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A study on pointwise approximation by double singular integral operators

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Dedicated to the memory of Prof. Akif Gadjiev

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Abstract
In the present work we prove the pointwise convergence and the rate of pointwise convergence for a family of singular integral operators with radial kernel in two-dimensional setting in the following form: $L_\lambda(f;x,y) = \iint_D f(t,s)H_\lambda(t-x,s-y)\,dt\,ds$, $(x,y) \in D$, where $D = \langle a,b\rangle \times \langle c,d\rangle$ is an arbitrary closed, semi-closed or open region in $\mathbb{R}^2$ and $\lambda \in \Lambda$, Λ is a set of non-negative numbers with accumulation point λ_0. Also we provide an example to justify the theoretical results.
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1 Introduction

Taberski [1] analyzed both the pointwise convergence of functions in $L_1(-\pi,\pi)$, where $L_1(-\pi,\pi)$ is the collection of all measurable functions f for which $|f|$ is integrable on $(-\pi,\pi)$ and the approximation properties of their derivatives by a two parameter family of convolution type singular integral operators $U_\lambda(f;x)$ of the form

$$U_\lambda(f;x) = \int_{-\pi}^{\pi} f(t)K_\lambda(t-x)\,dt, \quad x \in (-\pi,\pi). \tag{1.1}$$

Here, $K_\lambda(t)$ denotes a kernel fulfilling appropriate conditions with $\lambda \in \Lambda$, where Λ is a given set of non-negative numbers with accumulation point λ_0. Following this work, Gadjiev [2] proved the pointwise convergence of operators of type (1.1) at a generalized Lebesgue point and established the pertinent convergence order. Rydzewska [3] extended these results to approximation at a μ-generalized Lebesgue point. Karsli and Ibikli [4, 5] proceeded to the study of the more general integral operators defined by

$$T_\lambda(f;x) = \int_a^b f(t)K_\lambda(t-x)\,dt, \quad x \in \langle a,b\rangle, \lambda \in \Lambda \in \mathbb{R}, \tag{1.2}$$

with functions in $L_1\langle a,b\rangle$ where $\langle a,b\rangle$ is an arbitrary interval in $\mathbb{R}$ such as $[a,b]$, (a,b), $[a,b)$ or $(a,b]$.

The convergence of the other operators have been studied at characteristic points such as a generalized Lebesgue point, m-Lebesgue point, and so on, by other workers: a family of nonlinear singular integral operators [6, 7], a family of nonlinear m-singular integral operators [8], Fejer-Type singular integrals [9], moment type operators [10], a family of nonlinear Mellin type convolution operators [11], nonlinear integral operators with homogeneous kernels [12] and a family of Mellin type nonlinear m-singular integral operators [13].

Taberski [14] stepped up his analysis to two-dimensional singular integrals of the form

$$T_\lambda(f;x,y) = \iint_Q f(t,s)K_\lambda(t-x,s-y)\,dt\,ds, \quad (x,y)\in Q, \tag{1.3}$$

where Q denotes a given rectangle. His findings were later used by Siudut [15, 16] rendering significant results. Yilmaz *et al.* [17] replaced K_λ in (1.3) by a radial function H_λ as follows:

$$L_\lambda(f;x,y) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(t,s)H_\lambda(t-x,s-y)\,dt\,ds, \quad (x,y)\in\langle-\pi,\pi\rangle\times\langle-\pi,\pi\rangle. \tag{1.4}$$

The new operator approaches $f(x_0,y_0)$ as (x,y,λ) tends to (x_0,y_0,λ_0). In [18], the function $f\in L_1(\langle-\pi,\pi\rangle\times\langle-\pi,\pi\rangle)$ became $f\in L_p(D)$ where $D=\langle a,b\rangle\times\langle c,d\rangle$ is an arbitrary closed, semi-closed or open region in $\mathbb{R}^2$.

The current manuscript presents a continuation and further generalization of [18]. The main purpose is to investigate the pointwise convergence and the rate of convergence of the operators in the following form:

$$L_\lambda(f;x,y) = \iint_D f(t,s)H_\lambda(t-x,s-y)\,ds\,dt, \quad (x,y)\in D, \tag{1.5}$$

where $D=\langle a,b\rangle\times\langle c,d\rangle$ is an arbitrary closed, semi-closed or open region in $\mathbb{R}^2$, at a μ-generalized Lebesgue point of $f\in L_1(D)$ as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$. Here $L_1(D)$ is the collection of all measurable functions f for which $|f|$ is integrable on D and the kernel function $H_\lambda(s,t)$ is a radial function. As concerns the study of linear singular operators in several settings, the reader may see also *e.g.* [19–23].

