

RESEARCH

Open Access

# On function spaces with fractional Fourier transform in weighted Lebesgue spaces

Erdem Toksoy and Ayşe Sandıkçı\*

Dedicated to Professor Ravi P Agarwal.

\*Correspondence:  
ayses@omu.edu.tr  
Department of Mathematics,  
Faculty of Arts and Sciences,  
Ondokuz Mayıs University, Samsun,  
Turkey

## Abstract

Let  $w$  and  $\omega$  be weight functions on  $\mathbb{R}^d$ . In this work, we define  $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$  to be the vector space of  $f \in L_w^1(\mathbb{R}^d)$  such that the fractional Fourier transform  $F_\alpha f$  belongs to  $L_\omega^p(\mathbb{R}^d)$  for  $1 \leq p < \infty$ . We endow this space with the sum norm  $\|f\|_{A_{\alpha,p}^{w,\omega}} = \|f\|_{1,w} + \|F_\alpha f\|_{p,\omega}$  and show that  $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$  becomes a Banach space and invariant under time-frequency shifts. Further we show that the mapping  $y \rightarrow T_y f$  is continuous from  $\mathbb{R}^d$  into  $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ , the mapping  $z \rightarrow M_z f$  is continuous from  $\mathbb{R}^d$  into  $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$  and  $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$  is a Banach module over  $L_w^1(\mathbb{R}^d)$  with  $\Theta$  convolution operation. At the end of this work, we discuss inclusion properties of these spaces.

**Keywords:** fractional Fourier transform; convolution; Banach module

## 1 Introduction

In this work, for any function  $f: \mathbb{R}^d \rightarrow \mathbb{C}$ , the translation and modulation operator are defined as  $T_x f(t) = f(t - x)$  and  $M_w f(t) = e^{iwt} f(t)$  for all  $y, w \in \mathbb{R}^d$ , respectively. Also we write the Lebesgue space  $(L^p(\mathbb{R}^d), \|\cdot\|_p)$ , for  $1 \leq p < \infty$ . Let  $w$  be a weight function on  $\mathbb{R}^d$ , that is, a measurable and locally bounded function  $w$  satisfying  $w(x) \geq 1$  and  $w(x + y) \leq w(x)w(y)$  for all  $x, y \in \mathbb{R}^d$ . We define, for  $1 \leq p < \infty$ ,

$$L_w^p(\mathbb{R}^d) = \{f | fw \in L^p(\mathbb{R}^d)\}.$$

It is well known that  $L_w^p(\mathbb{R}^d)$  is a Banach space under the norm  $\|f\|_{p,w} = \|fw\|_p$ .

Let  $w_1$  and  $w_2$  are two weight functions. We say that  $w_1 < w_2$  if there exists  $c > 0$ , such that  $w_1(x) \leq cw_2(x)$  for all  $x \in \mathbb{R}^d$  [1, 2].

The Fourier transform  $\hat{f}$  (or  $\mathcal{F}f$ ) of  $f \in L^1(\mathbb{R})$  is given by

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-iwt} dt.$$

The fractional Fourier transform is a generalization of the Fourier transform with a parameter  $\alpha$  and can be interpreted as a rotation by an angle  $\alpha$  in the time-frequency plane. The fractional Fourier transform with angle  $\alpha$  of a function  $f$  is defined by

$$\mathcal{F}_\alpha f(u) = \int_{-\infty}^{+\infty} K_\alpha(u, t) f(t) dt,$$

where

$$K_{\alpha}(u, t) = \begin{cases} \sqrt{\frac{1-i \cot \alpha}{2\pi}} e^{i(\frac{u^2+t^2}{2}) \cot \alpha - iut \operatorname{cosec} \alpha}, & \text{if } \alpha \text{ is not multiple of } \pi, \\ \delta(t-u), & \text{if } \alpha = 2k\pi, k \in \mathbb{Z}, \\ \delta(t+u), & \text{if } \alpha = (2k+1)\pi, k \in \mathbb{Z}, \end{cases}$$

and  $\delta$  is a Dirac delta function. The fractional Fourier transform with  $\alpha = \frac{\pi}{2}$  corresponds to the Fourier transform [3–9].

The fractional Fourier transform can be extended to higher dimensions as [9]:

$$\begin{aligned} & (\mathcal{F}_{\alpha_1, \dots, \alpha_n} f)(u_1, \dots, u_n) \\ &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} K_{\alpha_1, \dots, \alpha_n}(u_1, \dots, u_n; t_1, \dots, t_n) f(t_1, \dots, t_n) dt_1 \cdots dt_n, \end{aligned}$$

or shortly

$$\mathcal{F}_{\alpha} f(u) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} K_{\alpha}(u, t) f(t) dt,$$

where

$$K_{\alpha}(u, t) = K_{\alpha_1, \dots, \alpha_n}(u_1, \dots, u_n; t_1, \dots, t_n) = K_{\alpha_1}(u_1, t_1) K_{\alpha_2}(u_2, t_2) \cdots K_{\alpha_n}(u_n, t_n).$$

In this work we define the function spaces with fractional Fourier transform in weighted Lebesgue spaces and discuss some properties of these spaces.

