{E}^k, \|\cdot\|_k) : k \in \mathbf{N})$ is a multi-Banach space.

Now, we present two examples (see [8]).

Example 2.3 The sequence $(\|\cdot\|_k : k \in \mathbf{N})$ on $\{\mathcal{E}^k : k \in \mathbf{N}\}$ defined by

$$\big\|(x_1, \ldots, x_k)\big\|_k := \max_{i \in \mathbf{N}_k} \|x_i\|$$

for any $x_1, \ldots, x_k \in \mathcal{E}$ is a multi-norm, which is called the *minimum multi-norm*.

Example 2.4 Let $\{(\|\cdot\|_k^\alpha : k \in \mathbf{N}) : \alpha \in A\}$ be the (non-empty) family of all multi-norms on $\{\mathcal{E}^k : k \in \mathbf{N}\}$. For each $k \in \mathbf{N}$, set

$$\|(x_1,\dots,x_k)\|_k := \sup_{\alpha \in A} \|(x_1,\dots,x_k)\|_k^\alpha$$

for any $x_1,\dots,x_k \in \mathcal{E}$. Then $(\|\cdot\|_k : k \in \mathbf{N})$ is a multi-norm on $\{\mathcal{E}^k : k \in \mathbf{N}\}$, which is called the *maximum multi-norm*.

Now, we need the following observation which can easily be deduced from Lemma 2.2(2) of multi-norms.

Lemma 2.5 *Suppose that* $k \in \mathbf{N}$ *and* $(x_1,\dots,x_k) \in \mathcal{E}^k$. *For each* $j \in \{1,\dots,k\}$, *let* (x_n^j) *be a sequence in* $\mathcal{E}$ *such that* $\lim_{n\to\infty} x_n^j = x_j$. *Then, for each* $(y_1,\dots,y_k) \in \mathcal{E}^k$,

$$\lim_{n\to\infty} (x_n^1 - y_1,\dots,x_n^k - y_k) = (x_1 - y_1,\dots,x_k - y_k).$$

Definition 2.6 Let $((\mathcal{E}^k, \|\cdot\|_k) : k \in \mathbf{N})$ be a multi-normed space. A sequence (x_n) in $\mathcal{E}$ is a *multi-null sequence* if, for each $\epsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that

$$\sup_{k\in\mathbf{N}} \|(x_n,\dots,x_{n+k-1})\|_k < \epsilon$$

for any $n \geq n_0$. Let $x \in \mathcal{E}$. We say that the sequence (x_n) is *multi-convergent* to $x \in \mathcal{E}$ and write

$$\lim_{n\to\infty} x_n = x$$

if $(x_n - x)$ is a multi-null sequence.

Definition 2.7 ([8, 11]) Let $(A, \|\cdot\|)$ be a normed algebra such that $((A^k, \|\cdot\|_k) : k \in \mathbf{N})$ is a multi-normed space. Then $((A^k, \|\cdot\|_k) : k \in \mathbf{N})$ is called a *multi-normed algebra* if

$$\|(a_1b_1,\dots,a_kb_k)\|_k \leq \|(a_1,\dots,a_k)\|_k \cdot \|(b_1,\dots,b_k)\|_k$$

for all $k \in \mathbf{N}$ and $a_1,\dots,a_k,b_1,\dots,b_k \in A$. Further, the multi-normed algebra $((A^k, \|\cdot\|_k) : k \in \mathbf{N})$ is a *multi-Banach algebra* if $((A^k, \|\cdot\|_k) : k \in \mathbf{N})$ is a multi-Banach space.

Example 2.8 ([8, 11]) Let p, q with $1 \leq p \leq q < \infty$ and let $A = \ell^p$. The algebra A is a Banach sequence algebra with respect to a coordinate-wise multiplication of sequences (see [12]). Let $(\|\cdot\|_k : k \in \mathbf{N})$ be the standard (p,q)-multi-norm on $\{A^k : k \in \mathbf{N}\}$. Then $((A^k, \|\cdot\|_k) : k \in \mathbf{N})$ is a multi-Banach algebra.

Definition 2.9 Let $((A^k, \|\cdot\|_k) : k \in \mathbf{N})$ be a multi-Banach algebra. A *multi-C^*-algebra* is a complex multi-Banach algebra $((A^k, \|\cdot\|_k) : k \in \mathbf{N})$ with an involution $*$ satisfying

$$\|(a_1^*a_1,\dots,a_k^*a_k)\|_k = \|(a_1,\dots,a_k)\|_k^2$$

for all $k \in \mathbf{N}$ and $a_1,\dots,a_k \in A$.

Definition 2.10 Let $((A^k, \|\cdot\|_k) : k \in \mathbf{N})$ be a multi-Banach space. A *multi-C^*-ternary algebra* is a complex multi-Banach space $((A^k, \|\cdot\|_k) : k \in \mathbf{N})$ equipped with a ternary product.

3 Approximation of homomorphisms in multi-Banach algebras

Throughout this paper, assume that A, B are C^*-ternary algebras.

For a given mapping $f : A \to B$, we define

$$C_\mu f(x_1, \ldots, x_p, y_1, \ldots, y_d) := 2f\left(\frac{\sum_{j=1}^p \mu x_j}{2} + \sum_{j=1}^d \mu y_j\right) - \sum_{j=1}^p \mu f(x_j) - 2\sum_{j=1}^d \mu f(y_j)$$

for all $\mu \in \mathbf{T}^1 := \{\lambda \in \mathbf{C} : |\lambda| = 1\}$ and $x_1, \ldots, x_p, y_1, \ldots, y_d \in A$.

One can easily show that a mapping $f : A \to B$ satisfies

$$C_\mu f(x_1, \ldots, x_p, y_1, \ldots, y_d) = 0$$

for all $\mu \in \mathbf{T}^1$ and all $x_1, \ldots, x_p, y_1, \ldots, y_d \in A$ if and only if

$$f(\mu x + \lambda y) = \mu f(x) + \lambda f(y)$$

for all $\mu, \lambda \in \mathbf{T}^1$ and $x, y \in A$.

Lemma 3.1 ([13]) *Let $f : A \to B$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in A$ and $\mu \in \mathbf{T}^1$. Then the mapping f is $\mathbf{C}$-linear.*

Lemma 3.2 *Let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be the convergent sequences in A. Then the sequence $\{[x_n, y_n, z_n]\}$ is convergent in A.*

Proof Let $x, y, z \in A$ be such that

$$\lim_{n\to\infty} x_n = x, \qquad \lim_{n\to\infty} y_n = y, \qquad \lim_{n\to\infty} z_n = z.$$

Since

$$\begin{aligned} &[x_n, y_n, z_n] - [x, y, z] \\ &\quad = [x_n - x, y_n - y, z_n, z] + [x_n, y_n, z] + [x, y_n - y, z_n] + [x_n, y, z_n - z] \end{aligned}$$

for all $n \geq 1$, we get

$$\begin{aligned} \big\| [x_n, y_n, z_n] - [x, y, z] \big\| &= \|x_n - x\| \|y_n - y\| \|z_n - z\| + \|x_n - x\| \|y_n\| \|z\| \\ &\quad + \|x\| \|y_n - y\| \|z_n\| + \|x_n\| \|y\| \|z_n - z\| \end{aligned}$$

for all $n \geq 1$, and so

$$\lim_{n\to\infty} [x_n, y_n, z_n] = [x, y, z].$$

This completes the proof. □

Using Theorem 1.1, we approximate homomorphisms in multi-C^*-ternary algebras for the functional equation $C_\mu f(x_1,\ldots,x_m) = 0$.

Theorem 3.3 *Let $((B^k,\|\cdot\|_k): k\in\mathbf{N})$ be a multi-C^*-ternary algebra. Let $f: A\to B$ be a mapping for which there are functions $\varphi: A^{(p+d)k}\to[0,\infty)$ and $\psi: A^{3k}\to[0,\infty)$ such that*

$$\begin{aligned}
&\lim_{n\to\infty}\gamma^{-n}\varphi\big(\gamma^n x_{11},\ldots,\gamma^n x_{1p},\gamma^n y_{11},\ldots,\gamma^n y_{1p},\\
&\quad\ldots,\gamma^n x_{k1},\ldots,\gamma^n x_{kp},\ldots,\gamma^n y_{k1},\ldots,\gamma^n y_{kd}\big)=0, &(1)\\
&\big\|\big(c_\mu f(x_{11},\ldots,x_{1p},y_{11},\ldots,y_{1d}),\ldots,c_\mu f(x_{k1},\ldots,x_{kp},y_{k1},\ldots,y_{kd})\big)\big\|_k\\
&\quad\le\varphi(x_{11},\ldots,x_{1p},y_{11},\ldots,y_{1d},\ldots,x_{k1},\ldots,x_{kp},y_{k1},\ldots,y_{kd}), &(2)\\
&\big\|\big(f\big([x_1,y_1,z_1]\big)-\big[f(x_1),f(y_1),f(z_1)\big],\\
&\qquad\ldots,f\big([x_k,y_k,z_k]\big)-\big[f(x_k),f(y_k),f(z_k)\big]\big)\big\|_k\\
&\quad\le\psi(x_1,y_1,z_1,\ldots,x_k,y_k,z_k), &(3)\\
&\lim_{n\to\infty}\gamma^{-3n}\psi\big(\gamma^n x_1,\gamma^n y_1,\gamma^n z_1,\ldots,\gamma^n x_k,\gamma^n y_k,\gamma^n z_k\big)=0, &(4)\\
&\lim_{n\to\infty}\gamma^{-2n}\psi\big(\gamma^n x_1,\gamma^n y_1,z_1,\ldots,\gamma^n x_k,\gamma^n y_k,z_k\big)=0 &(5)
\end{aligned}$$

