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On a probability inequality of van Dam

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Abstract

This note derives an interesting probability inequality between the expectation of a conditional variance and the variance of a conditional expectation for a function of two independent random variables. In the special case of finite discrete random variables, the inequality coincides with an inequality by Feng and Tonge (2000), which extends a result by van Dam (1998).

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Keywords: probability inequality; random variable; van Dam's inequality

Let T be a continuous random variable having the probability density function ϕ defined on R. By definition

$$E(T) = \int_{-\infty}^{\infty} t\phi(t)\,dt \tag{1}$$

is the expectation of T, and

$$\sigma^2(T) = \int_{-\infty}^{\infty} \bigl(t - E(T)\bigr)^2 \phi(t)\,dt \tag{2}$$

is the variance $\sigma^2(T)$.

Let X, Y be two independent random variables with known distribution having the probability density function $\phi_1(x)$ and $\phi_2(y)$, respectively. Then the joint probability density function of X and Y is $\phi_1(x)\phi_2(y)$.

Let another random variable Z be a function of X and Y

$$Z = f(X, Y),$$

where $f(\cdot,\cdot) \in L^2(R^2)$. Then the expectation of Z is

$$E(Z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\phi_1(x)\phi_2(y)\,dx\,dy,$$

and the conditional probability density functions of Z, given the occurrence of the value x of X and y of Y, are equal to $\phi_2(y)$ and $\phi_1(x)$, respectively, such that the conditional expectations of Z are equal to

$$E(Z|X) = \int_{-\infty}^{\infty} f(x,y)\phi_2(y)\,dy, \qquad E(Z|Y) = \int_{-\infty}^{\infty} f(x,y)\phi_1(x)\,dx.$$

Furthermore, we have

$$\begin{aligned} E\bigl(E(Z|X)\bigr) &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x,y)\phi_2(y)\,dy\right)\phi_1(x)\,dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\phi_2(y)\phi_1(x)\,dx\,dy = E(Z), \\ E\bigl(Z\cdot E(Z|X)\bigr) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\left(\int_{-\infty}^{\infty} f(x,t)\phi_2(t)\,dt\right)\phi_1(x)\phi_2(y)\,dx\,dy \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x,y)\phi_2(y)\,dy\right)^2\phi_1(x)\,dx = E\bigl(\bigl[E(Z|X)\bigr]^2\bigr), \end{aligned}$$

and

$$\begin{aligned} &E\bigl(E(Z|X)\cdot E(Z|Y)\bigr) \\ &\quad= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(s,y)\phi_1(s)\,ds\right)\left(\int_{-\infty}^{\infty} f(x,t)\phi_2(t)\,dt\right)\phi_1(x)\phi_2(y)\,dx\,dy \\ &\quad= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s,y)\phi_1(s)\phi_2(y)\,ds\,dy\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,t)\phi_1(x)\phi_2(t)\,dx\,dt = \bigl[E(Z)\bigr]^2. \end{aligned}$$

Shortly,

$$E\bigl(E(Z|X)\bigr) = E(Z), \tag{3}$$

$$E\bigl(Z\cdot E(Z|X)\bigr) = E\bigl(\bigl[E(Z|X)\bigr]^2\bigr), \tag{4}$$

$$E\bigl(E(Z|X)\cdot E(Z|Y)\bigr) = \bigl[E(Z)\bigr]^2. \tag{5}$$

Theorem 1 *Let X, Y be two independent random variables and $Z = f(X,Y)$ be a function of the random variables X and Y. Then*

$$E\bigl(\sigma^2(Z|X)\bigr) \le \sigma^2\bigl(E(Z|Y)\bigr). \tag{6}$$

Proof For convenience, we set $Z_1 = E(Z|X)$ and $Z_2 = E(Z|Y)$. It follows from (3) that

$$E(Z - Z_1 - Z_2) = -E(Z).$$

Since $\sigma^2(Z - Z_1 - Z_2) = E([Z - Z_1 - Z_2]^2) - [E(Z - Z_1 - Z_2)]^2 \ge 0$,