## 2 On function spaces with fractional Fourier transform in weighted Lebesgue spaces

**Definition 1** Let  $w$  and  $\omega$  be weight functions on  $\mathbb{R}^d$  and  $1 \leq p < \infty$ . The space  $A_{\alpha, p}^{w, \omega}(\mathbb{R}^d)$  consist of all  $f \in L_w^1(\mathbb{R}^d)$  such that  $\mathcal{F}_{\alpha} f \in L_{\omega}^p(\mathbb{R}^d)$ . The norm on the vector space  $A_{\alpha, p}^{w, \omega}(\mathbb{R}^d)$  is

$$\|f\|_{A_{\alpha, p}^{w, \omega}} = \|f\|_{1, w} + \|\mathcal{F}_{\alpha} f\|_{p, \omega}.$$

**Theorem 2**  $(A_{\alpha, p}^{w, \omega}(\mathbb{R}^d), \|\cdot\|_{A_{\alpha, p}^{w, \omega}})$  is a Banach space for  $1 \leq p < \infty$ .

*Proof* Let  $(f_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $A_{\alpha, p}^{w, \omega}(\mathbb{R}^d)$ . Thus  $(f_n)_{n \in \mathbb{N}}$  and  $(\mathcal{F}_{\alpha} f_n)_{n \in \mathbb{N}}$  are Cauchy sequences in  $L_w^1(\mathbb{R}^d)$  and  $L_{\omega}^p(\mathbb{R}^d)$ , respectively. Since  $L_w^1(\mathbb{R}^d)$  and  $L_{\omega}^p(\mathbb{R}^d)$  are Banach spaces, there exist  $f \in L_w^1(\mathbb{R}^d)$  and  $g \in L_{\omega}^p(\mathbb{R}^d)$  such that  $\|f_n - f\|_{1, w} \rightarrow 0$ ,  $\|\mathcal{F}_{\alpha} f_n - g\|_{p, \omega} \rightarrow 0$  and hence  $\|f_n - f\|_1 \rightarrow 0$  and  $\|\mathcal{F}_{\alpha} f_n - g\|_p \rightarrow 0$ . Then  $(\mathcal{F}_{\alpha} f_n)_{n \in \mathbb{N}}$  has a subsequence  $(\mathcal{F}_{\alpha} f_{n_k})_{n_k \in \mathbb{N}}$  that converges pointwise to  $g$  almost everywhere. Also it is easy to see that  $\|f_{n_k} - f\|_1 \rightarrow 0$ . Then we have

$$\begin{aligned} |\mathcal{F}_{\alpha} f(u) - g(u)| &\leq |\mathcal{F}_{\alpha}(f_{n_k} - f)(u)| + |\mathcal{F}_{\alpha} f_{n_k}(u) - g(u)| \\ &\leq \prod_{j=1}^d \left| \sqrt{\frac{1-i \cot \alpha_j}{2\pi}} \right| \end{aligned}$$

$$\begin{aligned}
& \times \int_{\mathbb{R}^d} |(f_{n_k} - f)(t)| \left| e^{\sum_{j=1}^d (\frac{i}{2}(u_j^2 + t_j^2) \cot \alpha_j - i u_j t_j \operatorname{cosec} \alpha_j)} \right| dt \\
& + |\mathcal{F}_\alpha f_{n_k}(u) - g(u)| \\
& = \prod_{j=1}^d \left| \sqrt{\frac{1 - i \cot \alpha_j}{2\pi}} \right| \|f_{n_k} - f\|_1 + |\mathcal{F}_\alpha f_{n_k}(u) - g(u)|.
\end{aligned}$$

From this inequality, we obtain  $\mathcal{F}_\alpha f = g$  almost everywhere. Thus  $\|f_n - f\|_{A_{\alpha,p}^{w,\omega}} \rightarrow 0$  and  $f \in A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ . Hence  $(A_{\alpha,p}^{w,\omega}(\mathbb{R}^d), \|\cdot\|_{A_{\alpha,p}^{w,\omega}})$  is a Banach space.  $\square$

The following proposition is generalization of the one-dimensional and two-dimensional versions. The proof of this proposition is very similar to the proofs of one-dimensional and two-dimensional versions in [3, 5, 10, 11], and we omit the details.

**Proposition 3** Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ , where  $\alpha_i \neq k\pi$  for each index  $i$  with  $1 \leq i \leq d$  and  $k \in \mathbb{Z}$ . Then

$$(1) \quad \mathcal{F}_\alpha(T_y f)(u) = e^{\sum_{j=1}^d (\frac{i}{2} y_j^2 \sin \alpha_j \cos \alpha_j - i u_j y_j \sin \alpha_j)} \mathcal{F}_\alpha f(u_1 - y_1 \cos \alpha_1, \dots, u_d - y_d \cos \alpha_d) \quad (1)$$

for all  $f \in L^1(\mathbb{R}^d)$  and  $y \in \mathbb{R}^d$ ;

$$(2) \quad \mathcal{F}_\alpha(M_v f)(u) = e^{\sum_{j=1}^d (-\frac{i}{2} v_j^2 \sin \alpha_j \cos \alpha_j + i u_j v_j \cos \alpha_j)} \mathcal{F}_\alpha f(u_1 - v_1 \sin \alpha_1, \dots, u_d - v_d \sin \alpha_d)$$

for all  $f \in L^1(\mathbb{R}^d)$  and  $v \in \mathbb{R}^d$ .

**Theorem 4** Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ , where  $\alpha_i \neq k\pi$  for each index  $i$  with  $1 \leq i \leq d$  and  $k \in \mathbb{Z}$ .

- (1) Let  $1 \leq p < \infty$ ,  $w$  and  $\omega$  be weight functions on  $\mathbb{R}^d$ . Then the space  $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$  is translation invariant.
- (2) Let  $\omega$  be a bounded weight function on  $\mathbb{R}^d$ . Then the mapping  $y \rightarrow T_y f$  of  $\mathbb{R}^d$  into  $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$  is continuous.