for all $\mu\in\mathbf{T}^1$ and $x_{11},\ldots,x_{1p},y_{11},\ldots,y_{1d},\ldots,x_{k1},\ldots,x_{kp},y_{k1},\ldots,y_{kd},x_1,\ldots,x_k,y_1,\ldots,y_k,z_1,\ldots,z_k\in A$, where $\gamma=\frac{p+2d}{2}$. If there exists a constant $L<1$ such that

$$\begin{aligned}
&\varphi\big(\overbrace{\gamma x_1,\ldots,\gamma x_1}^{p+d},\overbrace{\gamma x_2,\ldots,\gamma x_2}^{p+d},\ldots,\overbrace{\gamma x_k,\ldots,\gamma x_k}^{p+d}\big)\\
&\quad\le\gamma L\varphi\big(\overbrace{x_1,\ldots,x_1}^{p+d},\overbrace{x_2,\ldots,x_2}^{p+d},\ldots,\overbrace{x_k,\ldots,x_k}^{p+d}\big) &(6)
\end{aligned}$$

for all $x_1,x_2,\ldots,x_k\in A$, then there exists a unique homomorphism $H: A\to B$ such that

$$\begin{aligned}
&\big\|\big(f(x_1)-H(x_1),\ldots,f(x_k)-H(x_k)\big)\big\|_k\\
&\quad\le\frac{1}{(1-L)2\gamma}\varphi\big(\overbrace{x_1,\ldots,x_1}^{p+d},\overbrace{x_2,\ldots,x_2}^{p+d},\ldots,\overbrace{x_k,\ldots,x_k}^{p+d}\big) &(7)
\end{aligned}$$

for all $x_1,\ldots,x_k\in A$.

Proof Let $\mu=1$ and $x_{ij}=y_{ij}=x_i$ for $1\le i\le k$ in (2). Then we get

$$\begin{aligned}
&\big\|\big(f(\gamma x_1)-\gamma f(x_1),\ldots,f(\gamma x_k)-\gamma f(x_k)\big)\big\|_k\\
&\quad\le\frac{1}{2}\varphi\big(\overbrace{x_1,\ldots,x_1}^{p+d},\overbrace{x_2,\ldots,x_2}^{p+d},\ldots,\overbrace{x_k,\ldots,x_k}^{p+d}\big) &(8)
\end{aligned}$$

for all $x_1,\ldots,x_k\in A$. Consider the set

$$E:=\{g: A\to B\}$$

and introduce the *generalized metric* on E:

$$\begin{aligned} d(g,h) = \inf\bigl\{C \in \mathbf{R}_+ : \bigl\| \bigl(g(x_1) - h(x_1), \ldots, g(x_k) - h(x_k)\bigr) \bigr\|_k \\ \leq C\varphi\bigl(\overbrace{x_1,\ldots,x_1}^{p+d}, \overbrace{x_2,\ldots,x_2}^{p+d}, \ldots, \overbrace{x_k,\ldots,x_k}^{p+d}\bigr), \forall x_1,\ldots,x_k \in A\bigr\}. \end{aligned}$$

It is easy to see that (E,d) is complete (see also [9]).

First we show that d is metric on E. It is obvious $d(g,g) = 0$ for all $g \in E$. If $d(g,h) = 0$, then, for every fixed $x_1,\ldots,x_k \in A$,

$$\bigl\| \bigl(g(x_1) - h(x_1), \ldots, g(x_k) - h(x_k)\bigr) \bigr\|_k = 0$$

and therefore $g = h$. If $d(g,h) = a < \infty$ and $d(h,l) = b < \infty$ for all $g,h,l \in E$, then

$$\begin{aligned} &\bigl\| \bigl(g(x_1) - l(x_1), \ldots, g(x_k) - l(x_k)\bigr) \bigr\|_k \\ &\quad = \bigl\| \bigl(g(x_1) - h(x_1) + h(x_1) - l(x_1), \ldots, g(x_k) - h(x_k) + h(x_k) - l(x_k)\bigr) \bigr\|_k \\ &\quad \leq \bigl\| \bigl(g(x_1) - h(x_1), \ldots, g(x_k) - h(x_k)\bigr) \bigr\|_k + \bigl\| \bigl(h(x_1) - l(x_1), \ldots, h(x_k) - l(x_k)\bigr) \bigr\|_k \\ &\quad \leq a\varphi\bigl(\overbrace{x_1,\ldots,x_1}^{p+d}, \overbrace{x_2,\ldots,x_2}^{p+d}, \ldots, \overbrace{x_k,\ldots,x_k}^{p+d}\bigr) + b\varphi\bigl(\overbrace{x_1,\ldots,x_1}^{p+d}, \overbrace{x_2,\ldots,x_2}^{p+d}, \ldots, \overbrace{x_k,\ldots,x_k}^{p+d}\bigr) \\ &\quad = (a+b)\varphi\bigl(\overbrace{x_1,\ldots,x_1}^{p+d}, \overbrace{x_2,\ldots,x_2}^{p+d}, \ldots, \overbrace{x_k,\ldots,x_k}^{p+d}\bigr). \end{aligned}$$

So we have $d(g,l) \leq d(g,h) + d(h,l)$.

Let $\{g_n\}$ be a Cauchy sequence in (E,d). Then for all $\epsilon > 0$ there exists N such that $d(g_n,g_i) < \epsilon$, if $n,i \geq N$, Let $n,i \geq N$. Since $d(g_n,g_i) < \epsilon$ there exists $C \in [0,\epsilon)$ such that

$$\begin{aligned} &\bigl\| \bigl(g_n(x_1) - g_i(x_1), \ldots, g_n(x_k) - g_i(x_k)\bigr) \bigr\|_k \\ &\quad \leq C\varphi\bigl(\overbrace{x_1,\ldots,x_1}^{p+d}, \overbrace{x_2,\ldots,x_2}^{p+d}, \ldots, \overbrace{x_k,\ldots,x_k}^{p+d}\bigr) \\ &\quad \leq \epsilon\varphi\bigl(\overbrace{x_1,\ldots,x_1}^{p+d}, \overbrace{x_2,\ldots,x_2}^{p+d}, \ldots, \overbrace{x_k,\ldots,x_k}^{p+d}\bigr) \end{aligned} \tag{9}$$

for all $x_1,\ldots,x_k \in A$, so for each $x_1,\ldots,x_k \in A$, $\{g_n(x_1,\ldots,x_k)\}$ is a Cauchy sequence in B. Since B is complete, there exists $g(x_1,\ldots,x_k) \in B$ such that $g_n(x_1,\ldots,x_k) \to g(x_1,\ldots,x_k)$ as $n \to \infty$. Thus, we have $g \in E$. Taking the limit as $i \to \infty$ in (9) we obtain, for $n \geq N$,

$$\bigl\| \bigl(g_n(x_1) - g(x_1), \ldots, g_n(x_k) - g(x_k)\bigr) \bigr\|_k \leq \epsilon\varphi\bigl(\overbrace{x_1,\ldots,x_1}^{p+d}, \overbrace{x_2,\ldots,x_2}^{p+d}, \ldots, \overbrace{x_k,\ldots,x_k}^{p+d}\bigr).$$

Therefore $d(g_n,g) \leq \epsilon$. Hence $g_n \to g$ as $n \to \infty$, so (E,d) is complete. Now, we consider the linear mapping $\Lambda : E \to E$ such that

$$\Lambda g(x) := \frac{1}{\gamma} g(\gamma x)$$

for all $x \in A$. From Theorem 3.1 of [14] (also see Lemma 3.2 of [9]),

$$d(\Lambda g, \Lambda h) \leq Ld(g,h)$$

for all $g,h \in E$. Let $g,h \in E$ and let $C \in [0,\infty]$ be an arbitrary constant with $d(g,h) \leq C$. From the definition of d, we have

$$\left\| \left(g(x_1) - h(x_1), \ldots, g(x_k) - h(x_k)\right) \right\|_k \leq C\varphi\bigl(\overbrace{x_1, \ldots, x_1}^{p+d}, \ldots, \overbrace{x_k, \ldots, x_k}^{p+d}\bigr)$$

for all $x_1, \ldots, x_k \in A$. From our assumption and the last inequality, we have

$$\begin{aligned}
&\left\| \left(\Lambda g(x_1) - \Lambda h(x_1), \ldots, \Lambda g(x_k) - \Lambda h(x_k)\right) \right\|_k \\
&\quad = \frac{1}{\gamma} \left\| \left(g(\gamma x_1) - h(\gamma x_1), \ldots, g(\gamma x_k) - h(\gamma x_k)\right) \right\|_k \\
&\quad \leq \frac{C}{\gamma} \varphi\bigl(\overbrace{\gamma x_1, \ldots, \gamma x_1}^{p+d}, \ldots, \overbrace{\gamma x_k, \ldots, \gamma x_k}^{p+d}\bigr) \\
&\quad \leq CL\varphi\bigl(\overbrace{x_1, \ldots, x_1}^{p+d}, \ldots, \overbrace{x_k, \ldots, x_k}^{p+d}\bigr)
\end{aligned}$$