The paper is organized as follows: In Section 2, we introduce the fundamental definitions. In Section 3, we give a theorem concerning the existence of the operator of type (1.5). In Section 4, we prove two theorems about the pointwise convergence of $L_\lambda(f;x,y)$ to $f(x_0,y_0)$ whenever (x_0,y_0) is a μ-generalized Lebesgue point of f in bounded region and unbounded region. In Section 5, we establish the rate of convergence of operators of type (1.5) to $f(x_0,y_0)$ as (x,y,λ) tends to (x_0,y_0,λ_0) and the paper is ended with an example to support our results.

2 Preliminaries

In this section we introduce the main definitions used in this paper.

Definition 1 A function $H\in L_1(\mathbb{R}^2)$ is said to be radial, if there exists a function $K:\mathbb{R}_0^+\to\mathbb{R}$ such that $H(t,s)=K(\sqrt{t^2+s^2})$ a.e. [24].

Definition 2 A point $(x_0,y_0)\in D$ is called a μ-generalized Lebesgue point of function $f\in L_1(D)$ if

$$\lim_{(h,k)\to(0,0)}\frac{1}{\mu_1(h)\mu_2(k)}\int_0^h\int_0^k\left|f(t+x_0,s+y_0)-f(x_0,y_0)\right|dt\,ds=0,$$

where $\mu_1(t) : \mathbb{R} \to \mathbb{R}$, absolutely continuous on $[-\delta_0, \delta_0]$, increasing on $[0, \delta_0]$ and $\mu_1(0) = 0$ and also $\mu_2(s) : \mathbb{R} \to \mathbb{R}$, absolutely continuous on $[-\delta_0, \delta_0]$, increasing on $[0, \delta_0]$ and $\mu_2(0) = 0$. Here $0 < h, k < \delta_0$ [25].

The following two examples are simple applications to a generalized Lebesgue point and μ-generalized Lebesgue point of some functions that belong to $L_1(\mathbb{R}^2)$.

Example 1 Let $g : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$g(t,s) = \begin{cases} 1, & \text{if } (t,s) = (0,0), \\ \frac{1}{\sqrt{|t|}(1+|t|)\sqrt{|s|}(1+|s|)}, & \text{if } (t,s) \in \mathbb{R}^2 \setminus (0,0). \end{cases}$$

Now, if $\mu_1(t) = t^{\frac{1}{4}} e^t$ and $\mu_2(s) = s^{\frac{1}{4}} e^s$, then the origin is a μ-generalized Lebesgue point of $g \in L_1(\mathbb{R}^2)$ but not a generalized Lebesgue point.

Example 2 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(t,s) = \begin{cases} e^{-(t+s)}, & \text{if } (t,s) \in (0,1] \times (0,1], \\ 0, & \text{if } (t,s) \in \mathbb{R}^2 \setminus (0,1] \times (0,1]. \end{cases}$$

If we take $\mu_1(t) = t^{\frac{1}{4}+1}$ and $\mu_2(s) = s^{\frac{1}{4}+1}$, then the origin is a μ-generalized Lebesgue point of $f \in L_1(\mathbb{R}^2)$. On the other hand, if we take $\alpha = \frac{1}{4}$ and $p = 1$, then the origin is also a generalized Lebesgue point. Clearly, this example shows that generalized Lebesgue points are also μ-generalized Lebesgue points.

Definition 3 (Class A) Let $H_\lambda : \mathbb{R}^2 \times \Lambda \to \mathbb{R}$ be a radial function *i.e.*, there exists a function $K_\lambda : \mathbb{R}_0^+ \times \Lambda \to \mathbb{R}$ such that the following equality holds for $(t,s) \in \mathbb{R}^2$ a.e.:

$$H_\lambda(t,s) := K_\lambda\left(\sqrt{t^2 + s^2}\right),$$

where Λ is a given set of non-negative numbers with accumulation point λ_0.