*Proof* (1) Let  $f \in A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ . Then  $f \in L_w^1(\mathbb{R}^d)$  and  $\mathcal{F}_\alpha f \in L_\omega^p(\mathbb{R}^d)$ . It is well known that the space  $L_w^1(\mathbb{R}^d)$  is translation invariant and holds  $\|T_y f\|_{1,w} \leq w(y) \|f\|_{1,w}$  for all  $y \in \mathbb{R}^d$  [12]. Let  $b = (y_1 \cos \alpha_1, \dots, y_d \cos \alpha_d)$ . By using the equality (1), we get

$$\begin{aligned}
\|\mathcal{F}_\alpha(T_y f)\|_{p,\omega} &= \left( \int_{\mathbb{R}^d} |\mathcal{F}_\alpha(T_y f)(u)|^p \omega^p(u) du \right)^{\frac{1}{p}} \\
&= \left( \int_{\mathbb{R}^d} |\mathcal{F}_\alpha f(u_1 - y_1 \cos \alpha_1, \dots, u_d - y_d \cos \alpha_d)|^p \right. \\
&\quad \times \left. |e^{\sum_{j=1}^d (\frac{i}{2} y_j^2 \sin \alpha_j \cos \alpha_j - i u_j y_j \sin \alpha_j)}|^p \omega^p(u) du \right)^{\frac{1}{p}} \\
&\leq \left( \int_{\mathbb{R}^d} |\mathcal{F}_\alpha f(u - b)|^p \omega^p(u - b) \omega^p(b) du \right)^{\frac{1}{p}} \\
&= \omega(b) \|\mathcal{F}_\alpha f\|_{p,\omega}
\end{aligned}$$

for all  $y \in \mathbb{R}^d$ . Hence, we have

$$\|T_y f\|_{A_{\alpha,p}^{w,\omega}} \leq w(y)\|f\|_{1,w} + \omega(b)\|\mathcal{F}_\alpha f\|_{p,\omega} < \infty.$$

This means that  $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$  is translation invariant.

(2) Let  $f \in A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ . We will show that if  $\lim_{n \rightarrow \infty} y_n = 0$  for any sequence  $(y_n)_{n \in \mathbb{N}} \subset \mathbb{R}^d$ , then  $\lim_{n \rightarrow \infty} T_{y_n} f = f$ , which will complete the proof. It is well known that the mapping  $y \rightarrow T_y f$  is continuous from  $\mathbb{R}^d$  into  $L_w^1(\mathbb{R}^d)$  (see [12]). Thus, we have

$$\|T_{y_n} f - f\|_{1,w} \rightarrow 0 \quad (2)$$

as  $n \rightarrow \infty$ . Also,

$$\begin{aligned} \|\mathcal{F}_\alpha(T_{y_n} f - f)\|_{p,\omega} &= \|\mathcal{F}_\alpha(T_{y_n} f) - \mathcal{F}_\alpha f\|_{p,\omega} \\ &= \left\| e^{\sum_{j=1}^d \left(\frac{i}{2}(y_n^j)^2 \sin \alpha_j \cos \alpha_j - i u_j y_n^j \sin \alpha_j\right)} T_{(y_n^1 \cos \alpha_1, \dots, y_n^d \cos \alpha_d)}(\mathcal{F}_\alpha f) - \mathcal{F}_\alpha f \right\|_{p,\omega} \\ &\leq \left\| (T_{(y_n^1 \cos \alpha_1, \dots, y_n^d \cos \alpha_d)}(\mathcal{F}_\alpha f) - \mathcal{F}_\alpha f) \right\|_{p,\omega} \\ &\quad + \left\| \left( e^{\sum_{j=1}^d \left(\frac{i}{2}(y_n^j)^2 \sin \alpha_j \cos \alpha_j - i u_j y_n^j \sin \alpha_j\right)} - 1 \right) \mathcal{F}_\alpha f \right\|_{p,\omega}. \end{aligned}$$

Since  $\mathcal{F}_\alpha f \in L_\omega^p(\mathbb{R}^d)$ , the mapping  $y \rightarrow T_y(\mathcal{F}_\alpha f)$  is continuous from  $\mathbb{R}^d$  into  $L_\omega^p(\mathbb{R}^d)$  for all  $y \in \mathbb{R}^d$  [12]. Then we obtain  $\|T_{(y_n^1 \cos \alpha_1, \dots, y_n^d \cos \alpha_d)}(\mathcal{F}_\alpha f) - \mathcal{F}_\alpha f\|_{p,\omega} \rightarrow 0$  as  $n \rightarrow \infty$ . Now let  $h_{y_n}(u) = |e^{\sum_{j=1}^d \left(\frac{i}{2}(y_n^j)^2 \sin \alpha_j \cos \alpha_j - i u_j y_n^j \sin \alpha_j\right)} - 1| |\mathcal{F}_\alpha f(u)|$ . Since  $\lim_{n \rightarrow \infty} y_n = 0$  and  $\omega$  is a bounded weight function on  $\mathbb{R}^d$ , we see that  $\lim_{n \rightarrow \infty} h_{y_n}^p(u) \omega^p(u) = 0$  for all  $u \in \mathbb{R}^d$ . Also, since

$$h_{y_n}(u) = \left| e^{\sum_{j=1}^d \left(\frac{i}{2}(y_n^j)^2 \sin \alpha_j \cos \alpha_j - i u_j y_n^j \sin \alpha_j\right)} - 1 \right| |\mathcal{F}_\alpha f(u)| \leq 2 |\mathcal{F}_\alpha f(u)|$$