for all $x_1, \ldots, x_k \in A$ and so

$$\begin{aligned}
&\left\| \left(\Lambda f(x_1) - f(x_1), \ldots, \Lambda f(x_k) - f(x_k)\right) \right\|_k \\
&\quad = \left\| \left(\frac{1}{\gamma} f(\gamma x_1) - f(x_1), \ldots, \frac{1}{\gamma} f(\gamma x_k) - f(x_k)\right) \right\|_k \\
&\quad = \frac{1}{\gamma} \left\| \left(f(\gamma x_1) - \gamma f(x_1), \ldots, f(\gamma x_k) - \gamma f(x_k)\right) \right\|_k \\
&\quad \leq \frac{1}{2\gamma} \varphi\bigl(\overbrace{x_1, \ldots, x_1}^{p+d}, \ldots, \overbrace{x_k, \ldots, x_k}^{p+d}\bigr)
\end{aligned}$$

for all $x_1, \ldots, x_k \in A$. Hence $d(\Lambda f, f) \leq \frac{1}{2\gamma}$. From Theorem 1.1, the sequence $\{\Lambda^n f\}$ converges to a fixed point H of Λ, *i.e.*, $H : A \to B$ is a mapping defined by

$$H(x) = \lim_{n\to\infty} \left(\Lambda^n f\right)(x) = \lim_{n\to\infty} \frac{1}{\gamma^n} f\left(\gamma^n x\right) \tag{10}$$

and $H(\gamma x) = \gamma H(x)$ for all $x \in A$. Also, H is the unique fixed point of Λ in the set $E' = \{g \in E : d(f,g) < \infty\}$ and

$$d(H,f) \leq \frac{1}{1-L} d(\Lambda f, f) \leq \frac{1}{(1-L)2\gamma},$$

i.e., the inequality (7) hold for all $x_1, \ldots, x_k \in A$. Thus it follows from the definition of H, (1), and (2) that

$$\begin{aligned}
&\left\| \left(2H\left(\frac{\sum_{j=1}^p \mu x_{1j}}{2} + \sum_{j=1}^d \mu y_{1j}\right) - \sum_{j=1}^p \mu H(x_{1j}) - 2\sum_{j=1}^d \mu H(y_{1j}), \right.\right. \\
&\qquad \left.\left. \ldots, 2H\left(\frac{\sum_{j=1}^p \mu x_{kj}}{2} + \sum_{j=1}^d \mu y_{kj}\right) - \sum_{j=1}^p \mu H(x_{kj}) - 2\sum_{j=1}^d \mu H(y_{kj})\right) \right\|_k
\end{aligned}$$

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{1}{\gamma^n} \left\| \left(2f\left(\gamma^n \frac{\sum_{j=1}^{p} \mu x_{1j}}{2} + \gamma^n \sum_{j=1}^{d} \mu y_{1j} \right) - \sum_{j=1}^{p} \mu f(\gamma^n x_{1j}) - 2\sum_{j=1}^{d} \mu f(\gamma^n y_{1j}), \right.\right. \\
&\quad \left.\left. \dots, 2f\left(\gamma^n \frac{\sum_{j=1}^{p} \mu x_{kj}}{2} + \gamma^n \sum_{j=1}^{d} \mu y_{kj} \right) - \sum_{j=1}^{p} \mu f(\gamma^n x_{kj}) - 2\sum_{j=1}^{d} \mu f(\gamma^n y_{kj}) \right) \right\|_k \\
&\le \lim_{n\to\infty} \frac{1}{\gamma^n} \big\| \big(C_\mu f(\gamma^n x_{11}, \dots, \gamma^n x_{1p}, \gamma^n y_{11}, \dots, \gamma^n y_{1d}), \\
&\quad \dots, C_\mu f(\gamma^n x_{k1}, \dots, \gamma^n x_{kp}, \gamma^n y_{k1}, \dots, \gamma^n y_{kd}) \big) \big\|_k \\
&\le \lim_{n\to\infty} \frac{1}{\gamma^n} \varphi(\gamma^n x_{11}, \dots, \gamma^n x_{1p}, \gamma^n y_{11}, \dots, \gamma^n y_{1d}, \\
&\quad \dots, \gamma^n x_{k1}, \dots, \gamma^n x_{kp}, \gamma^n y_{k1}, \dots, \gamma^n y_{kd}) = 0
\end{aligned}$$

for all $\mu \in \mathbf{T}^1$ and $x_{11}, \dots, x_{1p}, y_{11}, \dots, y_{1d}, \dots, x_{k1}, \dots, x_{kp}, y_{k1}, \dots, y_{kd} \in A$. Hence we have

$$2H\left(\frac{\sum_{j=1}^{p} \mu x_{ij}}{2} + \sum_{j=1}^{d} \mu y_{ij} \right) = \sum_{j=1}^{p} \mu H(x_{ij}) + 2\sum_{j=1}^{d} \mu H(y_{ij})$$

for all $\mu \in \mathbf{T}^1$, $x_{11}, \dots, x_{1p}, y_{11}, \dots, y_{1d}, \dots, x_{k1}, \dots, x_{kp}, y_{k1}, \dots, y_{kd} \in A$ and $1 \le i \le k$ and so $H(\lambda x + \mu y) = \lambda H(x) + \mu H(y)$ for all $\lambda, \mu \in \mathbf{T}^1$ and $x, y \in A$. Therefore, by Lemma 3.1, the mapping $H : A \to B$ is $\mathbf{C}$-linear.

Also it follows from (3) and (4) that

$$\begin{aligned}
&\big\| \big(H([x_1, y_1, z_1]) - [H(x_1), H(y_1), H(z_1)], \dots, H([x_k, y_k, z_k]) - [H(x_k), H(y_k), H(z_k)] \big) \big\|_k \\
&= \lim_{n\to\infty} \frac{1}{\gamma^{3n}} \big\| \big(f([\gamma^n x_1, \gamma^n y_1, \gamma^n z_1]) - [f(\gamma^n x_1), f(\gamma^n y_1), f(\gamma^n z_1)], \\
&\quad \dots, f([\gamma^n x_k, \gamma^n y_k, \gamma^n z_k]) - [f(\gamma^n x_k), f(\gamma^n y_k), f(\gamma^n z_k)] \big) \big\|_k \\
&\le \lim_{n\to\infty} \frac{1}{\gamma^{3n}} \psi(\gamma^n x_1, \gamma^n y_1, \gamma^n z_1, \dots, \gamma^n x_k, \gamma^n y_k, \gamma^n z_k) = 0
\end{aligned}$$

for all $x_1, y_1, z_1, \dots, x_k, y_k, z_k \in A$. Thus we have

$$H([x, y, z]) = [H(x), H(y), H(z)]$$

for all $x, y, z \in A$. Thus $H : A \to B$ is a homomorphism satisfying (7).

Now, let $T : A \to B$ be another C^*-ternary-algebras homomorphism satisfying (7). Since $d(f, T) \le \frac{1}{(1-L)2\gamma}$ and T is $\mathbf{C}$-linear, we get $T \in E'$ and $(\Lambda T)(x) = \frac{1}{\gamma}(T\gamma x) = T(x)$ for all $x \in A$, *i.e.*, T is a fixed point of Λ. Since H is the unique fixed point of $\Lambda \in E'$, we get $H = T$. This completes the proof. □

Theorem 3.4 *Let* $((B^k, \|\cdot\|_k) : k \in \mathbf{N})$ *be a multi-*C^**-ternary algebra. Let* $f : A \to B$ *be a mapping for which there are the functions* $\varphi : A^{(p+d)k} \to [0, \infty)$ *and* $\psi : A^{3k} \to [0, \infty)$ *satisfying the inequalities* (2) *and* (3) *such that*

$$\lim_{n\to\infty} \gamma^n \varphi\left(\frac{x_{11}}{\gamma^n}, \dots, \frac{x_{1p}}{\gamma^n}, \frac{y_{11}}{\gamma^n}, \dots, \frac{y_{1p}}{\gamma^n}, \dots, \frac{x_{k1}}{\gamma^n}, \dots, \frac{x_{kp}}{\gamma^n}, \dots, \frac{y_{k1}}{\gamma^n}, \dots, \frac{y_{kd}}{\gamma^n} \right) = 0, \tag{11}$$

$$\lim_{n\to\infty}\gamma^{3n}\psi\left(\frac{x_1}{\gamma^n},\frac{y_1}{\gamma^n},\frac{z_1}{\gamma^n},\dots,\frac{x_k}{\gamma^n},\frac{y_k}{\gamma^n},\frac{z_k}{\gamma^n}\right)=0, \tag{12}$$

$$\lim_{n\to\infty}\gamma^{2n}\psi\left(\frac{x_1}{\gamma^n},\frac{y_1}{\gamma^n},z_1,\dots,\frac{x_k}{\gamma^n},\frac{y_k}{\gamma^n},z_k\right)=0 \tag{13}$$

for all $\mu\in\mathbf{T}^1$ *and* $x_{11},\dots,x_{1p},y_{11},\dots,y_{1d},\dots,x_{k1},\dots,x_{kp},y_{k1},\dots,y_{kd},x_1,\dots,x_k,y_1,\dots,y_k,z_1,\dots,z_k\in A$, *where* $\gamma=\frac{p+2d}{2}$. *If the constant* $L<1$ *exists such that*

$$\begin{aligned}&\varphi\Bigl(\overbrace{\frac{x_1}{\gamma},\dots,\frac{x_1}{\gamma}}^{p+d},\overbrace{\frac{x_2}{\gamma},\dots,\frac{x_2}{\gamma}}^{p+d},\dots,\overbrace{\frac{x_k}{\gamma},\dots,\frac{x_k}{\gamma}}^{p+d}\Bigr)\\&\quad\le\frac{L}{\gamma}\varphi(\overbrace{x_1,\dots,x_1}^{p+d},\overbrace{x_2,\dots,x_2}^{p+d},\dots,\overbrace{x_k,\dots,x_k}^{p+d})\end{aligned} \tag{14}$$

for all $x_1,x_2,\dots,x_k\in A$, *then there exists a unique homomorphism* $H:A\to B$ *such that*

$$\begin{aligned}&\bigl\|\bigl(f(x_1)-H(x_1),\dots,f(x_k)-H(x_k)\bigr)\bigr\|_k\\&\quad\le\frac{1}{(1-L)2\gamma}\varphi(\overbrace{x_1,\dots,x_1}^{p+d},\overbrace{x_2,\dots,x_2}^{p+d},\dots,\overbrace{x_k,\dots,x_k}^{p+d})\end{aligned} \tag{15}$$

for all $x_1,\dots,x_k\in A$.