$H_\lambda(t,s)$ belongs to class A, if the following conditions are satisfied:

(a) $H_\lambda(t,s) = K_\lambda(\sqrt{t^2 + s^2})$ is even, non-negative and integrable as a function of (s,t) on $\mathbb{R}^2$ for each fixed $\lambda \in \Lambda$.
(b) For fixed $(x_0, y_0) \in D$, $K_\lambda(\sqrt{x_0^2 + y_0^2})$ tends to infinity as λ tends to λ_0.
(c) $\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)} \iint_{\mathbb{R}^2} K_\lambda(\sqrt{(t-x)^2 + (s-y)^2})\, dt\, ds = 1$.
(d) $\lim_{\lambda\to\lambda_0} [\sup_{\xi \leq \sqrt{t^2+s^2}} K_\lambda(\sqrt{t^2 + s^2})] = 0$, $\forall \xi > 0$.
(e) $\lim_{\lambda\to\lambda_0} \iint_{\xi \leq \sqrt{t^2+s^2}} K_\lambda(\sqrt{t^2 + s^2})\, dt\, ds = 0$, $\forall \xi > 0$.
(f) $K_\lambda(\sqrt{t^2 + s^2})$ is monotonically increasing with respect to t on $(-\infty, 0]$ and similarly $K_\lambda(\sqrt{t^2 + s^2})$ is monotonically increasing with respect to s on $(-\infty, 0]$ for any $\lambda \in \Lambda$. Analogously, $K_\lambda(\sqrt{t^2 + s^2})$ is bimonotonically increasing with respect to (t,s) on $[0,\infty) \times [0,\infty)$ and $(-\infty, 0] \times (-\infty, 0]$ and bimonotonically decreasing with respect to (t,s) on $[0,\infty) \times (-\infty, 0]$ and $(-\infty, 0] \times [0,\infty)$ for any $\lambda \in \Lambda$.

Throughout this paper we assume that the kernel $H_\lambda(t,s)$ belongs to class A.

3 Existence of the operator

Lemma 1 *If $f \in L_1(D)$, then the operator $L_\lambda(f;x,y)$ defines a continuous transformation over $L_1(D)$* [26].

4 Pointwise convergence

The following theorem gives a pointwise approximation of the integral operators of type (1.5) to the function f at a μ-generalized Lebesgue point of $f \in L_1(D)$ where $D = \langle a,b\rangle \times \langle c,d\rangle$ is a bounded region in $\mathbb{R}^2$, which is closed, semi-closed or open.

Theorem 1 *If (x_0,y_0) is a μ-generalized Lebesgue point of $f \in L_1(D)$, then*

$$\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)} L_\lambda(f;x,y) = f(x_0,y_0)$$

on any set Z on which the functions

$$\int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta} K_\lambda\bigl(\sqrt{(t-x)^2+(s-y)^2}\bigr)\bigl|\bigl\{\mu_1\bigl(|t-x_0|\bigr)\bigr\}'_t\bigr|\bigl|\bigl\{\mu_2\bigl(|s-y_0|\bigr)\bigr\}'_s\bigr|\,dt\,ds \tag{4.1}$$

and

$$K_\lambda(0)\mu_1\bigl(|x-x_0|\bigr) \quad \textit{and} \quad K_\lambda(0)\mu_2\bigl(|y-y_0|\bigr) \tag{4.2}$$

are bounded as (x,y,λ) tends to (x_0,y_0,λ_0).

Proof Suppose that $(x_0,y_0) \in D$ is a μ-generalized Lebesgue point of $f \in L_1(D)$. Therefore, for all given $\varepsilon > 0$, there exists $\delta > 0$ such that for all h, k satisfying $0 < h, k \le \delta$, the following inequality holds:

$$\int_{x_0}^{x_0+\delta}\int_{y_0-\delta}^{y_0}\bigl|f(t,s)-f(x_0,y_0)\bigr|\,dt\,ds < \varepsilon\mu_1(h)\mu_2(k). \tag{4.3}$$

If we follow the same strategy as used in the proof of Theorem 4.1 in [18], then we obtain

$$\begin{aligned}\bigl|L_\lambda(f;x,y)-f(x_0,y_0)\bigr| &\le \iint_D\bigl|f(t,s)-f(x_0,y_0)\bigr|K_\lambda\bigl(\sqrt{(t-x)^2+(s-y)^2}\bigr)\,dt\,ds\\ &\quad+\bigl|f(x_0,y_0)\bigr|\left|\iint_{\mathbb{R}^2}K_\lambda\bigl(\sqrt{(t-x)^2+(s-y)^2}\bigr)\,dt\,ds-1\right|\\ &\quad+\bigl|f(x_0,y_0)\bigr|\iint_{\mathbb{R}^2\setminus D}K_\lambda\bigl(\sqrt{(t-x)^2+(s-y)^2}\bigr)\,dt\,ds\\ &= I_1+I_2+I_3.\end{aligned}$$