and  $\mathcal{F}_\alpha f \in L_\omega^p(\mathbb{R}^d)$ , we can write  $h_{y_n}^p(u) \omega^p(u) \leq 2^p |\mathcal{F}_\alpha f(u)|^p \omega^p(u)$ . Thus, by the Lebesgue dominated convergence theorem,

$$\left\| \left( e^{\sum_{j=1}^d \left(\frac{i}{2}(y_n^j)^2 \sin \alpha_j \cos \alpha_j - i u_j y_n^j \sin \alpha_j\right)} - 1 \right) \mathcal{F}_\alpha f \right\|_{p,\omega} \rightarrow 0$$

as  $\lim_{n \rightarrow \infty} y_n = 0$ . Hence,

$$\|T_{y_n} f - f\|_{A_{\alpha,p}^{w,\omega}} \rightarrow 0 \quad (3)$$

as  $n \rightarrow \infty$ . Combining (2) and (3),

$$\|T_{y_n} f - f\|_{A_{\alpha,p}^{w,\omega}} = \|T_{y_n} f - f\|_{1,w} + \|\mathcal{F}_\alpha(T_{y_n} f - f)\|_{p,\omega} \rightarrow 0$$

as  $n \rightarrow \infty$ . This is the desired result.  $\square$

**Theorem 5** Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ , where  $\alpha_i \neq k\pi$  for each index  $i$  with  $1 \leq i \leq d$  and  $k \in \mathbb{Z}$ .

- (1) Let  $1 \leq p < \infty$ ,  $w$  and  $\omega$  be weight functions on  $\mathbb{R}^d$ . Then  $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$  is invariant under modulations.
- (2) Let  $\omega$  be a bounded weight function on  $\mathbb{R}^d$ . Then the mapping  $z \rightarrow M_z f$  is continuous from  $\mathbb{R}^d$  into  $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ .

*Proof* (1) Let  $f \in A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ . Then  $f \in L_w^1(\mathbb{R}^d)$  and  $\mathcal{F}_\alpha f \in L_\omega^p(\mathbb{R}^d)$ . It is easy to see that  $\|M_\eta f\|_{1,w} = \|f\|_{1,w}$  and  $M_\eta f \in L_w^1(\mathbb{R}^d)$ . Let  $c = (\eta_1 \sin \alpha_1, \dots, \eta_d \sin \alpha_d) \in \mathbb{R}^d$ . Thus by Proposition 3, we have

$$\begin{aligned} \|\mathcal{F}_\alpha(M_\eta f)\|_{p,\omega} &= \left( \int_{\mathbb{R}^d} |\mathcal{F}_\alpha(M_\eta f)(u)|^p \omega^p(u) du \right)^{\frac{1}{p}} \\ &= \left( \int_{\mathbb{R}^d} |\mathcal{F}_\alpha f(u_1 - \eta_1 \sin \alpha_1, \dots, u_d - \eta_d \sin \alpha_d)|^p \right. \\ &\quad \times \left. |e^{\sum_{j=1}^d (-\frac{i}{2} \eta_j^2 \sin \alpha_j \cos \alpha_j + i u_j \eta_j \cos \alpha_j)}|^p \omega^p(u) du \right)^{\frac{1}{p}} \\ &\leq \left( \int_{\mathbb{R}^d} |\mathcal{F}_\alpha f(u - c)|^p \omega^p(u - c) \omega^p(c) du \right)^{\frac{1}{p}} \\ &= \omega(c) \|\mathcal{F}_\alpha f\|_{p,\omega} \end{aligned}$$

for all  $\eta \in \mathbb{R}^d$ . Hence, we get

$$\|M_\eta f\|_{A_{\alpha,p}^{w,\omega}} \leq \|f\|_{1,w} + \omega(c) \|\mathcal{F}_\alpha f\|_{p,\omega} < \infty.$$

(2) The proof technique of this part is the same as that of Theorem 4(2). So, for the sake of brevity, we will not prove it.  $\square$

The following definition is an extension of the convolution in [13, 14] of two functions to  $n$  dimensions.

**Definition 6** Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ , where  $\alpha_i \neq k\pi$  for each index  $i$  with  $1 \leq i \leq d$  and  $k \in \mathbb{Z}$ . Then the convolution of two functions  $f, g \in L^1(\mathbb{R}^d)$  is the function  $f \Theta g$  defined by

$$(f \Theta g)(x) = \int_{\mathbb{R}^d} f(y) g(x - y) e^{\sum_{j=1}^d i y_j (y_j - x_j) \cot \alpha_j} dy.$$

It is easy to see that  $f \Theta g$  belongs to  $L^1(\mathbb{R}^d)$  by Fubini's theorem.

**Theorem 7** Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ , where  $\alpha_i \neq k\pi$  for each index  $i$  with  $1 \leq i \leq d$  and  $k \in \mathbb{Z}$ , and  $f, g \in L^1(\mathbb{R}^d)$ . Then

$$\mathcal{F}_\alpha(f \Theta g)(u) = \left[ \prod_{j=1}^d \sqrt{\frac{2\pi}{1 - i \cot \alpha_j}} \right] e^{\sum_{j=1}^d -\frac{i}{2} u_j^2 \cot \alpha_j} \mathcal{F}_\alpha f(u) \mathcal{F}_\alpha g(u),$$

where  $\mathcal{F}_\alpha f$  and  $\mathcal{F}_\alpha g$  are the fractional Fourier transforms of functions  $f$  and  $g$ , respectively.

