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Strong law for linear processes

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Abstract

In this paper, the authors obtain the strong approximations, the laws of the single logarithm, the functional laws of the single logarithm, and the limit law of the single logarithm for linear processes generated by arrays of independent and identically distributed random variables, and they show that they are all equivalent under the same moment conditions.

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1 Introduction and main result

Throughout the paper, we assume that $\{\varepsilon_i, -\infty < i < \infty\}$ (or $\{\varepsilon_{n,i}, -\infty < i < \infty, n \geq 1\}$) is a sequence (or an array) of independent and identically distributed (i.i.d.) random variables with the same distribution as the random variable ε, $\{a_i, -\infty < i < \infty\}$ is a sequence of real constants with

$$\sum_{i=-\infty}^{\infty} |a_i| < \infty, \quad a = \sum_{i=-\infty}^{\infty} a_i \neq 0.$$

Define

$$X_n = \sum_{i=-\infty}^{\infty} a_i \varepsilon_{n-i}, \quad n \geq 1 \quad \left(\text{or } X_{nk} = \sum_{i=-\infty}^{\infty} a_i \varepsilon_{n,k-i}, k \geq 1, n \geq 1\right). \tag{1.1}$$

We call $\{X_n, n \geq 1\}$ (or $\{X_{nk}, k \geq 1, n \geq 1\}$) the linear process of $\{\varepsilon_i, -\infty < i < \infty\}$ (or $\{\varepsilon_{ni}, -\infty < i < \infty, n \geq 1\}$). In particular if $a_i = 0$ for $i = -1, -2, \ldots$, we call $\{X_n, n \geq 1\}$ (or $\{X_{nk}, k \geq 1, n \geq 1\}$) the one-side linear processes. Denote by $\mathcal{K}$ the set of all absolutely continuous f on $[0, 1]$ with $f(0) = 0$ and $\int_0^1 (f'(x))^2 \, dx \leq 1$; it is well known that $\mathcal{K}$ is a compact, convex, and symmetric subset of the Banach space $C[0, 1]$, which is the set of real continuous functions on $[0, 1]$ with the uniform norm.

For the one-side linear processes, Philipps and Solo [1] established the strong law of large numbers and the law of the iterated logarithm, Wang *et al.* [2] established the strong approximation to a Gaussian process. Lu and Qiu [3] obtained the functional law of the iterated logarithm for the one-side linear processes as follows.

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Theorem A *Suppose that* $E\varepsilon = 0$, $E\varepsilon^2 = 1$, *and* $\sum_{i=1}^{\infty} i|a_i| < \infty$. *Then the stochastic process sequence* $\{\frac{S(nt)}{\sqrt{2a^2 n \log\log n}}, 0 \le t \le 1, n \ge 1\}$ *converges to and clusters throughout* $\mathcal{K}$ *in the uniform norm with probability* 1, *that is,*

$$
\begin{cases}
\lim_{n\to\infty} \inf_{f\in\mathcal{K}} \sup_{0\le t\le 1} |\frac{S(nt)}{\sqrt{2a^2 n \log\log n}} - f(t)| = 0 \quad a.s. \quad and \\
\liminf_{n\to\infty} \sup_{0\le t\le 1} |\frac{S(nt)}{\sqrt{2a^2 n \log\log n}} - f(t)| = 0 \quad a.s. \quad for\ every\ f \in \mathcal{K},
\end{cases}
$$

where

$$
S(nt) = \sum_{j=1}^{[nt]} X_j + (nt - [nt]) X_{[nt]+1}, \quad 0 \le t \le 1, n \ge 1,
$$

$[\cdot]$ *is the greatest integer function, and* $\log x = \ln(\max\{e, x\})$ *for* $x > 0$.

By the same argument as Strassen [4], Theorem A implies the classical Hartman-Wintner law of the iterated logarithm (*cf.* Hartman and Wintner [5])

$$
\limsup_{n\to\infty} \frac{\sum_{j=1}^{n} X_j}{\sqrt{2a^2 n \log\log n}} = 1 \quad a.s. \quad and \quad \liminf_{n\to\infty} \frac{\sum_{j=1}^{n} X_j}{\sqrt{2a^2 n \log\log n}} = -1 \quad a.s.
$$

Recently, Tan *et al.* [6] proved the above-mentioned conclusions without the condition $\sum_{i=1}^{\infty} i|a_i| < \infty$.