In view of conditions (c) and (d) of class A, $I_2 \to 0$, and $I_3 \to 0$ as $\lambda \to \lambda_0$, respectively,

$$\begin{aligned}I_1 &= \left\{\iint_{D\setminus B_\delta}+\iint_{B_\delta}\right\}\bigl|f(t,s)-f(x_0,y_0)\bigr|K_\lambda\bigl(\sqrt{(t-x)^2+(s-y)^2}\bigr)\,dt\,ds\\ &= I_{11}+I_{12},\end{aligned}$$

where $B_\delta := \{(s,t) : (s-x_0)^2+(t-y_0)^2 < \delta^2, (x_0,y_0) \in D\}$.

Since $K_\lambda(\sqrt{t^2+s^2})$ is monotonically decreasing on $D\backslash B_\delta$, the inequality

$$I_{11} \le K_\lambda\big((\sqrt{2}-1)\delta/\sqrt{2}\big)\big(\|f\|_{L_1(D)} + \big|f(x_0,y_0)\big||b-a||d-c|\big)$$

holds. Hence by condition (d) of class A, $I_{11} \to 0$ as $(x,y,\lambda) \to (x_0,y_0,\lambda_0)$.

Now, we prove that I_{12} tends to zero as (x,y,λ) tends to (x_0,y_0,λ_0). It is easy to see that the following inequality holds for I_{12}, *i.e.*:

$$\begin{aligned} I_{12} &\le \left\{\int_{x_0}^{x_0+\delta}\int_{y_0-\delta}^{y_0} + \int_{x_0-\delta}^{x_0}\int_{y_0-\delta}^{y_0}\right\}\big|f(t,s)-f(x_0,y_0)\big|K_\lambda\big(\sqrt{(t-x)^2+(s-y)^2}\big)\,dt\,ds \\ &\quad + \left\{\int_{x_0-\delta}^{x_0}\int_{y_0}^{y_0+\delta} + \int_{x_0}^{x_0+\delta}\int_{y_0}^{y_0+\delta}\right\}\big|f(t,s)-f(x_0,y_0)\big|K_\lambda\big(\sqrt{(t-x)^2+(s-y)^2}\big)\,dt\,ds \\ &= I_{121}+I_{122}+I_{123}+I_{124}. \end{aligned}$$

Let us consider the integral I_{121}. In view of (4.3), for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{x_0}^{x_0+h}\int_{y_0-k}^{y_0}\big|f(t,s)-f(x_0,y_0)\big|\,dt\,ds < \varepsilon\mu_1(h)\mu_2(k)$$

holds for all $0 < h,k \le \delta$.

Let us define a new function by

$$F(t,s) := \int_{x_0}^{t}\int_{s}^{y_0}\big|f(u,v)-f(x_0,y_0)\big|\,du\,dv. \tag{4.4}$$

For all t and s satisfying $0 < t-x_0 \le \delta$ and $0 < y_0-s \le \delta$ we have

$$\big|F(t,s)\big| \le \varepsilon\mu_1(t-x_0)\mu_2(y_0-s). \tag{4.5}$$

In view of (4.4) and (4.5) and applying the method of bivariate integration by parts to I_{121} (see Theorem 2.2, p.100 in [14]) we have

$$\begin{aligned} I_{121} &\le \varepsilon\int_{x_0}^{x_0+\delta}\int_{y_0-\delta}^{y_0}\mu_1(t-x_0)\mu_2(y_0-s)\big|dK_\lambda\big(\sqrt{(t-x)^2+(s-y)^2}\big)\big| \\ &\quad + \varepsilon\mu_2(\delta)\int_{x_0}^{x_0+\delta}\mu_1(t-x_0)\big|dK_\lambda\big(\sqrt{(t-x)^2+(y_0-\delta-y)^2}\big)\big| \\ &\quad + \varepsilon\mu_1(\delta)\int_{y_0-\delta}^{y_0}\mu_2(y_0-s)\big|dK_\lambda\big(\sqrt{(x_0+\delta-x)^2+(s-y)^2}\big)\big| \\ &\quad + \varepsilon\mu_1(\delta)\mu_2(\delta)K_\lambda\big(\sqrt{(x_0+\delta-x)^2+(y_0-\delta-y)^2}\big). \end{aligned}$$