*Proof* Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ , where  $\alpha_i \neq k\pi$  for each index  $i$  with  $1 \leq i \leq d$  and  $k \in \mathbb{Z}$ , and  $f, g \in L^1(\mathbb{R}^d)$ . We can write from the definition of the fractional Fourier transform

$$\begin{aligned}\mathcal{F}_\alpha(f \ominus g)(u) &= \left[ \prod_{j=1}^d \sqrt{\frac{1-i \cot \alpha_j}{2\pi}} \right] \int_{\mathbb{R}^d} (f \ominus g)(t) e^{\sum_{j=1}^d \left( \frac{i}{2} (u_j^2 + t_j^2) \cot \alpha_j - i u_j t_j \operatorname{cosec} \alpha_j \right)} dt \\ &= \left[ \prod_{j=1}^d \sqrt{\frac{1-i \cot \alpha_j}{2\pi}} \right] \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y) g(t-y) e^{\sum_{j=1}^d i y_j (y_j - t_j) \cot \alpha_j} \\ &\quad \times e^{\sum_{j=1}^d \left( \frac{i}{2} (u_j^2 + t_j^2) \cot \alpha_j - i u_j t_j \operatorname{cosec} \alpha_j \right)} dt dy.\end{aligned}$$

We make the substitution  $t - y = k$  and obtain

$$\begin{aligned}\mathcal{F}_\alpha(f \ominus g)(u) &= \left[ \prod_{j=1}^d \sqrt{\frac{1-i \cot \alpha_j}{2\pi}} \right] \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} f(y) e^{\sum_{j=1}^d \left( \frac{i}{2} (u_j^2 + y_j^2) \cot \alpha_j - i u_j y_j \operatorname{cosec} \alpha_j \right)} dy \right) \\ &\quad \times g(k) e^{\sum_{j=1}^d \left( \frac{i}{2} k_j^2 \cot \alpha_j - i u_j k_j \operatorname{cosec} \alpha_j \right)} dk \\ &= \left[ \prod_{j=1}^d \sqrt{\frac{2\pi}{1-i \cot \alpha_j}} \right] e^{\sum_{j=1}^d -\frac{i}{2} u_j^2 \cot \alpha_j} \left[ \prod_{j=1}^d \sqrt{\frac{1-i \cot \alpha_j}{2\pi}} \right]^2 \\ &\quad \times \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} f(y) e^{\sum_{j=1}^d \left( \frac{i}{2} (u_j^2 + y_j^2) \cot \alpha_j - i u_j y_j \operatorname{cosec} \alpha_j \right)} dy \right) \\ &\quad \times g(k) e^{\sum_{j=1}^d \left( \frac{i}{2} (k_j^2 + u_j^2) \cot \alpha_j - i u_j k_j \operatorname{cosec} \alpha_j \right)} dk \\ &= \left[ \prod_{j=1}^d \sqrt{\frac{2\pi}{1-i \cot \alpha_j}} \right] e^{\sum_{j=1}^d -\frac{i}{2} u_j^2 \cot \alpha_j} \left[ \prod_{j=1}^d \sqrt{\frac{1-i \cot \alpha_j}{2\pi}} \right] \\ &\quad \times \int_{\mathbb{R}^d} \mathcal{F}_\alpha f(u) g(k) e^{\sum_{j=1}^d \left( \frac{i}{2} (k_j^2 + u_j^2) \cot \alpha_j - i u_j k_j \operatorname{cosec} \alpha_j \right)} dk \\ &= \left[ \prod_{j=1}^d \sqrt{\frac{2\pi}{1-i \cot \alpha_j}} \right] e^{\sum_{j=1}^d -\frac{i}{2} u_j^2 \cot \alpha_j} \mathcal{F}_\alpha f(u) \mathcal{F}_\alpha g(u).\end{aligned}$$

□

**Theorem 8** Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ , where  $\alpha_i \neq k\pi$  for each index  $i$  with  $1 \leq i \leq d$  and  $k \in \mathbb{Z}$ .  $L_w^1(\mathbb{R}^d)$  is a Banach algebra under  $\ominus$  convolution.

*Proof* It is well known that  $L_w^1(\mathbb{R}^d)$  is a Banach space [2]. Let  $f, g \in L_w^1(\mathbb{R}^d)$ , then we have

$$\begin{aligned}\|f \ominus g\|_{1,w} &= \int_{\mathbb{R}^d} |f \ominus g| w(x) dy \\ &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(y) g(x-y) e^{\sum_{j=1}^d i y_j (y_j - x_j) \cot \alpha_j} dy \right| w(x) dx \\ &\leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |g(x-y)| w(x-y) dx \right) |f(y)| w(y) dy \\ &= \|g\|_{1,w} \int_{\mathbb{R}^d} |f(y)| w(y) dy \\ &= \|g\|_{1,w} \|f\|_{1,w}.\end{aligned}\tag{4}$$

It is easy to show that the other conditions of the Banach algebra are satisfied. □

**Theorem 9** Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ , where  $\alpha_i \neq k\pi$  for each index  $i$  with  $1 \leq i \leq d$  and  $k \in \mathbb{Z}$ .  $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$  is a Banach  $\Theta$ -convolution module over  $L_w^1(\mathbb{R}^d)$ .