There is a substantial difference between the a.s. limiting behavior for sequences of i.i.d. random variables and arrays of i.i.d. random variables. For example, if $E\varepsilon = 0$, $E\varepsilon^2 = 1$, $E\varepsilon^4 < \infty$, Hu and Weber [7] proved the following law of the single logarithm:

$$
\limsup_{n\to\infty} \frac{\sum_{j=1}^{n} \varepsilon_{n,j}}{\sqrt{2n \log n}} = 1 \quad a.s. \quad and \quad \liminf_{n\to\infty} \frac{\sum_{j=1}^{n} \varepsilon_{n,j}}{\sqrt{2n \log n}} = -1 \quad a.s. \tag{1.2}
$$

Hence the classical Hartman-Wintner law of the iterated logarithm (*cf.* Hartman and Wintner [5]) does not hold for arrays. The above result was improved by Li *et al.* [8] and by Qi [9] who simultaneously and independently proved that

$$
E\varepsilon = 0, \quad E\varepsilon^2 = 1, \quad and \quad E\frac{|\varepsilon|^4}{\log^2 |\varepsilon|} < \infty
$$

are necessary and sufficient for (1.2) to hold.

Afterwards, using the strong approximations for partial sums of i.i.d. random variables, Li *et al.* [10] obtained more general results as follows.

Theorem B *Let* $\alpha > 0$. *On a suitable probability space one can redefine* $\{\varepsilon, \varepsilon_{n,i}, 1 \le i \le [n^\alpha], n \ge 1\}$ *without changing the distribution, and there exist a sequence of independent real standard Wiener processes* $\{W_n(t), 0 \le t \le 1, n \ge 1\}$ *and a sequence of independent standard normal random variables* $\{Z_n, n \ge 1\}$, *such that the following are equivalent*:

$$
E\varepsilon = 0, \quad E\varepsilon^2 = 1, \quad and \quad E\frac{|\varepsilon|^{2p}}{\log^p |\varepsilon|} < \infty, \tag{1.3}
$$

$$\limsup_{n\to\infty} \frac{\sum_{j=1}^{[n^\alpha]} \varepsilon_{n,j}}{\sqrt{2n^\alpha \log n}} = 1 \quad a.s. \quad and \quad \liminf_{n\to\infty} \frac{\sum_{j=1}^{[n^\alpha]} \varepsilon_{n,j}}{\sqrt{2n^\alpha \log n}} = -1 \quad a.s., \tag{1.4}$$

$$\begin{cases} \lim_{n\to\infty} \inf_{f\in\mathcal{K}} \sup_{0\le t\le 1} |\frac{U_n(t)}{(2n^\alpha \log n)^{1/2}} - f(t)| = 0 \quad a.s. \quad and \\ \liminf_{n\to\infty} \sup_{0\le t\le 1} |\frac{U_n(t)}{(2n^\alpha \log n)^{1/2}} - f(t)| = 0 \quad a.s. \quad for\ every\ f \in \mathcal{K}, \end{cases} \tag{1.5}$$

$$\left| \frac{\sum_{j=1}^{[n^\alpha]} \varepsilon_{n,j}}{n^{\alpha/2}} - Z_n \right| = o\big((\log n)^{1/2}\big) \quad a.s., \tag{1.6}$$

$$\sup_{0\le t\le 1} \left| \frac{U_n(t)}{n^{\alpha/2}} - W_n(t) \right| = o\big((\log n)^{1/2}\big) \quad a.s., \tag{1.7}$$

where $p = 1 + 1/\alpha$ *and*

$$U_n(t) = \sum_{j=1}^{[[n^\alpha]t]} \varepsilon_{n,j} + \big([n^\alpha]t - [[n^\alpha]t]\big)\varepsilon_{n,[[n^\alpha]t]+1}, \quad 0 \le t \le 1, n \ge 1. \tag{1.8}$$

Remark 1.1 Theorem B also holds if (1.4) is replaced by

$$\limsup_{n\to\infty} \frac{\max_{1\le m\le[n^\alpha]} |\sum_{j=1}^m \varepsilon_{n,j}|}{\sqrt{2n^\alpha \log n}} = 1 \quad a.s. \tag{1.9}$$