$$\bigl[E(Z)\bigr]^2 \le E\bigl([Z - Z_1 - Z_2]^2\bigr). \tag{7}$$

From (4) and (5), we have

$$E(ZZ_1) = E\bigl(Z_1^2\bigr), \qquad E(ZZ_2) = E\bigl(Z_2^2\bigr)$$

and

$$E(Z_1Z_2) = \bigl[E(Z)\bigr]^2.$$

Therefore, we have

$$\begin{aligned}
&E\big([Z-Z_1-Z_2]^2\big)\\
&\quad = E\big(Z^2+Z_1^2+Z_2^2-2ZZ_1-2ZZ_2+2Z_1Z_2\big)\\
&\quad = E\big(Z^2\big)+E\big(Z_1^2\big)+E\big(Z_2^2\big)-2E\big(Z_1^2\big)-2E\big(Z_2^2\big)+2\big[E(Z)\big]^2\\
&\quad = E\big(Z^2\big)-E\big(Z_1^2\big)-E\big(Z_2^2\big)+2\big[E(Z)\big]^2.
\end{aligned}$$

Combining with (7), we get

$$E\big(Z_1^2\big)-\big[E(Z)\big]^2\le E\big(Z^2\big)-E\big(Z_2^2\big),$$

which is equivalent to the desired inequality. □

When X and Y are two finite discrete random variables, a discrete version of (6) is as follows:

$$\sum_{i=1}^m\left(\sum_{j=1}^n a_{ij}q_j\right)^2 p_i+\sum_{j=1}^n\left(\sum_{i=1}^m a_{ij}p_i\right)^2 q_j\le\left(\sum_{i=1}^m\sum_{j=1}^n a_{ij}p_iq_j\right)^2+\sum_{i=1}^m\sum_{j=1}^n a_{ij}^2p_iq_j, \tag{8}$$

where the nonnegative real numbers p_i $(1\le i\le m)$ and q_j $(1\le j\le n)$ satisfy $\sum_{i=1}^m p_i=1$ and $\sum_{j=1}^n q_i=1$, $A=(a_{ij})$ is a real $m\times n$ matrix. This discrete version is given by Feng and Tonge [1].

If we put $p_i=\frac{1}{m}$, $i=1,2,\ldots,m$ and $q_j=\frac{1}{n}$, $j=1,2,\ldots,\frac{1}{n}$, then (8) becomes the following van Dam inequality:

$$\sum_{i=1}^m\left(\sum_{j=1}^n a_{ij}\right)^2+\sum_{j=1}^n\left(\sum_{i=1}^m a_{ij}\right)^2\le\left(\sum_{i=1}^m\sum_{j=1}^n a_{ij}\right)^2+\sum_{i=1}^m\sum_{j=1}^n a_{ij}^2. \tag{9}$$

van Dam [2] applied his inequality to pose a related problem on the maximum irregularity of a directed graph with prescribed number of vertices and arcs.

In the special case of (0,1)-matrices, the inequality (9) reduces to the following Khinchin-type inequality:

$$m\sum_{i=1}^m r_i^2+n\sum_{j=1}^n c_j^2\le\sigma^2+mn\sigma, \tag{10}$$

where r_i, c_i, and σ denote row sums, column sums, and entries summing of an $m\times n$ (0,1) matrix, respectively, as presented by Matúš and Tuzar [3]. This inequality is an improvement (in the nonsquare case) of a result by Khinchin [4], who proved that $l\sum_{i=1}^m r_i^2+l\sum_{i=1}^m c_i^2\le\sigma^2+l^2\sigma$, where $l=\max\{m,n\}$. Khinchin [5] applied his inequality to prove a surprising number theoretic result.

Recently, Yan [6] presented another extension of (9). It is natural to ask whether the analog of (6) holds or not for three independent random variables. We have been unable to prove (or disprove) it.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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