*Proof* It is well known that  $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$  is a Banach space by Theorem 2. Let  $f \in A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$  and  $g \in L_w^1(\mathbb{R}^d)$ . By using the inequality (4), we get

$$\begin{aligned} \|\mathcal{F}_\alpha(f \Theta g)\|_{p,\omega} &= \left\| \left[ \prod_{j=1}^d \sqrt{\frac{2\pi}{1-i \cot \alpha_j}} \right] e^{\sum_{j=1}^d -\frac{i}{2} u_j^2 \cot \alpha_j} \mathcal{F}_\alpha f(u) \mathcal{F}_\alpha g(u) \right\|_{p,\omega} \\ &= \left| \prod_{j=1}^d \sqrt{\frac{2\pi}{1-i \cot \alpha_j}} \right| \left( \int_{\mathbb{R}^d} |\mathcal{F}_\alpha f(u)|^p |\mathcal{F}_\alpha g(u)|^p \omega^p(u) du \right)^{\frac{1}{p}} \\ &= \left| \prod_{j=1}^d \sqrt{\frac{2\pi}{1-i \cot \alpha_j}} \right| \left( \int_{\mathbb{R}^d} |\mathcal{F}_\alpha f(u)|^p \left| \prod_{j=1}^d \sqrt{\frac{1-i \cot \alpha_j}{2\pi}} \right|^p \right. \\ &\quad \times \left. \left| \int_{\mathbb{R}^d} g(t) e^{\sum_{j=1}^d (\frac{i}{2}(u_j^2 + t_j^2) \cot \alpha_j - i u_j t_j \operatorname{cosec} \alpha_j)} dt \right|^p \omega^p(u) du \right)^{\frac{1}{p}} \\ &\leq \left( \int_{\mathbb{R}^d} |\mathcal{F}_\alpha f(u)|^p \left( \int_{\mathbb{R}^d} |g(t)| dt \right)^p \omega^p(u) du \right)^{\frac{1}{p}} \\ &= \|g\|_1 \left( \int_{\mathbb{R}^d} |\mathcal{F}_\alpha f(u)|^p \omega^p(u) du \right)^{\frac{1}{p}} \\ &\leq \|g\|_{1,w} \|\mathcal{F}_\alpha f\|_{p,\omega}. \end{aligned} \quad (5)$$

Combining (4) and (5), we obtain

$$\begin{aligned} \|f \Theta g\|_{A_{\alpha,p}^{w,\omega}} &= \|f \Theta g\|_{1,w} + \|\mathcal{F}_\alpha(f \Theta g)\|_{p,\omega} \\ &\leq \|g\|_{1,w} \|f\|_{1,w} + \|g\|_{1,w} \|\mathcal{F}_\alpha f\|_{p,\omega} \\ &= \|f\|_{A_{\alpha,p}^{w,\omega}} \|g\|_{1,w}. \end{aligned}$$

This is the desired result. It is easy to see that the other conditions of the module are satisfied.  $\square$

### 3 Inclusion properties of the space $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$

**Proposition 10** For every  $0 \neq f \in A_{\alpha,p}^{w,1}(\mathbb{R}^d)$  there exists  $c(f) > 0$  such that

$$c(f)w(x) \leq \|T_{\mathcal{X}}f\|_{A_{\alpha,p}^{w,1}} \leq w(x)\|f\|_{A_{\alpha,p}^{w,1}}.$$

*Proof* Let  $0 \neq f \in A_{\alpha,p}^{w,1}(\mathbb{R}^d)$ . By [12], there exists  $c(f) > 0$  such that

$$c(f)w(x) \leq \|T_{\mathcal{X}}f\|_{1,w} \leq w(x)\|f\|_{1,w}. \quad (6)$$

By using (6) and the equality  $\|\mathcal{F}_\alpha(T_{\mathcal{X}}f)\|_p = \|\mathcal{F}_\alpha f\|_p$ , we obtain

$$\begin{aligned} c(f)w(x) &\leq \|T_{\mathcal{X}}f\|_{1,w} \leq \|T_{\mathcal{X}}f\|_{1,w} + \|\mathcal{F}_\alpha(T_{\mathcal{X}}f)\|_p \\ &\leq w(x)\|f\|_{1,w} + \|\mathcal{F}_\alpha f\|_p \end{aligned}$$

$$\begin{aligned} &\leq w(x) \|f\|_{1,w} + w(x) \|\mathcal{F}_\alpha f\|_p \\ &= w(x) \|f\|_{A_{\alpha,p}^{w,1}} \end{aligned}$$

for all  $f \in A_{\alpha,p}^{w,1}(\mathbb{R}^d)$ .  $\square$

**Lemma 11** Let  $w_1, w_2, \omega_1$  and  $\omega_2$  be weight functions on  $\mathbb{R}^d$ . If  $A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d) \subset A_{\alpha,p}^{w_2,\omega_2}(\mathbb{R}^d)$ , then  $A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d)$  is a Banach space under the norm  $\|f\| = \|f\|_{A_{\alpha,p}^{w_1,\omega_1}} + \|f\|_{A_{\alpha,p}^{w_2,\omega_2}}$ .