In fact, by the equivalence of (1.3) and (1.4), $E\varepsilon = 0$, $E\varepsilon^2 = 1$. By Markov's inequality, for fixed $\delta > 0$, when n is large enough,

$$\max_{1\le m\le[n^\alpha]-1} P\left\{ \left| \sum_{j=m+1}^{[n^\alpha]} \varepsilon_{n,j} \right| > (\delta/2)\sqrt{2n^\alpha \log n} \right\} \le 2\delta^{-2}\big(E|\varepsilon|^2\big)(\log n)^{-1} < 1/2.$$

Hence, by Ottaviani's inequality (see Lemma 6.2 in Ledoux and Talagrand [11]), when n is large enough,

$$P\left\{ \max_{1\le m\le[n^\alpha]} \left| \sum_{j=1}^m \varepsilon_{n,j} \right| > (1+\delta)\sqrt{2n^\alpha \log n} \right\}$$

$$= P\left\{ \max_{1\le m\le[n^\alpha]} \left| \sum_{j=1}^m \varepsilon_{n,j} \right| > (1+\delta/2)\sqrt{2n^\alpha \log n} + (\delta/2)\sqrt{2n^\alpha \log n} \right\}$$

$$\le \frac{P\{| \sum_{j=1}^{[n^\alpha]} \varepsilon_{n,j}| > (1+\delta/2)\sqrt{2n^\alpha \log n}\}}{1 - \max_{1\le m\le[n^\alpha]-1} P\{| \sum_{j=m+1}^{[n^\alpha]} \varepsilon_{n,j}| > (\delta/2)\sqrt{2n^\alpha \log n}\}}$$

$$\le 2P\left\{ \left| \sum_{j=1}^{[n^\alpha]} \varepsilon_{n,j} \right| > (1+\delta/2)\sqrt{2n^\alpha \log n} \right\}.$$

By the Borel-Cantelli lemma, (1.4) implies that

$$\sum_{n=1}^\infty P\left\{ \left| \sum_{j=1}^{[n^\alpha]} \varepsilon_{n,j} \right| > (1+\delta/2)\sqrt{2n^\alpha \log n} \right\} < \infty.$$

Hence

$$\sum_{n=1}^{\infty} P\left\{ \max_{1 \le m \le [n^{\alpha}]} \left| \sum_{j=1}^{m} \varepsilon_{n,j} \right| > (1+\delta)\sqrt{2n^{\alpha}\log n} \right\} < \infty.$$

By the Borel-Cantelli lemma again

$$\limsup_{n\to\infty} \frac{\max_{1 \le m \le [n^{\alpha}]} | \sum_{j=1}^{m} \varepsilon_{n,j}|}{\sqrt{2n^{\alpha}\log n}} \le 1 + \delta \quad \text{a.s.}$$

Note that $\delta > 0$ is arbitrary and

$$\limsup_{n\to\infty} \frac{\max_{1 \le m \le [n^{\alpha}]} | \sum_{j=1}^{m} \varepsilon_{n,j}|}{\sqrt{2n^{\alpha}\log n}} \ge \limsup_{n\to\infty} \frac{| \sum_{j=1}^{[n^{\alpha}]} \varepsilon_{n,j}|}{\sqrt{2n^{\alpha}\log n}} = 1 \quad \text{a.s.},$$

(1.9) holds.

Very few results for a moving average process of an array of i.i.d. random variables are known. In this paper, we will extend Theorem B to the linear processes as follows.

Theorem 1.1 *Let $\alpha > 0$. On a suitable probability space one can redefine $\{\varepsilon, \varepsilon_{n,i}, -\infty < i < \infty, n \ge 1\}$ without changing the distribution, and there exist a sequence of independent real standard Wiener processes $\{W_n(t), 0 \le t \le 1, n \ge 1\}$ and a sequence of independent standard normal random variables $\{Z_n, n \ge 1\}$, such that the following are equivalent*:

$$E\varepsilon = 0, \qquad E\varepsilon^2 = 1, \quad and \quad E\frac{|\varepsilon|^{2p}}{\log^p |\varepsilon|} < \infty, \tag{1.10}$$

$$\limsup_{n\to\infty} \frac{\sum_{j=1}^{[n^{\alpha}]} X_{n,j}}{\sqrt{2a^2 n^{\alpha}\log n}} = 1 \quad a.s. \quad and \quad \liminf_{n\to\infty} \frac{\sum_{j=1}^{[n^{\alpha}]} X_{n,j}}{\sqrt{2a^2 n^{\alpha}\log n}} = -1 \quad a.s., \tag{1.11}$$

