Şahiner et al. *Journal of Inequalities and Applications* (2015) 2015:77
DOI 10.1186/s13660-015-0594-7

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Sturm comparison theorems for some elliptic type equations with damping and external forcing terms

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Abstract
In this paper we establish Picone-type inequalities for a pair of a damped linear elliptic equation and a forced nonlinear elliptic equation with damped term and give some Sturm comparison theorems via the Picone-type inequality. An oscillation result is also given as an application.
MSC: 35B05

Keywords: Picone's inequality; half-linear equations; elliptic equations; Sobolev space; oscillation criteria

1 Introduction

The existence and location of the zeros of the solutions of differential equations are very important. Accordingly a large literature on this subject has arisen during the past century.

After Sturm's significant work [1] in 1836, Sturmian comparison theorems have been derived for differential equations of various types. In order to obtain Sturmian comparison theorems for ordinary differential equations of second order, Picone [2] established an identity, known as the Picone identity. In the latter years, Jaroš and Kusano [3] derived a Picone-type identity for half-linear differential equations of second order. They also developed Sturmian theory for both forced and unforced half-linear equations based on this identity [4, 5].

The Sturm-Picone theorem and much of the related theory should allow generalization to certain partial differential equations. There are many papers (or books) dealing with Sturm comparison (or oscillation results) for a pair of elliptic type operators. We refer to Kreith [6, 7], Swanson [8] for Sturmian comparison theorems for linear elliptic equations and to Allegretto [9], Allegretto and Huang [10, 11], Bognár and Dosly [12], Dunninger [13], Kusano *et al.* [14], Yoshida [15–17] for Picone identities, Sturmian comparison and/or oscillation theorems for half-linear elliptic equations. In particular, we mention the paper [13] by Dunninger which seems to be the first paper dealing with Sturmian comparison theorems for half-linear elliptic equations. Note that most of the work in the literature deal with the Sturm comparison results for elliptic equations that contain undamped terms for example see [6–13, 15, 17–20] and damped terms [21–23]. The forced differential equations and their oscillations has recently become more interesting research areas. Yoshida established results dealing with the forced oscillations of solutions of second order elliptic equations [17, 24].

We consider a damped linear elliptic operator

$$\ell(u) = \nabla \cdot \big(a(x)\nabla u\big) + 2b(x) \cdot \nabla u + c(x)u \tag{1.1}$$

with forced nonlinear elliptic operator with a damped term of the form

$$P_\alpha(v) = \nabla \cdot \big(A(x)|\nabla v|^{\alpha-1}\nabla v\big) + (\alpha+1)|\nabla v|^{\alpha-1}B(x) \cdot \nabla v + g(x,v), \tag{1.2}$$

$$g(x,v) = C(x)|v|^{\alpha-1}v + \sum_{i=1}^{\ell} D_i(x)|v|^{\beta_i-1}v + \sum_{j=1}^{m} E_j(x)|v|^{\gamma_j-1}v,$$

where $|\cdot|$ denotes the Euclidean length, $\cdot$ denotes the scalar product. It is assumed that $0 < \gamma_j < \alpha < \beta_i$ $(i = 1,2,\dots,\ell; j = 1,2,\dots,m)$.

Although there are plenty of results related to Sturm comparison (or oscillation results) of linear equations, there are only a few results dealing with nonlinear equations [21, 25]. By investigating the behavior of solutions of linear equations, we can establish considerable results for the behavior of solutions of nonlinear equations. Motivated by this idea, we give some Sturm comparison theorems via Picone-type inequalities for $\ell(u) = 0$ and $P_\alpha(v) = f(x)$. To the best of our knowledge, the above damped elliptic operators ℓ and P_α have not been studied.

Note that the principal part of (1.2) are reduced to the p-Laplacian $\nabla \cdot (|\nabla u)|^{p-2}\nabla u$ $(p = \alpha + 1)$. We know that a variety of physical phenomena are modeled by equations the p-Laplacian [26–31]. We refer the reader to Diaz [32] for detailed references on physical background of the p-Laplacian.

We organize this paper as follows: In Section 2, we establish Picone-type inequalities for a pair of $\{\ell, P_\alpha\}$. In Section 3 we present Sturmian comparison theorems and Section 4 is left for an application.

2 Picone-type inequalities

In this section, we establish Picone-type inequalities for a pair of differential equations $\ell(u) = 0$ and $P_\alpha(v) = f(x)$ defined by (1.1) and (1.2), respectively. Let G be a bounded domain in R^n with piecewise smooth boundary ∂G. We assume that $a(x) \in C(\bar{G}, R^+)$, $A(x) \in C(\bar{G}, R^+)$, $b(x) \in C(\bar{G}, R^n)$, $B(x) \in C(\bar{G}, R^n)$, $c(x) \in C(\bar{G}, R)$, $C(x) \in C(\bar{G}, R)$, $D_i(x) \in C(\bar{G}, R^+ \cup \{0\})$, $E_j(x) \in C(\bar{G}, R^+ \cup \{0\})$ $(i = 1,2,\dots,\ell; j = 1,2,\dots,m)$ and $f(x) \in C(\bar{G}, R)$.