*Proof* Let  $(f_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $(A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d), \|\cdot\|)$ . Then  $(f_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $(A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d), \|\cdot\|_{A_{\alpha,p}^{w_1,\omega_1}})$  and  $(A_{\alpha,p}^{w_2,\omega_2}(\mathbb{R}^d), \|\cdot\|_{A_{\alpha,p}^{w_2,\omega_2}})$ . As these spaces are Banach spaces, there exist  $f \in A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d)$  and  $g \in A_{\alpha,p}^{w_2,\omega_2}(\mathbb{R}^d)$  such that  $\|f_n - f\|_{A_{\alpha,p}^{w_1,\omega_1}} \rightarrow 0$ ,  $\|f_n - g\|_{A_{\alpha,p}^{w_2,\omega_2}} \rightarrow 0$ . Using the inequalities  $\|\cdot\|_1 \leq \|\cdot\|_{1,w_1} \leq \|\cdot\|_{A_{\alpha,p}^{w_1,\omega_1}}$  and  $\|\cdot\|_1 \leq \|\cdot\|_{1,w_2} \leq \|\cdot\|_{A_{\alpha,p}^{w_2,\omega_2}}$ , we obtain  $\|f_n - f\|_1 \rightarrow 0$  and  $\|f_n - g\|_1 \rightarrow 0$ . Also  $\|f - g\|_1 \leq \|f_n - f\|_1 + \|f_n - g\|_1$ , we have  $f = g$ . Hence  $\|f_n - f\| \rightarrow 0$  and  $f \in A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d)$ . That means  $(A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d), \|\cdot\|)$  is a Banach space.  $\square$

**Theorem 12** Let  $w_1$  and  $w_2$  be weight functions on  $\mathbb{R}^d$ . Then  $A_{\alpha,p}^{w_1,1}(\mathbb{R}^d) \subset A_{\alpha,p}^{w_2,1}(\mathbb{R}^d)$  if and only if  $w_2 < w_1$ .

*Proof* Suppose that  $w_2 < w_1$ . Thus there exists  $c_1 > 0$  such that  $w_2(x) \leq c_1 w_1(x)$  for all  $x \in \mathbb{R}^d$ . Also let  $f \in A_{\alpha,p}^{w_1,1}(\mathbb{R}^d)$ . Then we write

$$\|f\|_{1,w_2} \leq c_1 \|f\|_{1,w_1} < \infty.$$

Hence we have

$$\|f\|_{A_{\alpha,p}^{w_2,1}} = \|f\|_{1,w_2} + \|\mathcal{F}_\alpha f\|_p \leq c_1 \|f\|_{1,w_1} + c_1 \|\mathcal{F}_\alpha f\|_p = c_1 \|f\|_{A_{\alpha,p}^{w_1,1}}.$$

Therefore,  $A_{\alpha,p}^{w_1,1}(\mathbb{R}^d) \subset A_{\alpha,p}^{w_2,1}(\mathbb{R}^d)$ .

Conversely, suppose that  $A_{\alpha,p}^{w_1,1}(\mathbb{R}^d) \subset A_{\alpha,p}^{w_2,1}(\mathbb{R}^d)$ . For every  $f \in A_{\alpha,p}^{w_1,1}(\mathbb{R}^d)$ , we have  $f \in A_{\alpha,p}^{w_2,1}(\mathbb{R}^d)$ . By Proposition 10, there are constants  $c_1, c_2, c_3, c_4 > 0$  such that

$$c_1 w_1(x) \leq \|T_x f\|_{A_{\alpha,p}^{w_1,1}} \leq c_2 w_1(x) \quad (7)$$

and

$$c_3 w_2(x) \leq \|T_x f\|_{A_{\alpha,p}^{w_2,1}} \leq c_4 w_2(x) \quad (8)$$

for all  $x \in \mathbb{R}^d$ . It is well known from Lemma 11 that the space  $A_{\alpha,p}^{w_1,1}(\mathbb{R}^d)$  is a Banach space under the norm  $\|f\|, f \in A_{\alpha,p}^{w_1,1}(\mathbb{R}^d)$ . Then by the closed graph theorem the norms  $\|\cdot\|_{A_{\alpha,p}^{w_1,1}}$  and  $\|\cdot\|_{A_{\alpha,p}^{w_2,1}}$  are equivalent on  $A_{\alpha,p}^{w_1,1}(\mathbb{R}^d)$ . So, there exists  $c > 0$  such that  $\|f\|_{A_{\alpha,p}^{w_2,1}} \leq \|f\|_{A_{\alpha,p}^{w_1,1}}$  for all  $f \in A_{\alpha,p}^{w_1,1}(\mathbb{R}^d)$ . Moreover, as  $T_x f \in A_{\alpha,p}^{w_2,1}(\mathbb{R}^d)$ , we have

$$\|T_x f\|_{A_{\alpha,p}^{w_2,1}} \leq c \|T_x f\|_{A_{\alpha,p}^{w_1,1}}. \quad (9)$$



Then, combining (7), (8), and (9), we obtain

$$c_3 w_2(x) \leq \|T_x f\|_{A_{\alpha,p}^{w_2,1}} \leq c \|T_x f\|_{A_{\alpha,p}^{w_1,1}} \leq c c_2 w_1(x).$$

Thus,  $w_2(x) \leq \frac{c c_2}{c_3} w_1(x)$ . Let  $\frac{c c_2}{c_3} = k$ . Then we find  $w_2(x) \leq k w_1(x)$  for all  $x \in \mathbb{R}^d$ .  $\square$

**Proposition 13** *Let  $w_1, w_2, \omega_1$  and  $\omega_2$  be weight functions on  $\mathbb{R}^d$ . If  $w_2 \prec w_1$  and  $\omega_2 \prec \omega_1$ , then  $A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d) \subset A_{\alpha,p}^{w_2,\omega_2}(\mathbb{R}^d)$ .*