$$\begin{cases} \lim_{n\to\infty} \inf_{f\in\mathcal{K}} \sup_{0\le t\le 1} | \frac{S_n(t)}{(2a^2 n^{\alpha}\log n)^{1/2}} - f(t)| = 0 \quad a.s. \quad and \\ \liminf_{n\to\infty} \sup_{0\le t\le 1} | \frac{S_n(t)}{(2a^2 n^{\alpha}\log n)^{1/2}} - f(t)| = 0 \quad a.s. \quad for\ every\ f\in\mathcal{K}, \end{cases} \tag{1.12}$$

$$\left| \frac{\sum_{j=1}^{[n^{\alpha}]} X_{n,j}}{n^{\alpha/2}} - aZ_n \right| = o\big((\log n)^{1/2}\big) \quad a.s., \tag{1.13}$$

$$\sup_{0\le t\le 1} \left| \frac{S_n(t)}{n^{\alpha/2}} - aW_n(t) \right| = o\big((\log n)^{1/2}\big) \quad a.s., \tag{1.14}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{2a^2\log n}} \max_{1\le m\le n} \frac{| \sum_{j=1}^{[m^{\alpha}]} X_{m,j}|}{m^{\alpha/2}} = 1 \quad a.s., \tag{1.15}$$

where $p = 1 + 1/\alpha$ and

$$S_n(t) = \sum_{j=1}^{[[n^{\alpha}]t]} X_{n,j} + \big([n^{\alpha}]t - [[n^{\alpha}]t]\big)X_{n,[[n^{\alpha}]t]+1}, \quad 0 \le t \le 1, n \ge 1.$$

Remark 1.2 We call (1.15) the limit law of the single logarithm, an analog of the limit law of the iterated logarithm which is due to Chen [12] and developed by Li and Liang [13].

2 Proof of main result

To prove our main result, the following lemmas are needed.

Lemma 2.1 *Let $\alpha > 0$ and $\{Y, Y_{n,j}, 1 \leq j \leq [n^\alpha], n \geq 1\}$ be an array of identically distributed random variables (without independence assumption) with $EY = 0$ and $E\frac{|Y|^{2p}}{\log^p |Y|} < \infty$, $p = 1 + 1/\alpha$. Then*

$$\frac{1}{\sqrt{2n^\alpha \log n}} \max_{1 \leq j \leq [n^\alpha]} |Y_{n,j}| \to 0 \quad a.s. \tag{2.1}$$

Proof Note that $E\frac{|Y|^{2p}}{\log^p |Y|} < \infty$ is equivalent to

$$\sum_{n=1}^{\infty} n^\alpha P\{|Y| > \delta\sqrt{2n^\alpha \log n}\} < \infty, \quad \forall \delta > 0.$$

Hence $\forall \delta > 0$

$$\sum_{n=1}^{\infty} P\left\{\max_{1 \leq j \leq [n^\alpha]} |Y_{nj}| > \delta\sqrt{2n^\alpha \log n}\right\} \leq \sum_{n=1}^{\infty} n^\alpha P\{|Y| > \delta\sqrt{2n^\alpha \log n}\} < \infty,$$

which implies that (2.1) holds by the Borel-Cantelli lemma. □

Lemma 2.2 *Let $\alpha > 0$ and $\{Y, Y_{n,j}, 1 \leq j \leq [n^\alpha], n \geq 1\}$ be an array of i.i.d. random variables with $EY = 0$ and $E\frac{|Y|^{2p}}{\log^p |Y|} < \infty$, $p = 1 + 1/\alpha$. Then*

$$E \sup_{n \geq 1} \frac{\max_{1 \leq m \leq [n^\alpha]} |\sum_{j=1}^{m} Y_{n,j}|}{\sqrt{2n^\alpha \log n}} < \infty. \tag{2.2}$$

Proof Let $\{Y_n, n \geq 1\}$ be a sequence of independent random variables with common distribution as Y. By the same argument as Theorem 4 in Li and Spătaru [14] or Theorem 1.2 in Chen and Wang [15],

$$\int_M^{\infty} \sum_{n=1}^{\infty} P\left\{\left|\sum_{j=1}^{[n^\alpha]} Y_j\right| > y\sqrt{2n^\alpha \log n}\right\} dy < \infty$$

holds for some large enough $M > 0$. For all $x \geq 2M$, by Markov's inequality, when n is large enough,

$$\max_{1 \leq m \leq [n^\alpha]-1} P\left\{\left|\sum_{j=m+1}^{[n^\alpha]} Y_{n,j}\right| > x\sqrt{2n^\alpha \log n}\right\} \leq (2M)^{-2}(E|Y|^2)(2\log n)^{-1} \leq 1/2.$$