Proof If we replace x_i in (8) by $\frac{x_i}{\gamma}$ for $1\le i\le k$, then we get

$$\begin{aligned}&\left\|\left(f(x_1)-\gamma f\left(\frac{1}{x_1}\right),\dots,f(x_k)-\gamma f\left(\frac{1}{x_k}\right)\right)\right\|_k\\&\quad\le\frac{1}{2}\varphi\Bigl(\overbrace{\frac{1}{x_1},\dots,\frac{1}{x_1}}^{p+d},\overbrace{\frac{1}{x_2},\dots,\frac{1}{x_2}}^{p+d},\dots,\overbrace{\frac{1}{x_k},\dots,\frac{1}{x_k}}^{p+d}\Bigr)\end{aligned} \tag{16}$$

for all $x_1,\dots,x_k\in A$. Consider the set

$$E:=\{g:A\to B\}$$

and introduce the generalized metric on E:

$$\begin{aligned}d(g,h)=\inf\bigl\{&C\in\mathbf{R}_+:\bigl\|\bigl(g(x_1)-h(x_1),\dots,g(x_k)-h(x_k)\bigr)\bigr\|_k\\&\le C\varphi(\overbrace{x_1,\dots,x_1}^{p+d},\overbrace{x_2,\dots,x_2}^{p+d},\dots,\overbrace{x_k,\dots,x_k}^{p+d}),\forall x_1,\dots,x_k\in A\bigr\}.\end{aligned}$$

It is easy to see that (E,d) is complete (see [9]).

Now, we consider the linear mapping $\Lambda:E\to E$ such that

$$\Lambda g(x):=\gamma g\left(\frac{x}{\gamma}\right)$$

for all $x\in A$. From Theorem 3.1 of [14] (also see Lemma 3.2 of [9]),

$$d(\Lambda g,\Lambda h)\le Ld(g,h)$$

for all $g, h \in E$. Let $g, h \in E$ and let $C \in [0, \infty]$ be an arbitrary constant with $d(g,h) \leq C$. From the definition of d, we have

$$\left\| \left(g(x_1) - h(x_1), \ldots, g(x_k) - h(x_k)\right) \right\|_k \leq C\varphi\big(\overbrace{x_1, \ldots, x_1}^{p+d}, \ldots, \overbrace{x_k, \ldots, x_k}^{p+d}\big)$$

for all $x_1, \ldots, x_k \in A$. From our assumption and the last inequality, we have

$$\begin{aligned}
&\left\| \left(\Lambda g(x_1) - \Lambda h(x_1), \ldots, \Lambda g(x_k) - \Lambda h(x_k)\right) \right\|_k \\
&= \gamma \left\| \left(g\left(\frac{x_1}{\gamma}\right) - h\left(\frac{x_1}{\gamma}\right), \ldots, g\left(\frac{x_k}{\gamma}\right) - h\left(\frac{x_k}{\gamma}\right) \right) \right\|_k \\
&\leq C\gamma\varphi\left(\overbrace{\frac{x_1}{\gamma}, \ldots, \frac{x_1}{\gamma}}^{p+d}, \ldots, \overbrace{\frac{x_k}{\gamma}, \ldots, \frac{x_k}{\gamma}}^{p+d} \right) \\
&\leq CL\varphi\big(\overbrace{x_1, \ldots, x_1}^{p+d}, \ldots, \overbrace{x_k, \ldots, x_k}^{p+d}\big)
\end{aligned}$$

for all $x_1, \ldots, x_k \in A$ and so $d(\Lambda g, \Lambda h) \leq Ld(g,h)$ for any $g, h \in E$. It follows from (16) that $d(\Lambda f, f) \leq \frac{1}{2\gamma}$. Therefore, according to Theorem 1.1, the sequence $\{\Lambda^n f\}$ converges to a fixed point H of Λ, *i.e.*, $H : A \to B$ is a mapping defined by

$$H(x) = \lim_{n\to\infty} \left(\Lambda^n f\right)(x) = \lim_{n\to\infty} \gamma^n f\left(\frac{x}{\gamma^n}\right) \tag{17}$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 3.3 and so we omit it. This completes the proof. □

Theorem 3.5 *Let r and θ be non-negative real numbers such that $r \notin [1,3]$ and let $((B^k, \|\cdot\|_k) : k \in \mathbf{N})$ be a multi-C^*-ternary algebra. Let $f : A \to B$ be a mapping such that*

$$\begin{aligned}
&\left\| \left(C_\mu f(x_{11}, \ldots, x_{1p}, y_{11}, \ldots, y_{1d}), \ldots, C_\mu f(x_{k1}, \ldots, x_{kp}, y_{k1}, \ldots, y_{kd})\right) \right\|_k \\
&\leq \theta \left(\sum_{j=1}^{p} \|x_{1j}\|_A^r + \sum_{j=1}^{d} \|y_{1j}\|_A^r + \cdots + \sum_{j=1}^{p} \|x_{kj}\|_A^r + \sum_{j=1}^{d} \|y_{kj}\|_A^r \right)
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
&\left\| \left(f\left([x_1, y_1, z_1]\right) - \left[f(x_1), f(y_1), f(z_1)\right], \ldots, f\left([x_k, y_k, z_k]\right) - \left[f(x_k), f(y_k), f(z_k)\right]\right) \right\|_k \\
&\leq \theta \left(\|x_1\|_A^r \cdot \|y_1\|_A^r \cdot \|z_1\|_A^r + \cdots + \|x_k\|_A^r \cdot \|y_k\|_A^r \cdot \|z_k\|_A^r \right)
\end{aligned} \tag{19}$$

for all $\mu \in \mathbf{T}^1$ and $x_{11}, \ldots, x_{1p}, y_{11}, \ldots, y_{1d}, \ldots, x_{k1}, \ldots, x_{kp}, y_{k1}, \ldots, y_{kd}, x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k \in A$. Then there exists a unique C^*-ternary algebra homomorphism $H : A \to B$ such that

$$\begin{aligned}
&\left\| \left(f(x_1) - H(x_1), \ldots, f(x_k) - H(x_k)\right) \right\|_B \\
&\leq \frac{2^r (p+d)\theta}{|2(p+2d)^r - (p+2d)2^r|} \left(\|x_1\|_A^r + \cdots + \|x_k\|_A^r \right)
\end{aligned} \tag{20}$$

for all $x_1, \ldots, x_k \in A$.

Proof The proof follows from Theorem 3.3 by taking

$$\begin{aligned}
&\varphi(x_{11},\ldots,x_{1p},y_{11},\ldots,y_{1d},\ldots,x_{k1},\ldots,x_{kp},y_{k1},\ldots,y_{kd}) \\
&\quad := \theta\left(\sum_{j=1}^{p}\|x_{ij}\|_A^r + \sum_{j=1}^{d}\|y_{ij}\|_A^r + \cdots + \sum_{j=1}^{p}\|x_{kj}\|_A^r + \sum_{j=1}^{d}\|y_{kj}\|_A^r\right), \\
&\psi(x_1,y_1,z_1,\ldots,x_k,y_k,z_k) \\
&\quad := \theta\left(\|x_1\|_A^r\cdot\|y_1\|_A^r\cdot\|z_1\|_A^r + \cdots + \|x_k\|_A^r\cdot\|y_k\|_A^r\cdot\|z_k\|_A^r\right)
\end{aligned}$$

for all $\mu \in \mathbf{T}^1$ and $x_{11},\ldots,x_{1p},y_{11},\ldots,y_{1d},\ldots,x_{k1},\ldots,x_{kp},y_{k1},\ldots,y_{kd},x_1,\ldots,x_k,y_1,\ldots,y_k,z_1,\ldots,z_k \in A$. Then we can choose $L = 2^{1-r}(p+2d)^{r-1}$, when $0 < r < 1$, and $L = 2 - 2^{1-r}(p+2d)^{r-1}$, when $r > 3$, and so we get the desired result. This completes the proof. □