Let us define the variations:

$$B_1(u,v) := \begin{cases} \bigvee_u^{x_0+\delta-x}\bigvee_{y_0-\delta-y}^{v}(K_\lambda(\sqrt{t^2+s^2})), & x_0-x \le u < x_0+\delta-x, \\ & y_0-\delta-y < v \le y_0-y, \\ 0, & \text{otherwise}, \end{cases}$$

$$B_2(u) := \begin{cases} \bigvee_u^{x_0+\delta-x}(K_\lambda(\sqrt{t^2+(y_0-\delta-y)^2})), & x_0-x \le u < x_0+\delta-x, \\ 0, & \text{otherwise}, \end{cases}$$

$$B_3(v) := \begin{cases} \bigvee_{y_0-\delta-y}^{v}(K_\lambda(\sqrt{(x_0-x+\delta)^2+s^2})), & y_0-\delta-y < v \le y_0-y, \\ 0, & \text{otherwise.} \end{cases}$$

Taking the above variations into account and applying the method of bivariate integration by parts to the last inequality, we have

$$\begin{aligned} I_{121} &\le -\varepsilon \int_{x_0-x}^{x_0-x+\delta} \int_{y_0-y-\delta}^{y_0-y} \big[B_1(t,s) + B_2(t) + B_3(s) + K_\lambda\big(\sqrt{(x_0-x+\delta)^2+(y_0-\delta-y)^2}\big)\big] \\ &\quad \times \big\{\mu_1(t-x_0+x)\big\}_t'\big\{\mu_2(y_0-s-y)\big\}_s'\,dt\,ds \\ &= \varepsilon(i_1+i_2+i_3+i_4). \end{aligned}$$

Remark 1 If the function $g:\mathbb{R}^2 \to \mathbb{R}$ is bimonotonic on $[\alpha_1,\alpha_2]\times[\beta_1,\beta_2]\subset\mathbb{R}^2$ then the equality given by

$$V\big(g;[\alpha_1,\alpha_2]\times[\beta_1,\beta_2]\big) = \bigvee_{\alpha_1}^{\alpha_2}\bigvee_{\beta_1}^{\beta_2}\big(g(t,s)\big) = \big|g(\alpha_1,\beta_1)-g(\alpha_1,\beta_2)-g(\alpha_2,\beta_1)+g(\alpha_2,\beta_2)\big|$$

holds [14, 27].

Splitting i_1 into two parts yields

$$\begin{aligned} i_1 &= -\left\{\int_{x_0-x}^{x_0-x+\delta}\int_0^{y_0-y} + \int_{x_0-x}^{x_0-x+\delta}\int_{y_0-y-\delta}^{0}\right\} B_1(t,s)\big\{\mu_1(t-x_0+x)\big\}_t'\big\{\mu_2(y_0-s-y)\big\}_s'\,dt\,ds \\ &= i_{11}+i_{12}. \end{aligned}$$

Using Remark 1 and condition (f) of class A, we can write for i_{11}

$$\begin{aligned} i_{11} &= -\int_{x_0-x}^{x_0-x+\delta}\int_0^{y_0-y}\left[\left\{\bigvee_{t}^{x_0+\delta-x}\bigvee_{y_0-\delta-y}^{0} + \bigvee_{t}^{x_0+\delta-x}\bigvee_{0}^{s}\right\} K_\lambda\big(\sqrt{u^2+v^2}\big)\right] \\ &\quad \times \big\{\mu_1(t-x_0+x)\big\}_t'\big\{\mu_2(y_0-s-y)\big\}_s'\,dt\,ds \\ &= \int_{x_0-x}^{x_0-x+\delta}\int_0^{y_0-y}\big(K_\lambda\big(\sqrt{t^2+(y_0-\delta-y)^2}\big) - K_\lambda\big(\sqrt{s^2+(x_0+\delta-x)^2}\big) - 2K_\lambda\big(|t|\big) \\ &\quad + K_\lambda\big(\sqrt{t^2+s^2}\big) + 2K_\lambda\big(|x_0+\delta-x|\big) - K_\lambda\big(\sqrt{(y_0-\delta-y)^2+(x_0+\delta-x)^2}\big)\big) \\ &\quad \times \big\{\mu_1(t-x_0+x)\big\}_t'\big\{\mu_2(y_0-s-y)\big\}_s'\,dt\,ds. \end{aligned}$$