Hence by Ottaviani's inequality (see Lemma 6.2 in Ledoux and Talagrand [11]), for all $x \geq 2M$, when n is large enough,

$$P\left\{\max_{1 \leq m \leq [n^\alpha]} \left|\sum_{j=1}^{m} Y_{n,j}\right| > x\sqrt{2n^\alpha \log n}\right\} \leq 2P\left\{\left|\sum_{j=1}^{[n^\alpha]} Y_{nj}\right| > x\sqrt{2n^\alpha \log n}/2\right\}.$$

Then

$$E \sup_{n \geq 1} \frac{\max_{1 \leq m \leq [n^\alpha]} |\sum_{j=1}^{m} Y_{n,j}|}{\sqrt{2n^\alpha \log n}}$$

$$= \int_0^\infty P \left\{ \sup_{n \geq 1} \frac{\max_{1 \leq m \leq [n^\alpha]} |\sum_{j=1}^{m} Y_{n,j}|}{\sqrt{2n^\alpha \log n}} > x \right\} dx$$

$$\leq 2M + \int_{2M}^\infty \sum_{n=1}^\infty P \left\{ \max_{1 \leq m \leq [n^\alpha]} \left| \sum_{j=1}^{m} Y_{n,j} \right| > x\sqrt{2n^\alpha \log n} \right\} dx$$

$$\leq 2M + 2 \int_{2M}^\infty \sum_{n=1}^\infty P \left\{ \left| \sum_{j=1}^{[n^\alpha]} Y_{nj} \right| > x\sqrt{2n^\alpha \log n}/2 \right\} dx$$

$$= 2M + 4 \int_M^\infty \sum_{n=1}^\infty P \left\{ \left| \sum_{j=1}^{[n^\alpha]} Y_j \right| > y\sqrt{2n^\alpha \log n} \right\} dy \quad (x = 2y)$$

$$< \infty. \hspace{4cm} \square$$

Proof of Theorem 1.1 We only prove that $(1.10) \Rightarrow (1.14)$, $(1.11) \Rightarrow (1.10)$ and $(1.13) \Rightarrow (1.15) \Rightarrow (1.10)$, the proofs of $(1.14) \Rightarrow (1.13) \Rightarrow (1.11)$, $(1.14) \Rightarrow (1.12) \Rightarrow (1.11)$ are similar to those in Li *et al.* [10].

$(1.10) \Rightarrow (1.14)$. Note that

$$\sup_{0 \leq t \leq 1} \left| U_n(t) - \sum_{j=1}^{[[n^\alpha]t]} \varepsilon_{n,j} \right| \leq \max_{1 \leq j \leq [n^\alpha]} |\varepsilon_{n,j}|$$

and

$$\sup_{0 \leq t \leq 1} \left| S_n(t) - \sum_{j=1}^{[[n^\alpha]t]} X_{n,j} \right| \leq \max_{1 \leq j \leq [n^\alpha]} |X_{n,j}|,$$

where $U_n(t)$ is as Theorem B, and

$$\sup_{1 \leq t \leq 1} \left| \frac{S_n(t)}{n^{\alpha/2}} - a W_n(t) \right|$$

$$= \sup_{1 \leq t \leq 1} \left| \frac{1}{n^{\alpha/2}} \left[\left(S_n(t) - \sum_{j=1}^{[[n^\alpha]t]} X_{n,j} \right) + \left(\sum_{j=1}^{[[n^\alpha]t]} X_{n,j} - a \sum_{j=1}^{[[n^\alpha]t]} \varepsilon_{n,j} \right) \right. \right.$$

$$\left. \left. + a \left(\sum_{j=1}^{[[n^\alpha]t]} \varepsilon_{n,j} - U_n(t) \right) \right] + a \left(\frac{U_n(t)}{n^{\alpha/2}} - W_n(t) \right) \right|$$

$$\leq \frac{1}{n^{\alpha/2}} \max_{1 \leq j \leq [n^\alpha]} |X_{n,j}| + \frac{1}{n^{\alpha/2}} \sup_{1 \leq t \leq 1} \left| \sum_{j=1}^{[[n^\alpha]t]} X_{n,j} - a \sum_{j=1}^{[[n^\alpha]t]} \varepsilon_{n,j} \right|$$

$$+ \frac{|a|}{n^{\alpha/2}} \max_{1 \leq j \leq [n^\alpha]} |\varepsilon_{n,j}| + |a| \sup_{1 \leq t \leq 1} \left| \frac{U_n(t)}{n^{\alpha/2}} - W_n(t) \right|.$$