*Proof* Assume that  $w_2 \prec w_1$  and  $\omega_2 \prec \omega_1$ . Then there exist  $c_1, c_2 > 0$  such that  $w_2(x) \leq c_1 w_1(x)$  and  $\omega_2(x) \leq c_2 \omega_1(x)$  for all  $x \in \mathbb{R}^d$ . Let  $f \in A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d)$ . As  $f \in L_{w_1}^1(\mathbb{R}^d)$  and  $\mathcal{F}_\alpha f \in L_{\omega_1}^p(\mathbb{R}^d)$ , we have  $\|f\|_{1,w_2} \leq c_1 \|f\|_{1,w_1} < \infty$  and  $\|\mathcal{F}_\alpha f\|_{p,\omega_2} \leq c_2 \|\mathcal{F}_\alpha f\|_{p,\omega_1} < \infty$ . Hence, we obtain  $f \in A_{\alpha,p}^{w_2,\omega_2}(\mathbb{R}^d)$ , and then  $A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d) \subset A_{\alpha,p}^{w_2,\omega_2}(\mathbb{R}^d)$ .  $\square$

#### 4 Duality

Let the mapping  $\Phi : A_{\alpha,p}^{w,\omega}(\mathbb{R}^d) \rightarrow L_w^1(\mathbb{R}^d) \times L_\omega^p(\mathbb{R}^d)$  be defined by  $\Phi(f) = (f, \mathcal{F}_\alpha f)$  for  $1 \leq p < \infty$  and let  $H = \Phi(A_{\alpha,p}^{w,\omega}(\mathbb{R}^d))$ . Then

$$\|\Phi(f)\| = \|(f, \mathcal{F}_\alpha f)\| = \|f\|_{1,w} + \|\mathcal{F}_\alpha f\|_{p,\omega}$$

is a norm on  $H$  for all  $f \in A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ . Moreover, we define a set  $K$  as

$$K = \left\{ (\varphi, \psi) : ((\varphi, \psi) \in L_{w^{-1}}^\infty(\mathbb{R}^d) \times L_{\omega^{-1}}^{p'}(\mathbb{R}^d)), \right. \\ \left. \int_{\mathbb{R}^d} f(x) \varphi(x) dx + \int_{\mathbb{R}^d} \mathcal{F}_\alpha f(y) \psi(y) dy = 0 \text{ for all } (f, \mathcal{F}_\alpha f) \in H \right\},$$

where  $\frac{1}{p} + \frac{1}{p'} = 1$ .

The following proposition is proved by the duality theorem, Theorem 1.7 in [15].

**Proposition 14** *Let  $1 \leq p < \infty$ , and  $w$  and  $\omega$  be weight functions on  $\mathbb{R}^d$ . The dual space of  $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$  is isomorphic to  $L_{w^{-1}}^\infty(\mathbb{R}^d) \times L_{\omega^{-1}}^{p'}(\mathbb{R}^d)/K$  where  $\frac{1}{p} + \frac{1}{p'} = 1$ .*

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Received: 13 October 2014 Accepted: 24 February 2015 Published online: 06 March 2015

#### References

- Feichtinger, HG, Gürkanlı, AT: On a family of weighted convolution algebras. *Int. J. Math. Sci.* **13**, 517-526 (1990)
- Reiter, H: *Classical Harmonic Analysis and Locally Compact Group*. Oxford University Press, Oxford (1968)
- Namias, V: The fractional order of Fourier transform and its application in quantum mechanics. *IMA J. Appl. Math.* **25**, 241-265 (1980)
- Ozaktas, HM, Kutay, MA, Zalevsky, Z: *The Fractional Fourier Transform with Applications in Optics and Signal Processing*. Wiley, Chichester (2001)
- Almeida, LB: The fractional Fourier transform and time-frequency representations. *IEEE Trans. Signal Process.* **42**, 3084-3091 (1994)
- Zayed, AI: On the relationship between the Fourier and fractional Fourier transforms. *IEEE Signal Process. Lett.* **3**, 310-311 (1996)

7. Almeida, LB: Product and convolution theorems for the fractional Fourier transform. *IEEE Signal Process. Lett.* **4**(1), 15-17 (1997)
8. Zayed, AI: Fractional Fourier transform of generalized function. *Integral Transforms Spec. Funct.* **7**, 299-312 (1998)
9. Bultheel, A, Martinez, H: A shattered survey of the fractional Fourier transform. Report TW337, Department of Computer Science, K.U. Leuven (2002)
10. Sahin, A, Ozaktas, HM, Mendlovic, D: Optical implementations of two-dimensional fractional Fourier transforms and linear canonical transforms with arbitrary parameters. *Appl. Opt.* **37**(11), 2130-2141 (1998)
11. Sharma, VD: Operational calculus on generalized two-dimensional fractional Fourier transform. *Int. J. Eng. Innov. Technol.* **3**(2), 253-256 (2013)
12. Fischer, RH, Gürkanlı, AT, Liu, TS: On a family of weighted spaces. *Math. Slovaca* **46**(1), 71-82 (1996)
13. Singh, AK, Saxena, R: On convolution and product theorems for FRFT. *Wirel. Pers. Commun.* **65**, 189-201 (2012)
14. Zayed, AI: A convolution and product theorem for the fractional Fourier transform. *IEEE Signal Process. Lett.* **5**, 101-103 (1998)
15. Liu, T, Rooij, AV: Sums and intersections of normed linear spaces. *Math. Nachr.* **42**, 29-42 (1969)

**Submit your manuscript to a SpringerOpen<sup>®</sup> journal and benefit from:**

- Convenient online submission
- Rigorous peer review
- Immediate publication on acceptance
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

---

Submit your next manuscript at ► [springeropen.com](http://springeropen.com)

---