Using the same method for i_{12}, we have

$$\begin{aligned} i_{12} &= \int_{x_0-x}^{x_0-x+\delta}\int_{y_0-y-\delta}^{0}\big(K_\lambda\big(\sqrt{t^2+(y_0-\delta-y)^2}\big) + K_\lambda\big(\sqrt{s^2+(x_0+\delta-x)^2}\big) \\ &\quad - K_\lambda\big(\sqrt{t^2+s^2}\big) - K_\lambda\big(\sqrt{(y_0-\delta-y)^2+(x_0+\delta-x)^2}\big)\big) \\ &\quad \times \big\{\mu_1(t-x_0+x)\big\}_t'\big\{\mu_2(y_0-s-y)\big\}_s'\,dt\,ds. \end{aligned}$$

Making similar calculations for i_2 and i_3 and collecting the obtained terms, we may write

$$\begin{aligned} i_1 + i_2 + i_3 + i_4 = {} & -\int_{x_0-x}^{x_0-x+\delta}\int_{y_0-y-\delta}^{0} K_\lambda\bigl(\sqrt{t^2+s^2}\bigr)\bigl\{\mu_1(t-x_0+x)\bigr\}_t'\bigl\{\mu_2(y_0-s-y)\bigr\}_s'\,dt\,ds \\ & + \int_{x_0-x}^{x_0-x+\delta}\int_{0}^{y_0-y} \bigl(K_\lambda\bigl(\sqrt{t^2+s^2}\bigr) - 2K_\lambda\bigl(|t|\bigr)\bigr) \\ & \times \bigl\{\mu_1(t-x_0+x)\bigr\}_t'\bigl\{\mu_2(y_0-s-y)\bigr\}_s'\,dt\,ds. \end{aligned}$$

Hence the following inequality holds for I_{121}:

$$\begin{aligned} I_{121} \le {} & \varepsilon\int_{x_0}^{x_0+\delta}\int_{y_0-\delta}^{y_0} K_\lambda\bigl(\sqrt{(t-x)^2+(s-y)^2}\bigr)\bigl|\bigl\{\mu_1(t-x_0)\bigr\}_t'\bigr|\bigl|\bigl\{\mu_2(y_0-s)\bigr\}_s'\bigr|\,dt\,ds \\ & + 2\varepsilon K_\lambda(0)\mu_1(\delta)\mu_2\bigl(|y_0-y|\bigr). \end{aligned}$$

By a similar argument to the evaluation of the integral I_{121}, we can easily obtain the following inequalities for I_{122}, I_{123}, and I_{124}:

$$\begin{aligned} I_{122} \le {} & \varepsilon\int_{x_0-\delta}^{x_0}\int_{y_0-\delta}^{y_0} K_\lambda\bigl(\sqrt{(t-x)^2+(s-y)^2}\bigr)\bigl|\bigl\{\mu_1(x_0-t)\bigr\}_t'\bigr|\bigl|\bigl\{\mu_2(y_0-s)\bigr\}_s'\bigr|\,dt\,ds \\ & + 2\varepsilon K_\lambda(0)\bigl(\mu_1(\delta)\mu_2\bigl(|y_0-y|\bigr) + \mu_2(\delta)\mu_1\bigl(|x_0-x|\bigr)\bigr) \\ & + 4\varepsilon K_\lambda(0)\mu_1\bigl(|x_0-x|\bigr)\mu_2\bigl(|y_0-y|\bigr), \\ I_{123} \le {} & \varepsilon\int_{x_0-\delta}^{x_0}\int_{y_0}^{y_0+\delta} K_\lambda\bigl(\sqrt{(t-x)^2+(s-y)^2}\bigr)\bigl|\bigl\{\mu_1(x_0-t)\bigr\}_t'\bigr|\bigl|\bigl\{\mu_2(s-y_0)\bigr\}_s'\bigr|\,dt\,ds \\ & + 2\varepsilon K_\lambda(0)\mu_2(\delta)\mu_1\bigl(|x_0-x|\bigr), \\ I_{124} \le {} & \varepsilon\int_{x_0}^{x_0+\delta}\int_{y_0}^{y_0+\delta} K_\lambda\bigl(\sqrt{(t-x)^2+(s-y)^2}\bigr)\bigl|\bigl\{\mu_1(t-x_0)\bigr\}_t'\bigr|\bigl|\bigl\{\mu_2(s-y_0)\bigr\}_s'\bigr|\,dt\,ds. \end{aligned}$$