Therefore, to prove (1.14), by Lemma 2.1 and Theorem B it is enough to prove that

$$\frac{1}{\sqrt{2n^\alpha \log n}} \sup_{0 \le t \le 1} \left| \sum_{j=1}^{[[n^\alpha]t]} X_{n,j} - a \sum_{j=1}^{[[n^\alpha]t]} \varepsilon_{n,j} \right| \to 0 \quad \text{a.s.} \tag{2.3}$$

Given $m > 0$, set

$$Y_{nm}(t) = \sum_{j=1}^{[[n^\alpha]t]} \sum_{i=-m}^{m} a_i \varepsilon_{n,j-i},$$

$$\tilde{a}_m = 0, \qquad \tilde{a}_i = \sum_{k=i+1}^{m} a_k, \quad i = 0, \ldots, m-1,$$

$$\tilde{\tilde{a}}_{-m} = 0, \qquad \tilde{\tilde{a}}_i = \sum_{k=-m}^{i-1} a_k, \quad i = -m+1, -m+2, \ldots, 0,$$

$$\tilde{\varepsilon}_{nj} = \sum_{i=1}^{m} \tilde{a}_i \varepsilon_{n,j-i}, \qquad \tilde{\tilde{\varepsilon}}_{nj} = \sum_{i=-m}^{0} \tilde{\tilde{a}}_i \varepsilon_{n,j-i}.$$

Then

$$Y_{nm}(t) = \left(\sum_{i=-m}^{m} a_i \right) \sum_{j=1}^{[[n^\alpha]t]} \varepsilon_{n,j} + \left(\tilde{\varepsilon}_{n0} - \tilde{\varepsilon}_{n,[[n^\alpha]t]} + \tilde{\tilde{\varepsilon}}_{n,[[n^\alpha]t]+1} - \tilde{\tilde{\varepsilon}}_{n1} \right) \tag{2.4}$$

and

$$\sum_{j=1}^{[[n^\alpha]t]} X_{n,j} = Y_{nm}(t) + \sum_{j=1}^{[[n^\alpha]t]} \sum_{|i|>m} a_i \varepsilon_{n,j-i}. \tag{2.5}$$

For every i, by Lemma 2.1,

$$\frac{1}{\sqrt{2n^\alpha \log n}} \sup_{0 \le t \le 1} |\varepsilon_{n,[[n^\alpha]t]-i}| \le \frac{1}{\sqrt{2n^\alpha \log n}} \max_{0 \le m \le [n^\alpha]} |\varepsilon_{n,m-i}| \to 0 \quad \text{a.s.}$$

So

$$\frac{1}{\sqrt{2n^\alpha \log n}} \sup_{0 \le t \le 1} |\tilde{\varepsilon}_{n,[[n^\alpha]t]}| \to 0 \quad \text{a.s.}$$

and

$$\frac{1}{\sqrt{2n^\alpha \log n}} \sup_{0 \le t \le 1} |\tilde{\tilde{\varepsilon}}_{n,[[n^\alpha]t]+1}| \to 0 \quad \text{a.s.}$$

Furthermore

$$\lim_{n \to \infty} \frac{|\tilde{\varepsilon}_{n0}|}{\sqrt{2n^\alpha \log n}} = \lim_{n \to \infty} \frac{|\tilde{\tilde{\varepsilon}}_{n1}|}{\sqrt{2n^\alpha \log n}} = 0 \quad \text{a.s.}$$

Hence

$$\frac{1}{\sqrt{2n^\alpha \log n}} \sup_{0 \le t \le 1} |\widetilde{\varepsilon}_{n0} - \widetilde{\varepsilon}_{n,[[n^\alpha]t]} + \widetilde{\widetilde{\varepsilon}}_{n,[[n^\alpha]t]+1} - \widetilde{\widetilde{\varepsilon}}_{n1}| \to 0 \quad \text{a.s.} \tag{2.6}$$