The domain $D_\ell(G)$ of ℓ is defined to be set of all functions u of class $C^1(\bar{G}, R)$ with the property that $a(x)\nabla u \in C^1(\bar{G}, R^n) \cap C(\bar{G}, R^n)$. The domain $D_{P_\alpha}(G)$ of P_α is defined to be the set of all functions v with the property that $A(x)|\nabla v|^{\alpha-1}\nabla v \in C^1(\bar{G}, R^n) \cap C(\bar{G}, R^n)$.

Let $N = \min\{\ell, m\}$ and

$$\mathfrak{G}\big(\beta, \alpha, D(x), f(x)\big) = \left(\frac{\beta}{\alpha}\right)\left(\frac{\beta-\alpha}{\alpha}\right)^{\frac{\alpha-\beta}{\beta}} \big(D(x)\big)^{\frac{\alpha}{\beta}} \big|f(x)\big|^{\frac{\beta-\alpha}{\beta}}.$$

We need the following lemma in order to give the proof of our results.

Lemma 2.1 *The inequality*

$$|\xi|^{\alpha+1} + \alpha|\eta|^{\alpha+1} - (\alpha+1)|\eta|^{\alpha-1}\xi \cdot \eta \geq 0$$

is valid for any $\xi \in R^n$ and $\eta \in R^n$, where the equality holds if and only if $\xi = \eta$.

For the proof of the lemma see [12], Lemma 2.1.

Theorem 2.1 *If $u \in D_\ell(G)$ of $\ell(u) = 0$, $v \in D_{P_\alpha}(G)$, and $v \neq 0$ in G and $vf(x) \leq 0$ in G, then for any $u \in C^1(G,R)$ the following Picone-type inequality holds*:

$$\begin{aligned}
&\nabla \cdot \left(\frac{u}{\varphi(v)}\bigl[\varphi(v)a(x)\nabla u - \varphi(u)A(x)|\nabla v|^{\alpha-1}\nabla v\bigr]\right) \\
&\quad\geq -A(x)\left|\nabla u - \frac{uB(x)}{A(x)}\right|^{\alpha+1} + (\nabla u)^T\bigl(a(x) - \bigl|b(x)\bigr|\bigr)(\nabla u) - \bigl(\bigl|b(x)\bigr| + c(x)\bigr)(u)^2 \\
&\qquad + C_1(x)|u|^{\alpha+1} - \frac{u\varphi(u)}{\varphi(v)}\bigl[P_\alpha(v) - f(x)\bigr] \\
&\qquad + A(x)\left[\left|\nabla u - \frac{uB(x)}{A(x)}\right|^{\alpha+1} + \alpha\left|\frac{u}{v}\nabla v\right|^{\alpha+1} - (\alpha+1)\left(\nabla u - \frac{uB(x)}{A(x)}\right)\Phi\left(\frac{u}{v}\nabla v\right)\right].
\end{aligned} \tag{2.1}$$

Here $\varphi(s) = |s|^{\alpha-1}s$, $s \in R$, $\Phi(\xi) = |\xi|^{\alpha-1}\xi$, $\xi \in R^n$, and

$$C_1(x) = C(x) + \sum_{i=1}^{N} \mathfrak{G}\bigl(\beta_i, \alpha, D_i(x), f(x)\bigr).$$

Proof We easily see that

$$\nabla \cdot \bigl(ua(x)\nabla u\bigr) = (\nabla u)^T a(x)(\nabla u) - 2b(x)u \cdot \nabla u - c(x)u^2, \tag{2.2}$$

and using Young's inequality we have

$$2ub(x) \cdot \nabla u \leq \bigl|b(x)\bigr|\bigl(u^2 + (\nabla u)^2\bigr). \tag{2.3}$$

Using (2.2) and (2.3), we obtain the following inequality:

$$\nabla \cdot \left(\frac{u}{\varphi(v)}\bigl[\varphi(v)a(x)\nabla u\bigr]\right) \geq (\nabla u)^T\bigl(a(x) - \bigl|b(x)\bigr|\bigr)(\nabla u) - \bigl(\bigl|b(x)\bigr| + c(x)\bigr)u^2. \tag{2.4}$$