Theorem 3.6 *Let* $((B^k, \|\cdot\|_k) : k\mathbf{N})$ *be a multi-*C^**-ternary algebra. Let* $f : A \to B$ *be a mapping for which there are functions* $\varphi : A^{(p+d)k} \to [0,\infty)$ *and* $\psi : A^{3k} \to [0,\infty)$ *such that*

$$\begin{aligned}
&\lim_{n\to\infty} d^{-n}\varphi\big(d^n x_{11},\ldots,d^n x_{1p},d^n y_{11},\ldots,d^n y_{1p}, \\
&\quad \ldots,d^n x_{k1},\ldots,d^n x_{kp},\ldots,d^n y_{k1},\ldots,d^n y_{kd}\big) = 0, && (21)\\
&\big\|\big(c_\mu f(x_{11},\ldots,x_{1p},y_{11},\ldots,y_{1d}),\ldots,c_\mu f(x_{k1},\ldots,x_{kp},y_{k1},\ldots,y_{kd})\big)\big\|_k \\
&\quad \le \varphi(x_{11},\ldots,x_{1p},y_{11},\ldots,y_{1d},\ldots,x_{k1},\ldots,x_{kp},y_{k1},\ldots,y_{kd}), && (22)\\
&\big\|\big(f([x_1,y_1,z_1]) - [f(x_1),f(y_1),f(z_1)], \\
&\qquad \ldots,f([x_k,y_k,z_k]) - [f(x_k),f(y_k),f(z_k)]\big)\big\|_k \\
&\quad \le \psi(x_1,y_1,z_1,\ldots,x_k,y_k,z_k), && (23)\\
&\lim_{n\to\infty} d^{-3n}\psi\big(d^n x_1,d^n y_1,d^n z_1,\ldots,d^n x_k,d^n y_k,d^n z_k\big) = 0, && (24)\\
&\lim_{n\to\infty} d^{-2n}\psi\big(d^n x_1,d^n y_1,z_1,\ldots,d^n x_k,d^n y_k,z_k\big) = 0 && (25)
\end{aligned}$$

for all $\mu \in \mathbf{T}^1$ *and* $x_{11},\ldots,x_{1p},y_{11},\ldots,y_{1d},\ldots,x_{k1},\ldots,x_{kp},y_{k1},\ldots,y_{kd},x_1,\ldots,x_k,y_1,\ldots,y_k,z_1,\ldots,z_k \in A$, *where* $\gamma = \frac{p+2d}{2}$. *If there exists the constant* $L < 1$ *such that*

$$\begin{aligned}
&\varphi\big(\overbrace{dx_1,\ldots,dx_1}^{p+d},\overbrace{dx_2,\ldots,dx_2}^{p+d},\ldots,\overbrace{dx_k,\ldots,dx_k}^{p+d}\big) \\
&\quad \le dL\varphi\big(\overbrace{0,\ldots,0}^{p},\overbrace{x_1,\ldots,x_1}^{d},\overbrace{0,\ldots,0}^{p},\overbrace{x_2,\ldots,x_2}^{d},\ldots,\overbrace{0,\ldots,0}^{p},\overbrace{x_k,\ldots,x_k}^{d}\big) && (26)
\end{aligned}$$

for all $x_1,x_2,\ldots,x_k \in A$, *then there exists a unique homomorphism* $H : A \to B$ *such that*

$$\begin{aligned}
&\big\|\big(f(x_1) - H(x_1),\ldots,f(x_k) - H(x_k)\big)\big\|_k \\
&\quad \le \frac{1}{(1-L)2d}\varphi\big(\overbrace{0,\ldots,0}^{p},\overbrace{x_1,\ldots,x_1}^{d},\overbrace{0,\ldots,0}^{p},\overbrace{x_2,\ldots,x_2}^{d},\ldots,\overbrace{0,\ldots,0}^{p},\overbrace{x_k,\ldots,x_k}^{d}\big) && (27)
\end{aligned}$$

for all $x_1,\ldots,x_k \in A$.

Proof Let $\mu = 1$ and $x_{ij} = 0$, $y_{ij} = x_i$ for $1 \le i \le k$ in (22). Then we get

$$\begin{aligned} &\big\| \big(f(dx_1) - df(x_1), \dots, f(dx_k) - df(x_k)\big) \big\|_k \\ &\quad \le \frac{1}{2}\varphi\big(\overbrace{0,\dots,0}^{p}, \overbrace{x_1,\dots,x_1}^{d}, \overbrace{0,\dots,0}^{p}, \overbrace{x_2,\dots,x_2}^{d}, \dots, \overbrace{0,\dots,0}^{p}, \overbrace{x_k,\dots,x_k}^{d}\big) \end{aligned} \tag{28}$$

for all $x_1, \dots, x_k \in A$. Consider the set

$$E := \{g : A \to B\}$$

and introduce the generalized metric on E:

$$\begin{aligned} d(g,h) = \inf\Big\{ &C \in \mathbf{R}_+ : \big\| \big(g(x_1) - h(x_1), \dots, g(x_k) - h(x_k)\big) \big\|_k \\ &\le C\varphi\big(\overbrace{0,\dots,0}^{p}, \overbrace{x_1,\dots,x_1}^{d}, \overbrace{0,\dots,0}^{p}, \overbrace{x_2,\dots,x_2}^{d}, \dots, \overbrace{0,\dots,0}^{p}, \overbrace{x_k,\dots,x_k}^{d}\big), \\ &\forall x_1, \dots, x_k \in A \Big\}. \end{aligned}$$

It is easy to see that (E, d) is complete (see [9]).

Now, we consider the linear mapping $\Lambda : E \to E$ such that

$$\Lambda g(x) := \frac{1}{d} g(dx)$$

for all $x \in A$. From Theorem 3.1 of [14] (also see Lemma 3.2 of [9]),

$$d(\Lambda g, \Lambda h) \le L d(g,h)$$

for all $g, h \in E$. Let $g, h \in E$ and let $C \in [0, \infty]$ be an arbitrary constant with $d(g,h) \le C$. From the definition of d, we have

$$\big\| \big(g(x_1) - h(x_1), \dots, g(x_k) - h(x_k)\big) \big\|_k \le C\varphi\big(\overbrace{0,\dots,0}^{p}, \overbrace{x_1,\dots,x_1}^{d}, \dots, \overbrace{0,\dots,0}^{p}, \overbrace{x_k,\dots,x_k}^{d}\big)$$

for all $x_1, \dots, x_k \in A$. From our assumption and the last inequality, we have

$$\begin{aligned} &\big\| \big(\Lambda g(x_1) - \Lambda h(x_1), \dots, \Lambda g(x_k) - \Lambda h(x_k)\big) \big\|_k \\ &\quad = \frac{1}{d} \big\| \big(g(dx_1) - h(dx_1), \dots, g(dx_k) - h(dx_k)\big) \big\|_k \\ &\quad \le \frac{C}{d}\varphi\big(\overbrace{0,\dots,0}^{p}, \overbrace{dx_1,\dots,dx_1}^{d}, \dots, \overbrace{0,\dots,0}^{p}, \overbrace{dx_k,\dots,dx_k}^{d}\big) \\ &\quad \le CL\varphi\big(\overbrace{0,\dots,0}^{p}, \overbrace{x_1,\dots,x_1}^{d}, \dots, \overbrace{0,\dots,0}^{p}, \overbrace{x_k,\dots,x_k}^{d}\big) \end{aligned}$$

for all $x_1, \dots, x_k \in A$. Thus we have

$$\begin{aligned} &\big\| \big(\Lambda f(x_1) - f(x_1), \dots, \Lambda f(x_k) - f(x_k)\big) \big\|_k \\ &\quad = \bigg\| \bigg(\frac{1}{d} f(dx_1) - f(x_1), \dots, \frac{1}{d} f(dx_k) - f(x_k)\bigg) \bigg\|_k \end{aligned}$$

$$= \frac{1}{d} \left\| \left(f(dx_1) - df(x_1), \ldots, f(dx_k) - df(x_k) \right) \right\|_k$$
$$\leq \frac{1}{2d} \varphi\big(\overbrace{0, \ldots, 0}^{p}, \overbrace{x_1, \ldots, x_1}^{d}, \ldots, \overbrace{0, \ldots, 0}^{p}, \overbrace{x_k, \ldots, x_k}^{d}\big)$$

for all $x_1, \ldots, x_k \in A$. Hence $d(\Lambda f, f) \leq \frac{1}{2d}$. From Theorem 1.1, the sequence $\{\Lambda^n f\}$ converges to a fixed point H of Λ, *i.e.*, $H : A \to B$ is a mapping defined by

$$H(x) = \lim_{n\to\infty} \left(\Lambda^n f\right)(x) = \lim_{n\to\infty} \frac{1}{d^n} f\left(d^n x\right) \tag{29}$$

and $H(dx) = dH(x)$ for all $x \in A$. Also, H is the unique fixed point of Λ in the set $E' = \{g \in E : d(f, g) < \infty\}$ and

$$d(H, f) \leq \frac{1}{1 - L} d(\Lambda f, f) \leq \frac{1}{(1 - L)2d},$$

i.e., the inequality (27) hold for all $x_1, \ldots, x_k \in A$. It follows from the definition of H, (21), and (22) that