By (2.4)-(2.6) and Remark 1.2

$$\limsup_{n \to \infty} \frac{1}{\sqrt{2n^\alpha \log n}} \sup_{0 \le t \le 1} \left| \sum_{j=1}^{[[n^\alpha]t]} X_{n,j} - a \sum_{j=1}^{[[n^\alpha]t]} \varepsilon_{n,j} \right|$$

$$= \limsup_{n \to \infty} \frac{1}{\sqrt{2n^\alpha \log n}} \sup_{0 \le t \le 1} \left| \sum_{|i|>m} a_i \sum_{j=1}^{[[n^\alpha]t]} \varepsilon_{n,j-i} - \sum_{|i|>m} a_i \sum_{j=1}^{[[n^\alpha]t]} \varepsilon_{n,j} \right|$$

$$\le \limsup_{n \to \infty} \frac{1}{\sqrt{2n^\alpha \log n}} \sup_{0 \le t \le 1} \left| \sum_{|i|>m} a_i \sum_{j=1}^{[[n^\alpha]t]} \varepsilon_{n,j-i} \right|$$

$$+ \limsup_{n \to \infty} \frac{1}{\sqrt{2n^\alpha \log n}} \sup_{0 \le t \le 1} \left| \sum_{|i|>m} a_i \sum_{j=1}^{[[n^\alpha]t]} \varepsilon_{n,j} \right|$$

$$\le \sum_{|i|>m} |a_i| \sup_{n \ge 1} \frac{1}{\sqrt{2n^\alpha \log n}} \max_{1 \le m \le [n^\alpha]} \left| \sum_{j=1}^{m} \varepsilon_{n,j-i} \right| + \left| \sum_{|i|>m} a_i \right|. \tag{2.7}$$

By the stationarity of $\{\varepsilon, \varepsilon_{n,i}, -\infty < i < \infty, n \ge 1\}$ and Lemma 2.2

$$E \sum_{i=-\infty}^{\infty} |a_i| \sup_{n \ge 1} \frac{1}{\sqrt{2n^\alpha \log n}} \max_{1 \le m \le [n^\alpha]} \left| \sum_{j=1}^{m} \varepsilon_{n,j-i} \right|$$

$$\le \sum_{i=-\infty}^{\infty} |a_i| E \sup_{n \ge 1} \frac{1}{\sqrt{2n^\alpha \log n}} \max_{1 \le m \le [n^\alpha]} \left| \sum_{j=1}^{m} \varepsilon_{n,j-i} \right|$$

$$= \left(\sum_{i=-\infty}^{\infty} |a_i| \right) E \sup_{n \ge 1} \frac{1}{\sqrt{2n^\alpha \log n}} \max_{1 \le m \le [n^\alpha]} \left| \sum_{j=1}^{m} \varepsilon_{n,j} \right|$$

$$< \infty.$$

Hence

$$\sum_{i=-\infty}^{\infty} |a_i| \sup_{n \ge 1} \frac{1}{\sqrt{2n^\alpha \log n}} \max_{1 \le m \le [n^\alpha]} \left| \sum_{j=1}^{m} \varepsilon_{n,j-i} \right| < \infty \quad \text{a.s.}$$

Therefore letting $m \to \infty$ in (2.7)

$$\limsup_{n \to \infty} \frac{1}{\sqrt{2n^\alpha \log n}} \sup_{0 \le t \le 1} \left| \sum_{j=1}^{[[n^\alpha]t]} X_{n,j} - a \sum_{j=1}^{[[n^\alpha]t]} \varepsilon_{n,j} \right| = 0 \quad \text{a.s.,}$$

i.e. (2.3) holds.

(1.11) $\Rightarrow$ (1.10). Set $a_{n,i} = \sum_{j=1}^{[n^\alpha]} a_{j-i}$. Note that $\sum_{j=1}^{[n^\alpha]} X_{n,j} = \sum_{i=-\infty}^{\infty} a_{n,i} \varepsilon_{n,i}$, and $\{\sum_{j=1}^{[n^\alpha]} X_{n,j}, n \ge 1\}$ is a sequence of independent random variables, thus by the Borel-Cantelli lemma,