On the other hand we observe that the following identity holds:

$$\begin{aligned}
&-\nabla \cdot \left(u\varphi(u)\frac{A(x)|\nabla v|^{\alpha-1}\nabla v}{\varphi(v)}\right) \\
&\quad= -A(x)\left|\nabla u - \frac{uB(x)}{A(x)}\right|^{\alpha+1} + \frac{u\varphi(u)}{\varphi(v)}\bigl(g(x,v) - f(x)\bigr) - \frac{u\varphi(u)}{\varphi(v)}\bigl[P_\alpha(v) - f(x)\bigr] \\
&\qquad + A(x)\left[\left|\nabla u - \frac{uB(x)}{A(x)}\right|^{\alpha+1} + \alpha\left|\frac{u}{v}\nabla v\right|^{\alpha+1} - (\alpha+1)\left(\nabla u - \frac{uB(x)}{A(x)}\right)\Phi\left(\frac{u}{v}\nabla v\right)\right]
\end{aligned} \tag{2.5}$$

and it is clear that

$$\frac{u\varphi(u)}{\varphi(v)}\left(C(x)|v|^{\alpha-1}v+\sum_{i=1}^{\ell}D_i(x)|v|^{\beta_i-1}v+\sum_{j=1}^{m}E_j(x)|v|^{\gamma_j-1}v-f(x)\right)$$
$$\geq\left(C(x)+\sum_{i=1}^{\ell}D_i(x)|v|^{\beta_i-\alpha}v-\frac{f(x)}{|v|^{\alpha-1}v}\right)|u|^{\alpha+1}.$$

It can be shown that by using $vf(x)\leq 0$ and Young's inequality

$$C(x)+\sum_{i=1}^{\ell}D_i(x)|v|^{\beta_i-\alpha}v-\frac{f(x)}{|v|^{\alpha-1}v}=C(x)+\sum_{i=1}^{\ell}D_i(x)|v|^{\beta_i-\alpha}v-\frac{|f(x)|}{|v|^{\alpha}}\geq C_1(x).$$

This implies

$$\frac{u\varphi(u)}{\varphi(v)}\bigl(g(x,v)-f(x)\bigr)\geq C_1(x)|u|^{\alpha+1} \tag{2.6}$$

Combining (2.4), (2.5), and (2.6) we get the desired inequality (2.1). □

Theorem 2.2 *If $v\in D_{P_\alpha}(G)$ of $\ell(u)=0$, $v\neq 0$ in G, and $vf(x)\leq 0$ in G, then for any $u\in C^1(G,R)$ the following Picone-type inequality holds*:

$$-\nabla\cdot\left(\frac{u\varphi(u)}{\varphi(v)}A(x)|\nabla v|^{\alpha-1}\nabla v\right)$$
$$\geq -A(x)\left|\nabla u-\frac{uB(x)}{A(x)}\right|^{\alpha+1}+C_1(x)|u|^{\alpha+1}-\frac{u\varphi(u)}{\varphi(v)}\bigl[P_\alpha(v)-f(x)\bigr]$$
$$+A(x)\left[\left|\nabla u-\frac{uB(x)}{A(x)}\right|^{\alpha+1}+\alpha\left|\frac{u}{v}\nabla v\right|^{\alpha+1}-(\alpha+1)\left(\nabla u-\frac{uB(x)}{A(x)}\right)\Phi\left(\frac{u}{v}\nabla v\right)\right], \tag{2.7}$$

where $\varphi(s)=|s|^{\alpha-1}s$, $s\in R$, $\Phi(\xi)=|\xi|^{\alpha-1}\xi$, $\xi\in R^n$, and $C_1(x)$ is defined as in Theorem 2.1.

Proof Combining (2.5) with (2.6) yields the desired inequality (2.7). □

By using the ideas in [33], the condition on $f(x)$ can be removed if we impose another condition on v, as $|v|\geq k_0$. The proofs of the following theorems are similar to the proofs of Theorems 2.1 and 2.2 and the proof of the Lemma 1 in [33], hence omitted.

Theorem 2.3 *If $u\in D_\ell(G)$ of $\ell(u)=0$, $v\in D_{P_\alpha}(G)$, and $|v|\geq k_0$ then the following Picone-type inequality holds for any $u\in C^1(G,R)$*:

$$\nabla\cdot\left(\frac{u}{\varphi(v)}\bigl[\varphi(v)a(x)\nabla u-\varphi(u)A(x)|\nabla v|^{\alpha-1}\nabla v\bigr]\right)$$
$$\geq -A(x)\left|\nabla u-\frac{uB(x)}{A(x)}\right|^{\alpha+1}+(\nabla u)^T\bigl(a(x)-\bigl|b(x)\bigr|\bigr)(\nabla u)-\bigl(\bigl|b(x)\bigr|+c(x)\bigr)(u)^2$$
$$+\bigl(C_2(x)-k_0^{-\alpha}\bigl|f(x)\bigr|\bigr)|u|^{\alpha+1}-\frac{u\varphi(u)}{\varphi(v)}\bigl[P_\alpha(v)-f(x)\bigr]$$
$$+A(x)\left[\left|\nabla u-\frac{uB(x)}{A(x)}\right|^{\alpha+1}+\alpha\left|\frac{u}{v}\nabla v\right|^{\alpha+1}-(\alpha+1)\left(\nabla u-\frac{uB(x)}{A(x)}\right)\Phi\left(\frac{u}{v}\nabla v\right)\right], \tag{2.8}$$