$$\begin{aligned}
&\left\| 2H\left(\frac{\sum_{j=1}^{p} \mu x_{1j}}{2} + \sum_{j=1}^{d} \mu y_{1j} \right) - \sum_{j=1}^{p} \mu H(x_{1j}) - 2\sum_{j=1}^{d} \mu H(y_{1j}), \right. \\
&\quad \left. \ldots, 2H\left(\frac{\sum_{j=1}^{p} \mu x_{kj}}{2} + \sum_{j=1}^{d} \mu y_{kj} \right) - \sum_{j=1}^{p} \mu H(x_{kj}) - 2\sum_{j=1}^{d} \mu H(y_{kj}) \right\|_k \\
&= \lim_{n\to\infty} \frac{1}{d^n} \left\| 2f\left(d^n \frac{\sum_{j=1}^{p} \mu x_{1j}}{2} + d^n \sum_{j=1}^{d} \mu y_{1j} \right) - \sum_{j=1}^{p} \mu f\left(d^n x_{1j}\right) - 2\sum_{j=1}^{d} \mu f\left(d^n y_{1j}\right), \right. \\
&\quad \left. \ldots, 2f\left(d^n \frac{\sum_{j=1}^{p} \mu x_{kj}}{2} + d^n \sum_{j=1}^{d} \mu y_{kj} \right) - \sum_{j=1}^{p} \mu f\left(d^n x_{kj}\right) - 2\sum_{j=1}^{d} \mu f\left(d^n y_{kj}\right) \right\|_k \\
&\leq \lim_{n\to\infty} \frac{1}{d^n} \left\| \left(C_\mu f\left(d^n x_{11}, \ldots, d^n x_{1p}, d^n y_{11}, \ldots, d^n y_{1d}\right), \right. \right. \\
&\quad \left. \left. \ldots, C_\mu f\left(d^n x_{k1}, \ldots, d^n x_{kp}, d^n y_{k1}, \ldots, d^n y_{kd}\right) \right) \right\|_k \\
&\quad + \lim_{n\to\infty} \frac{1}{d^n} \varphi\left(d^n x_{11}, \ldots, d^n x_{1p}, d^n y_{11}, \ldots, d^n y_{1d}, \right. \\
&\quad \left. \ldots, d^n x_{k1}, \ldots, d^n x_{kp}, d^n y_{k1}, \ldots, d^n y_{kd}\right) = 0
\end{aligned}$$

for all $\mu \in \mathbf{T}^1$ and $x_{11}, \ldots, x_{1p}, y_{11}, \ldots, y_{1d}, \ldots, x_{k1}, \ldots, x_{kp}, y_{k1}, \ldots, y_{kd} \in A$. Hence we have

$$2H\left(\frac{\sum_{j=1}^{p} \mu x_{ij}}{2} + \sum_{j=1}^{d} \mu y_{ij} \right) = \sum_{j=1}^{p} \mu H(x_{ij}) + 2\sum_{j=1}^{d} \mu H(y_{ij})$$

for all $\mu \in \mathbf{T}^1$, $x_{11}, \ldots, x_{1p}, y_{11}, \ldots, y_{1d}, \ldots, x_{k1}, \ldots, x_{kp}, y_{k1}, \ldots, y_{kd} \in A$ and $1 \leq i \leq k$ and so $H(\lambda x + \mu y) = \lambda H(x) + \mu H(y)$ for all $\lambda, \mu \in \mathbf{T}^1$ and all $x, y \in A$. Therefore, by Lemma 3.1, the mapping $H : A \to B$ is $\mathbf{C}$-linear.

Also it follows from (23) and (24) that

$$\begin{aligned}
&\big\| H([x_1,y_1,z_1]) - [H(x_1),H(y_1),H(z_1)],\dots,H([x_k,y_k,z_k]) - [H(x_k),H(y_k),H(z_k)]\big\|_k \\
&= \lim_{n\to\infty}\frac{1}{d^{3n}}\big\| f([d^n x_1,d^n y_1,d^n z_1]) - [f(d^n x_1),f(d^n y_1),f(d^n z_1)], \\
&\quad \dots,f([d^n x_k,d^n y_k,d^n z_k]) - [f(d^n x_k),f(d^n y_k),f(d^n z_k)]\big\|_k \\
&\le \lim_{n\to\infty}\frac{1}{d^{3n}}\psi(d^n x_1,d^n y_1,d^n z_1,\dots,d^n x_k,d^n y_k,d^n z_k) = 0
\end{aligned}$$

for all $x_1,y_1,z_1,\dots,x_k,y_k,z_k \in A$. Thus

$$H([x,y,z]) = [H(x),H(y),H(z)]$$

for all $x,y,z \in A$.Thus $H : A \to B$ is a homomorphism satisfying (26).

Now, let $T : A \to B$ be another C^*-ternary algebras homomorphism satisfying (27). Since $d(f,T) \le \frac{1}{(1-L)2d}$ and T is **C**-linear, we get $T \in E'$ and $(\Lambda T)(x) = \frac{1}{d}(T\gamma x) = T(x)$ for all $x \in A$, *i.e.*, T is a fixed point of Λ. Since H is the unique fixed point of $\Lambda \in E'$, we get $H = T$. This completes the proof. □

Theorem 3.7 *Let* r, s, *and* θ *be non-negative real numbers such that* $0 < r \ne 1$, $0 < s \ne 3$, *and let* $d \ge 2$. *Suppose that* $f : A \to B$ *is a mapping with* $f(0) = 0$ *satisfying* (18) *and*

$$\begin{aligned}
&\big\| \big(f([x_1,y_1,z_1]) - [f(x_1),f(y_1),f(z_1)],\dots,f([x_k,y_k,z_k]) - [f(x_k),f(y_k),f(z_k)]\big)\big\| \\
&\quad\le \theta\big(\|x_1\|_A^s\cdot\|y_1\|_A^s\cdot\|z_1\|_A^s + \cdots + \|x_k\|_A^s\cdot\|y_k\|_A^s\cdot\|z_k\|_A^s\big)
\end{aligned} \tag{30}$$

for all $\mu \in \mathbf{T}^1$ *and* $x_1,\dots,x_k,y_1,\dots,y_k,z_1,\dots,z_k \in A$. *Then there exists a unique* C^*-*ternary algebra homomorphism* $H : A \to B$ *such that*

$$\begin{aligned}
&\big\| \big(f(x_1) - H(x_1),\dots,f(x_k) - H(x_k)\big)\big\|_K \\
&\quad\le \frac{d\theta}{2|d-d^r|}\big(\|x_1\|_A^r + \cdots + \|x_k\|_A^r\big)
\end{aligned} \tag{31}$$

for all $x_1,\dots,x_k \in A$.

Proof We only prove the theorem when $0 < r < 1$ and $0 < s < 3$. One can prove the theorem for the other cases in a similar way. The proof follows from Theorem 3.6 by taking

$$\begin{aligned}
&\varphi(x_{11},\dots,x_{1p},y_{11},\dots,y_{1d},\dots,x_{k1},\dots,x_{kp},y_{k1},\dots,y_{kd}) \\
&\quad:= \theta\left(\sum_{j=1}^{p}\|x_{1j}\|_A^r + \sum_{j=1}^{d}\|y_{1j}\|_A^r + \cdots + \sum_{j=1}^{p}\|x_{kj}\|_A^r + \sum_{j=1}^{d}\|y_{kj}\|_A^r\right), \\
&\psi(x_1,y_1,z_1,\dots,x_k,y_k,z_k) := \theta\big(\|x_1\|_A^s\cdot\|y_1\|_A^s\cdot\|z_1\|_A^s + \cdots + \|x_k\|_A^s\cdot\|y_k\|_A^s\cdot\|z_k\|_A^s\big)
\end{aligned}$$

for all $\mu \in \mathbf{T}^1$ and $x_{11},\dots,x_{1p},y_{11},\dots,y_{1d},\dots,x_{k1},\dots,x_{kp},y_{k1},\dots,y_{kd},x_1,\dots,x_k,y_1,\dots,y_k,z_1,\dots,z_k \in A$. Then we can choose $L = d^{r-1}$, when $0 < r < 1$ and $0 < s < 3$, and $L = 2 - d^{r-1}$, when $r > 1$ and $s > 3$, and so we get the desired result. □

Now, assume that A is a unital C^*-ternary algebra with norm $\|\cdot\|$ and unit e and B is a unital C^*-ternary algebra with norm $\|\cdot\|$ and unit e'.

We investigate homomorphisms in C^*-ternary algebras associated with the functional equation $C_\mu f(x_1,\ldots,x_p,y_1,\ldots,y_d)=0$.

Theorem 3.8 ([5]) *Let $r>1$ (resp., $r<1$) and θ be non-negative real numbers and let $f: A\to B$ be a bijective mapping satisfying* (18) *and*

$$f([x,y,z]) = [f(x),f(y),f(z)]$$

for all $x,y,z\in A$. If $\lim_{n\to\infty}\frac{(p+2d)^n}{2^n}f(\frac{2^n e}{(p+2d)^n}) = e'$ (resp., $\lim_{n\to\infty}\frac{2^n}{(p+2d)^n}f(\frac{(p+2d)^n}{2^n}e)=e'$), then the mapping $f:A\to B$ is a C^*-ternary algebra isomorphism.

Theorem 3.9 *Let $r<1$ and θ be non-negative real numbers and let $f:A\to B$ be a mapping satisfying* (18) *and* (19). *If there exist a real number $\lambda>1$ (resp., $0<\lambda<1$) and an element $x_0\in A$ such that $\lim_{n\to\infty}\frac{1}{\lambda^n}f(\lambda^n x_0)=e'$ (resp., $\lim_{n\to\infty}\lambda^n f(\frac{x_0}{\lambda^n})=e'$), then the mapping $f: A\to B$ is a multi-C^*-ternary algebra homomorphism.*

Proof By using the proof of Theorem 3.5, there exists a unique multi-C^*-ternary algebra homomorphism $H:A\to B$ satisfying (20). It follows from (20) that

$$H(x)=\lim_{n\to\infty}\frac{1}{\lambda^n}f(\lambda^n x) \qquad \left(\text{resp.}, H(x)=\lim_{n\to\infty}\lambda^n f\left(\frac{x}{\lambda^n}\right)\right)$$

for all $x\in A$ and $\lambda>1$ $(0<\lambda<1)$. Therefore, from our assumption, we get $H(x_0)=e'$.