Hence the following inequality is obtained for I_{12} *i.e.*:

$$\begin{aligned} I_{12} \le {} & \varepsilon\int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta} K_\lambda\bigl(\sqrt{(t-x)^2+(s-y)^2}\bigr)\bigl|\bigl\{\mu_1\bigl(|x_0-t|\bigr)\bigr\}_t'\bigr|\bigl|\bigl\{\mu_2\bigl(|y_0-s|\bigr)\bigr\}_s'\bigr|\,dt\,ds \\ & + 4\varepsilon K_\lambda(0)\bigl(\mu_1(\delta)\mu_2\bigl(|y_0-y|\bigr) + \mu_2(\delta)\mu_1\bigl(|x_0-x|\bigr)\bigr) \\ & + 4\varepsilon K_\lambda(0)\mu_1\bigl(|x_0-x|\bigr)\mu_2\bigl(|y_0-y|\bigr). \end{aligned}$$

The remaining part of the proof is obvious by the hypotheses (4.1) and (4.2). Hence $I_{12} \to 0$ as $\lambda \to \lambda_0$. Thus the proof is completed. □

The following theorem gives a pointwise approximation of the integral operators of type (1.5) to the function f at a μ-generalized Lebesgue point of $f \in L_1(\mathbb{R}^2)$.

Theorem 2 *Suppose that the hypothesis of Theorem* 1 *is satisfied for* $D = \mathbb{R}^2$. *If* (x_0, y_0) *is a* μ*-generalized Lebesgue point of* $f \in L_1(\mathbb{R}^2)$ *then*

$$\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)} L_\lambda(f;x,y) = f(x_0,y_0).$$

Proof The proof of this theorem is quite similar to the proof of Theorem 4.2 in [18] and thus is omitted. □

5 Rate of convergence

In this section, we give a theorem concerning the rate of pointwise convergence.

Theorem 3 *Suppose that the hypotheses of Theorem* 1 *and Theorem* 2 *are satisfied. Let*

$$\Delta(\lambda,\delta,x,y)=\int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta}K_\lambda\bigl(\sqrt{(t-x)^2+(s-y)^2}\bigr)\\ \times\bigl|\bigl\{\mu_1\bigl(|t-x_0|\bigr)\bigr\}_t'\bigr|\bigl|\bigl\{\mu_2\bigl(|s-y_0|\bigr)\bigr\}_s'\bigr|\,dt\,ds$$

for $\delta>0$ *and the following assumptions be satisfied*:

(i) $\Delta(\lambda,\delta,x,y)\to 0$ *as* $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$ *for some* $\delta>0$.

(ii) *For every* $\xi>0$

$$K_\lambda(\xi)=o\bigl(\Delta(\lambda,\delta,x,y)\bigr)$$

as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$.

(iii) *For every* $\xi>0$

$$\iint_{\xi\le\sqrt{s^2+t^2}}K_\lambda\bigl(\sqrt{t^2+s^2}\bigr)\,dt\,ds=o\bigl(\Delta(\lambda,\delta,x,y)\bigr)$$

as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$.

Then at each μ*-generalized Lebesgue point of* $f\in L_1(\mathbb{R}^2)$ *we have as* $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$

$$\bigl|L_\lambda(f;x,y)-f(x_0,y_0)\bigr|=o\bigl(\Delta(\lambda,\delta,x,y)\bigr).$$