(1.11) implies that

$$\sum_{n=1}^{\infty} P\left\{ \left| \sum_{i=-\infty}^{\infty} a_{n,i}\varepsilon_{n,i} \right| > M\sqrt{2a^2 n^\alpha \log n} \right\} < \infty \tag{2.8}$$

holds for any $M > 1$. Let $\{\varepsilon', \varepsilon'_{n,i}, -\infty < i < \infty, n \geq 1\}$ be an independent copy of $\{\varepsilon, \varepsilon_{n,i}, -\infty < i < \infty, n \geq 1\}$, Hence by (2.8), for $\{\varepsilon', \varepsilon'_{n,i}, -\infty < i < \infty, n \geq 1\}$

$$\sum_{n=1}^{\infty} P\left\{ \left| \sum_{i=-\infty}^{\infty} a_{n,i}\varepsilon'_{n,i} \right| > M\sqrt{2a^2 n^\alpha \log n} \right\} < \infty \tag{2.9}$$

also holds. Set $\varepsilon^s = \varepsilon - \varepsilon'$ and $\varepsilon^s_{n,i} = \varepsilon_{n,i} - \varepsilon'_{n,i}$, it is clear that $\{\varepsilon^s, \varepsilon^s_{n,i}, -\infty < i < \infty, n \geq 1\}$ is an array of independent symmetrical, identically distributed random variables. Then by (2.8) and (2.9)

$$\sum_{n=1}^{\infty} P\left\{ \left| \sum_{i=-\infty}^{\infty} a_{n,i}\varepsilon^s_{n,i} \right| > 2M\sqrt{2a^2 n^\alpha \log n} \right\} < \infty. \tag{2.10}$$

By the comparison principle (see Lemma 6.5 of Ledoux and Talagrand [11]), (2.10) implies that

$$\sum_{n=1}^{\infty} P\left\{ \left| \sum_{i=[n^\alpha/3]}^{[n^\alpha/2]} a_{n,i}\varepsilon^s_{n,i} \right| > 2M\sqrt{2a^2 n^\alpha \log n} \right\} < \infty. \tag{2.11}$$

Note that $\sum_{i=-\infty}^{\infty} a_i = a \neq 0$, then for n large enough, $|a_{n,i}| \geq |a|/2$ holds uniformly for $[n^\alpha/3] \leq i \leq [n^\alpha/2]$. By (2.11) and the comparison principle (see Lemma 6.5 of Ledoux [11]) again

$$\sum_{n=1}^{\infty} P\left\{ \left| \sum_{i=[n^\alpha/3]}^{[n^\alpha/2]} \varepsilon^s_{n,i} \right| > 4M\sqrt{2n^\alpha \log n} \right\} < \infty. \tag{2.12}$$

Then by the same argument as Li *et al.* (p.175 in [10]), (2.12) implies that $E\frac{|\varepsilon^s|^{2p}}{\log^p |\varepsilon^s|} < \infty$, and consequently $E\frac{|\varepsilon|^{2p}}{\log^p |\varepsilon|} < \infty$ holds. By (1.10) $\Rightarrow$ (1.11), we have

$$\limsup_{n\to\infty} \frac{|\sum_{i=-\infty}^{\infty} a_{n,i}(\varepsilon_{n,i} - E\varepsilon_{n,i})|}{\sqrt{2a^2 E(\varepsilon - E\varepsilon)^2 n^\alpha \log n}} = 1 \quad \text{a.s.} \tag{2.13}$$

Consequently

$$\lim_{n\to\infty} \frac{\sum_{i=-\infty}^{\infty} a_{n,i}(\varepsilon_{n,i} - E\varepsilon_{n,i})}{n^\alpha} = 0 \quad \text{a.s.}$$

By (1.11), we also have

$$\lim_{n\to\infty} \frac{\sum_{i=-\infty}^{\infty} a_{n,i}\varepsilon_{n,i}}{n^\alpha} = 0 \quad \text{a.s.,}$$

then $E\varepsilon = 0$. Comparing (2.13) and (1.11), we get $E\varepsilon^2 = 1$ immediately.

$(1.13) \Rightarrow (1.15)$. By Embrechts *et al.* ([16], p.176, line 10),

$$\lim_{n \to \infty} \frac{\max_{1 \le m \le n} Z_m}{\sqrt{2 \log n}} = 1 \quad \text{a.s.,}$$

which with (1.13) implies (1.15) immediately.

$(1.15) \Rightarrow (1.10)$. By (1.15),

$$\limsup_{n \to \infty} \frac{|\sum_{j=1}^{[n^\alpha]} X_{n,j}|}{\sqrt{2a^2 n^\alpha \log n}} \le 1 \quad \text{a.s.}$$

The rest proof is similar to that of $(1.11) \Rightarrow (1.10)$. The proof of the theorem is completed.

□

Authors' contributions
All authors have equal contributions. All authors read and approved the final manuscript.

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