Let $\lambda>1$ and $\lim_{n\to\infty}\frac{1}{\lambda^n}f(\lambda^n x_0)=e'$. It follows from (19) that

$$\begin{aligned}
&\big\|\big([H(x_1),H(y_1),H(z_1)]-[H(x_1),H(y_1),f(z_1)],\\
&\quad\ldots,[H(x_k),H(y_k),H(z_k)]-[H(x_k),H(y_k),f(z_k)]\big)\big\|\\
&=\big\|\big(H[x_1,y_1,z_1]-[H(x_1),H(y_1),f(z_1)],\\
&\quad\ldots,H[x_k,y_k,z_k]-[H(x_k),H(y_k),f(z_k)]\big)\big\|\\
&=\lim_{n\to\infty}\frac{1}{\lambda^{2n}}\big\|\big(f([\lambda^n x_1,\lambda^n y_1,z_1])-[f(\lambda^n x_1),f(\lambda^n y_1),f(z_1)],\\
&\quad\ldots,f([\lambda^n x_k,\lambda^n y_k,z_k])-[f(\lambda^n x_k),f(\lambda^n y_k),f(z_k)]\big)\big\|\\
&\le\lim_{n\to\infty}\frac{\lambda^{rn}}{\lambda^{3n}}\theta\big(\|x_1\|_A^r\cdot\|y_1\|_A^r\cdot\|z_1\|_A^r+\cdots+\|x_k\|_A^r\cdot\|y_k\|_A^r\cdot\|z_k\|_A^r\big)=0
\end{aligned}$$

for all $x_1,\ldots,x_k\in A$. Thus $[H(x),H(y),H(z)]=[H(x),H(y),f(z)]$ for all $x,y,z\in A$. Letting $x=y=x_0$ in the last equality, we get $f(z)=H(z)$ for all $z\in A$. Similarly, one can show that $H(x)=f(x)$ for all $x\in A$ when $0<\lambda<1$ and $\lim_{n\to\infty}\lambda^n f(\frac{x_0}{\lambda^n})=e'$.

Similarly, one can show the theorem for the case $\lambda>1$. Therefore, the mapping $f:A\to B$ is a multi-C^*-ternary algebra homomorphism. This completes the proof. □

Theorem 3.10 *Let $r>1$ and θ be non-negative real numbers and let $f:A\to B$ be a mapping satisfying* (18) *and* (19). *If there exist a real number $\lambda>1$ (resp., $0<\lambda<1$) and an element $x_0\in A$ such that $\lim_{n\to\infty}\frac{1}{\lambda^n}f(\lambda^n x_0)=e'$ (resp., $\lim_{n\to\infty}\lambda^n f(\frac{x_0}{\lambda^n})=e'$), then the mapping $f:A\to B$ is a multi-C^*-ternary algebra homomorphism.*

Proof The proof is similar to the proof of Theorem 3.9 and we omit it. □

4 Approximation of derivations on multi-C^*-ternary algebras

Throughout this section, assume that A is a C^*-ternary algebra with norm $\|\cdot\|$.

Park [5] studied approximation of derivations on C^*-ternary algebras for the functional equation $C_\mu f(x_1,\ldots,x_p,y_1,\ldots,y_d)=0$ (see also [5, 13, 15–59] and [60]).

For any mapping $f:A\to A$, let

$$\mathbf{D}f(x,y,z)=f\bigl([x,y,z]\bigr)-\bigl[f(x),y,z\bigr]-\bigl[x,f(y),z\bigr]-\bigl[x,y,f(z)\bigr]$$

for all $x,y,z\in A$.

Theorem 4.1 ([13]) *Let r and θ be non-negative real numbers such that $r\notin[1,3]$ and let $f:A\to A$ be a mapping satisfying* (19) *and*

$$\bigl\|\mathbf{D}f(x,y,z)\bigr\|\le\theta\bigl(\|x\|^r+\|y\|^r+\|z\|^r\bigr)$$

for all $x,y,z\in A$. Then there exists a unique C^*-ternary derivation $\delta:A\to A$ such that

$$\bigl\|f(x)-\delta(x)\bigr\|\le\frac{2^r(p+d)}{|2(p+2d)^r-(p+2d)2^r|}\theta\|x\|^r$$

for all $x\in A$.

In the following theorem, we generalize and improve the result in Theorem 4.1.

Theorem 4.2 *Let $((A^k,\|\cdot\|_k):k\in\mathbf{N})$ be a multi-C^*-ternary algebra. Let $f:A\to A$ be a mapping for which there are the functions $\varphi:A^{(p+d)k}\to[0,\infty)$ and $\psi:A^{3k}\to[0,\infty)$ satisfying the inequalities* (1), (2), *and* (4) *such that*

$$\bigl\|\bigl(\mathbf{D}f(x_1,y_1,z_1),\ldots,\mathbf{D}f(x_k,y_k,z_k)\bigr)\bigr\|\le\psi(x_1,y_1,z_1,\ldots,x_k,y_k,z_k)\tag{32}$$

for all $\mu\in\mathbf{T}^1$ and $x_{11},\ldots,x_{1p},y_{11},\ldots,y_{1d},\ldots,x_{k1},\ldots,x_{kp},y_{k1},\ldots,y_{kd},x_1,\ldots,x_k,y_1,\ldots,y_k,z_1,\ldots,z_k\in A$, where $\gamma=\frac{p+2d}{2}$. If the constant $L<1$ exists such that

$$\begin{aligned}&\varphi\bigl(\overbrace{\gamma x_1,\ldots,\gamma x_1}^{p+d},\overbrace{\gamma x_2,\ldots,\gamma x_2}^{p+d},\ldots,\overbrace{\gamma x_k,\ldots,\gamma x_k}^{p+d}\bigr)\\&\quad\le\gamma L\varphi\bigl(\overbrace{x_1,\ldots,x_1}^{p+d},\overbrace{x_2,\ldots,x_2}^{p+d},\ldots,\overbrace{x_k,\ldots,x_k}^{p+d}\bigr)\end{aligned}\tag{33}$$

for all $x_1,x_2,\ldots,x_k\in A$, then there exists a unique C^*-ternary derivation $\delta:A\to B$ such that

$$\begin{aligned}&\bigl\|\bigl(f(x_1)-\delta(x_1),\ldots,f(x_k)-\delta(x_k)\bigr)\bigr\|_k\\&\quad\le\frac{1}{(1-L)2\gamma}\varphi\bigl(\overbrace{x_1,\ldots,x_1}^{p+d},\overbrace{x_2,\ldots,x_2}^{p+d},\ldots,\overbrace{x_k,\ldots,x_k}^{p+d}\bigr)\end{aligned}\tag{34}$$

for all $x_1,\ldots,x_k\in A$.

Proof The same reasoning as in the proof of Theorem 3.3, guarantees there exists a unique **C**-linear mapping $\delta : A \to A$ satisfying (32). The mapping $\delta : A \to A$ is given by

$$\delta(x) = \lim_{n\to\infty} \left(\Lambda^n f\right)(x) = \lim_{n\to\infty} \frac{1}{\gamma^n} f\left(\gamma^n x\right) \tag{35}$$

and $\delta(\gamma x) = \gamma\delta(x)$ for all $x \in A$. Also, H is the unique fixed point of Λ in the set $E' = \{g \in E : d(f,g) < \infty\}$ and

$$d(\delta,f) \le \frac{1}{1-L} d(\Lambda f,f) \le \frac{1}{(1-L)2\gamma},$$

i.e., the inequality (6) holds for all $x_1,\dots,x_k \in A$. It follows from the definition of δ, (1) and (2), and (35) that

$$\begin{aligned}
&\left\| \left(C_\mu\delta(x_{11},\dots,x_{1p}y_{11},\dots,y_{1d}),\dots,C_\mu\delta(x_{k1},\dots,x_{kp}y_{k1},\dots,y_{kd})\right) \right\|_k \\
&\quad = \lim_{n\to\infty} \frac{1}{\gamma^n} \left\| \left(C_\mu f\left(\gamma^n x_{11},\dots,\gamma^n x_{1p},\gamma^n y_{11},\dots,\gamma^n y_{1d}\right),\right.\right. \\
&\quad\quad \left.\left.\dots,C_\mu f\left(\gamma^n x_{k1},\dots,\gamma^n x_{kp},\gamma^n y_{k1},\dots,\gamma^n y_{kd}\right)\right) \right\|_k \\
&\quad \le \lim_{n\to\infty} \frac{1}{\gamma^n} \varphi\left(\gamma^n x_{11},\dots,\gamma^n x_{1p}.\gamma^n y_{11},\dots,\gamma^n y_{1d},\right. \\
&\quad\quad \left.\dots,\gamma^n x_{k1},\dots,\gamma^n x_{kp},\gamma^n y_{k1},\dots,\gamma^n y_{kd}\right) = 0
\end{aligned}$$

for all $\mu \in \mathbf{T}^1$ and $x_{11},\dots,x_{1p},y_{11},\dots,y_{1d},\dots,x_{k1},\dots,x_{kp},y_{k1},\dots,y_{kd} \in A$. Hence we have

$$2\delta\left(\frac{\sum_{j=1}^p \mu x_{ij}}{2} + \sum_{j=1}^d \mu y_{ij}\right) = \sum_{j=1}^p \mu\delta(x_{ij}) + 2\sum_{j=1}^d \mu\delta(y_{ij})$$

for all $\mu \in \mathbf{T}^1$, $x_{11},\dots,x_{1p},y_{11},\dots,y_{1d},\dots,x_{k1},\dots,x_{kp},y_{k1},\dots,y_{kd} \in A$ and $1 \le i \le k$ and so $\delta(\lambda x + \mu y) = \lambda\delta(x) + \mu\delta(y)$ for all $\lambda,\mu \in \mathbf{T}^1$ and $x,y \in A$. Therefore, by Lemma 3.1, the mapping $\delta : A \to B$ is **C**-linear.