Proof Under the hypotheses of Theorem 1 and Theorem 2 we can write

$$\begin{aligned}&\bigl|L_\lambda(f;x,y)-f(x_0,y_0)\bigr|\\&\quad\le\varepsilon\int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta}K_\lambda\bigl(\sqrt{(t-x)^2+(s-y)^2}\bigr)\bigl|\bigl\{\mu_1\bigl(|x_0-t|\bigr)\bigr\}_t'\bigr|\bigl|\bigl\{\mu_2\bigl(|y_0-s|\bigr)\bigr\}_s'\bigr|\,dt\,ds\\&\qquad+4\varepsilon K_\lambda(0)\bigl(\mu_1(\delta)\mu_2\bigl(|y_0-y|\bigr)+\mu_2(\delta)\mu_1\bigl(|x_0-x|\bigr)\bigr)\\&\qquad+4\varepsilon K_\lambda(0)\mu_1\bigl(|x_0-x|\bigr)\mu_2\bigl(|y_0-y|\bigr)\\&\qquad+K_\lambda\bigl((\sqrt{2}-1)\delta/\sqrt{2}\bigr)\|f\|_{L_1(\mathbb{R}^2)}+\bigl|f(x_0,y_0)\bigr|\iint_{(\sqrt{2}-1)\delta/\sqrt{2}\le\sqrt{s^2+t^2}}K_\lambda\bigl(\sqrt{t^2+s^2}\bigr)\,dt\,ds\\&\qquad+\bigl|f(x_0,y_0)\bigr|\biggl|\iint_{\mathbb{R}^2}K_\lambda\bigl(\sqrt{(t-x)^2+(s-y)^2}\bigr)\,dt\,ds-1\biggr|.\end{aligned}$$

From (i)-(iii) and using conditions of class A, we have the desired result *i.e.*,

$$\bigl|L_\lambda(f;x,y)-f(x_0,y_0)\bigr|=o\bigl(\Delta(\lambda,\delta,x,y)\bigr).$$

□

Example 3 Let $\Lambda = (0,\infty)$, $\lambda_0 = 0$, and

$$H_\lambda(t,s) = \frac{1}{4\pi\lambda} e^{\frac{-(t^2+s^2)}{4\lambda}}.$$

To verify that $H_\lambda(t,s)$ satisfies the hypotheses of Theorem 1 and Theorem 2 see [15].

Let $(x_0,y_0) = (0,0)$, $\mu_1(t) = t$ and $\mu_2(s) = s$. Hence we obtain

$$\begin{aligned}\Delta(\lambda,\delta,x,y) &= \int_{-\delta}^{+\delta}\int_{-\delta}^{+\delta} \frac{1}{4\pi\lambda} e^{\frac{-((t-x)^2+(s-y)^2)}{4\lambda}}\,dt\,ds \\ &= \frac{1}{2}\left(Erf\left(\frac{\delta-x}{2\sqrt{\lambda}}\right) + Erf\left(\frac{x}{2\sqrt{\lambda}}\right)\right)\left(Erf\left(\frac{\delta-y}{2\sqrt{\lambda}}\right) + Erf\left(\frac{y}{2\sqrt{\lambda}}\right)\right).\end{aligned}$$

In order to find for which $\delta > 0$ the condition (i) in Theorem 3 is satisfied, let $\Delta(\lambda,\delta,x,y) \to 0$ as $(x,y,\lambda) \to (0,0,0)$. Hence

$$\lim_{(x,y,\lambda)\to(0,0,0)} \Delta(\lambda,\delta,x,y) = 0$$

if and only if $\delta^2 = o(\lambda)$. Consequently, the following equality holds:

$$\Delta(\lambda,\delta,x,y) = O(\lambda).$$

Finally, in order to get finite limit values from the expressions

$$\begin{aligned}\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)} K_\lambda(0)\mu_1\big(|x_0-x|\big) &= \lim_{(x,y,\lambda)\to(0,0,0)} \frac{1}{4\pi\lambda} e^{\frac{-(x^2+y^2)}{4\lambda}}|x|, \\ \lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)} K_\lambda(0)\mu_2\big(|y_0-y|\big) &= \lim_{(x,y,\lambda)\to(0,0,0)} \frac{1}{4\pi\lambda} e^{\frac{-(x^2+y^2)}{4\lambda}}|y|,\end{aligned}$$

the rates of convergence $\frac{1}{4\pi\lambda}e^{\frac{-(x^2+y^2)}{4\lambda}} \to \infty$ and $|x| \to 0$ and also $\frac{1}{4\pi\lambda}e^{\frac{-(x^2+y^2)}{4\lambda}} \to \infty$ and $|y| \to 0$ must be equivalent. Note that $|x| = |y| = O(\lambda)$.

Hence

$$\big|L_\lambda(f;x,y) - f(x_0,y_0)\big| = o\big(\Delta(\lambda,\delta,x,y)\big) = o(\lambda).$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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