Also it follows from (4) and (32) that

$$\begin{aligned}
&\left\| \left(\mathbf{D}\delta(x_1,y_1,z_1),\dots,\mathbf{D}\delta(x_k,y_k,z_k)\right) \right\|_k \\
&\quad = \lim_{n\to\infty} \frac{1}{\gamma^{3n}} \left\| f\left(\mathbf{D}f\left(\gamma^n x_1,\gamma^n y_1,\gamma^n z_1\right),\dots,f\left(\gamma^n x_k,\gamma^n y_k,\gamma^n z_k\right)\right) \right\| \\
&\quad \le \lim_{n\to\infty} \frac{1}{\gamma^{3n}} \psi\left(\gamma^n x_1,\gamma^n y_1,\gamma^n z_1,\dots,\gamma^n x_k,\gamma^n y_k,\gamma^n z_k\right) = 0
\end{aligned}$$

for all $x_1,y_1,z_1,\dots,x_k,y_k,z_k \in A$ and hence

$$\begin{aligned}
&\left(\delta\left([x_1,y_1,z_1]\right),\dots,\delta\left([x_k,y_k,z_k]\right)\right) \\
&\quad + \left(\left[\delta(x_1),(y_1),(z_1)\right] + \left[x_1,\delta(y_1),z_1\right] + \left[x_1,y_1,\delta(z_1)\right],\right. \\
&\quad \left.\dots,\left[\delta(x_k),(y_k),(z_k)\right] + \left[x_k,\delta(y_k),z_k\right] + \left[x_k,y_k,\delta(z_k)\right]\right)
\end{aligned} \tag{36}$$

for all $x, y, z \in A$ and so the mapping $\delta : A \to A$ is a C^*-ternary derivation. It follows from (32) and (4) that

$$\begin{aligned}
&\big\| \big(\delta[x_1, y_1, z_1] - \big[\delta(x_1), y_1, z_1\big] - \big[x_1, \delta(y_1), z_1\big] - \big[x, y, f(z_1)\big], \\
&\qquad \ldots, \delta[x_k, y_k, z_k] - \big[\delta(x_k), y_k, z_k\big] - \big[x_k, \delta(y_k), z_k\big] - \big[x, y, f(z_k)\big]\big) \big\| \\
&\quad = \lim_{n\to\infty} \frac{1}{\gamma^{2n}} \big\| \big(f\big[\gamma^n x_1, \gamma^n y_1, z_1\big] - \big[f\big(\gamma^n x_1\big), \gamma^n y_1, z_1\big] \\
&\qquad - \big[\gamma^n x_1, f\big(\gamma^n y_1\big), z_1\big] - \big[\gamma^n x_1, \gamma^n y_1, f(z_1)\big], \\
&\qquad \ldots, f\big[\gamma^n x_k, \gamma^n y_k, z_k\big] - \big[f\big(\gamma^n x_k\big), \gamma^n y_k, z_k\big] \\
&\qquad - \big[\gamma^n x_k, f\big(\gamma^n y_k\big), z_k\big] - \big[\gamma^n x_k, \gamma^n y_k, f(z_k)\big]\big) \big\| \\
&\quad \le \lim_{n\to\infty} \frac{1}{\gamma^{2n}} \psi\big(\gamma^n x_1, \gamma^n y_1, z_1, \ldots, \gamma^n x_k, \gamma^n y_k, z_k\big) = 0
\end{aligned}$$

for all $x_1, y_1, z_1, \ldots, x_k, y_k, z_k \in A$ and so we have

$$\big(\delta[x, y, z]\big) = \big[\delta(x), y, z\big] + \big[x, \delta(y), z\big] + \big[x, y, f(z)\big] \tag{37}$$

for all $x, y, z \in A$. Hence it follows from (36) and (37) that

$$\big[x, y, \delta(z)\big] = \big[x, y, f(z)\big] \tag{38}$$

for all $x, y, z \in A$. Letting $x = y = f(z) - \delta(z)$ in (38), we get

$$\big\|f(z) - \delta(z)\big\|^3 = \big\|\big[f(z) - \delta(z), f(z) - \delta(z), f(z) - \delta(z)\big]\big\| = 0 \tag{39}$$

for all $z_1, \ldots, z_k \in A$ and hence $f(z) = \delta(z)$ for all $z \in A$. Therefore, the mapping $f : A \to A$ is a C^*-ternary derivation. This completes the proof. □

Corollary 4.3 *Let $r < 1$, $s < 2$, and θ be non-negative real numbers and let $f : A \to A$ be a mapping satisfying* (18) *and*

$$\begin{aligned}
&\big\| \big(\mathbf{D}f(x_1, y_1, z_1), \ldots, \mathbf{D}f(x_k, y_k, z_k)\big) \big\| \\
&\quad \le \theta\big(\|x_1\|_A^s \cdot \|y_1\|_A^s \cdot \|z_1\|_A^s + \cdots + \|x_k\|_A^s \cdot \|y_k\|_A^s \cdot \|z_k\|_A^s\big)
\end{aligned}$$

for all $x_1, y_1, z_1, \ldots, x_k, y_k, z_k \in A$. Then the mapping $f : A \to A$ is a C^*-ternary derivation.

Proof Define

$$\begin{aligned}
&\varphi(x_{11}, \ldots, x_{1p}, y_{11}, \ldots, y_{1d}, \ldots, x_{k1}, \ldots, x_{kp}, y_{k1}, \ldots, y_{kd}) \\
&\quad = \theta\left(\sum_{j=1}^{p} \|x_{1j}\|_A^r + \sum_{j=1}^{d} \|y_{1j}\|_A^r, \ldots, \sum_{j=1}^{p} \|x_{kj}\|_A^r + \sum_{j=1}^{d} \|y_{kj}\|_A^r\right)
\end{aligned}$$

and

$$\begin{aligned}
&\psi(x_1, y_1, z_1, \ldots, x_k, y_k, z_k) \\
&\quad = \theta\big(\|x_1\|_A^s \cdot \|y_1\|_A^s \cdot \|z_1\|_A^s + \cdots + \|x_k\|_A^s \cdot \|y_k\|_A^s \cdot \|z_k\|_A^s\big)
\end{aligned}$$

for all $x_1, y_1, z_1, \dots, x_k, y_k, z_k, x_{11}, \dots, x_{1p}, y_{11}, \dots, y_{1d}, \dots, x_{k1}, \dots, x_{kp}, y_{k1}, \dots, y_{kd} \in A$ and applying Theorem 4.2, we get the desired result. □

Theorem 4.4 *Let $((A^k, \|\cdot\|_k) : k \in \mathbf{N})$ be a multi-C^*-ternary algebra. Let $f : A \to A$ be a mapping for which there are the functions $\varphi : A^{(p+d)k} \to [0,\infty)$ and $\psi : A^{3k} \to [0,\infty)$ satisfying the inequalities* (2), (11), (12), *and* (32) *for all $\mu \in \mathbf{T}^1$ and $x_{11}, \dots, x_{1p}, y_{11}, \dots, y_{1d}, \dots, x_{k1}, \dots, x_{kp}, y_{k1}, \dots, y_{kd}, x_1, \dots, x_k, y_1, \dots, y_k, z_1, \dots, z_k \in A$, where $\gamma = \frac{p+2d}{2}$. If there exists the constant $L < 1$ such that*

$$\begin{aligned}
&\varphi\Bigl(\overbrace{\frac{x_1}{\gamma},\dots,\frac{x_1}{\gamma}}^{p+d},\overbrace{\frac{x_2}{\gamma},\dots,\frac{x_2}{\gamma}}^{p+d},\dots,\overbrace{\frac{x_k}{\gamma},\dots,\frac{x_k}{\gamma}}^{p+d}\Bigr)\\
&\quad\le \frac{L}{\gamma}\varphi\bigl(\overbrace{x_1,\dots,x_1}^{p+d},\overbrace{x_2,\dots,x_2}^{p+d},\dots,\overbrace{x_k,\dots,x_k}^{p+d}\bigr)
\end{aligned} \tag{40}$$

for all $x_1, x_2, \dots, x_k \in A$, then there exists a unique homomorphism $\delta : A \to A$ such that

$$\begin{aligned}
&\bigl\|\bigl(f(x_1)-\delta(x_1),\dots,f(x_k)-\delta(x_k)\bigr)\bigr\|_k\\
&\quad\le \frac{1}{(1-L)2\gamma}\varphi\bigl(\overbrace{x_1,\dots,x_1}^{p+d},\overbrace{x_2,\dots,x_2}^{p+d},\dots,\overbrace{x_k,\dots,x_k}^{p+d}\bigr)
\end{aligned} \tag{41}$$

for all $x_1, \dots, x_k \in A$.

Proof The same reasoning as in the proof of Theorem 3.4 guarantees there exists a unique $\mathbf{C}$-linear mapping $\delta : A \to A$ satisfying (32). The rest of the proof is similar to the proof of Theorem 4.2 and so we omit it. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors carried out the proof. All authors conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

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