where $\varphi(s) = |s|^{\alpha-1}s$, $s \in R$, $\Phi(\xi) = |\xi|^{\alpha-1}\xi$, $\xi \in R^n$, *and*

$$C_2(x) = C(x) + \sum_{i=1}^{N} H\bigl(\beta_i, \alpha, \gamma_i, D_i(x), E_i(x)\bigr),$$

where

$$H\bigl(\beta, \alpha, \gamma, D(x), E(x)\bigr) = \frac{\beta-\gamma}{\alpha-\gamma}\left(\frac{\beta-\alpha}{\alpha-\gamma}\right)^{\frac{\alpha-\beta}{\beta-\gamma}} \bigl(C(x)\bigr)^{\frac{\alpha-\gamma}{\beta-\gamma}} \bigl(D(x)\bigr)^{\frac{\beta-\alpha}{\beta-\gamma}}.$$

Theorem 2.4 *If* $v \in D_{P_\alpha}(G)$ *and* $|v| \geq k_0$ *then the following Picone-type inequality holds for any* $u \in C^1(G, R)$:

$$\begin{aligned}
&\nabla \cdot \left(\frac{u\varphi(u)}{\varphi(v)}\bigl[A(x)|\nabla v|^{\alpha-1}\nabla v\bigr]\right) \\
&\quad \geq -A(x)\left|\nabla u - \frac{uB(x)}{A(x)}\right|^{\alpha+1} + \bigl(C_2(x) - k_0^{-\alpha}\bigl|f(x)\bigr|\bigr)|u|^{\alpha+1} \\
&\qquad + A(x)\left[\left|\nabla u - \frac{uB(x)}{A(x)}\right|^{\alpha+1} + \alpha\left|\frac{u}{v}\nabla v\right|^{\alpha+1} - (\alpha+1)\left(\nabla u - \frac{uB(x)}{A(x)}\right)\Phi\left(\frac{u}{v}\nabla v\right)\right] \\
&\qquad - \frac{u\varphi(u)}{\varphi(v)}\bigl[P_\alpha(v) - f(x)\bigr],
\end{aligned} \tag{2.9}$$

where $\varphi(s) = |s|^{\alpha-1}s$, $s \in R$, $\Phi(\xi) = |\xi|^{\alpha-1}\xi$, $\xi \in R^n$, *and* $C_2(x)$ *is defined before in Theorem* 2.3.

3 Sturmian comparison theorems

In this section we establish some Sturmian comparison results on the basis of the Picone-type inequalities obtained in Section 2. We begin with a theorem needed for comparison results.

Theorem 3.1 *If there is a nontrivial function* $u \in C^1(\bar{G}, R)$ *such that* $u = 0$ *on* ∂G *and*

$$M[u] := \int_G \left\{A(x)\left|\nabla u - \frac{uB(x)}{A(x)}\right|^{\alpha+1} - C_1(x)|u|^{\alpha+1}\right\} dx \leq 0, \tag{3.1}$$

then every solution $v \in D_{P_\alpha}(G)$ *of* $P_\alpha(v) = f(x)$ *satisfying* $vf(x) \leq 0$ *in* G *vanishes at some point of* $\bar{G}$. *Furthermore, if* $\partial G \in C^1$, *then every solution* $v \in D_{P_\alpha}(G)$ *of* $P_\alpha(v) = f(x)$ *satisfying* $vf(x) \leq 0$ *in* G *has one of the following properties*:

(1) v *has a zero in* G,

(2) $u = c_0 e^{\alpha(x)} v$, *where* $c_0 \neq 0$ *is a constant and* $\nabla\alpha(x) = \frac{B(x)}{A(x)}$.

Proof (The first statement) Suppose to the contrary that there exists a solution $v \in D_{P_\alpha}(G)$ of $P_\alpha(v) = f(x)$ satisfying $vf(x) \leq 0$ in G and $v \neq 0$ on $\bar{G}$. Then the inequality (2.7) of Theorem 2.2 holds. Integrating (2.7) over G and then using the divergence theorem, we get

$$\begin{aligned}
M[u] \geq \int_G A(x)\Biggl[&\left|\nabla u - \frac{uB(x)}{A(x)}\right|^{\alpha+1} + \alpha\left|\frac{u}{v}\nabla v\right|^{\alpha+1} \\
&- (\alpha+1)\left(\nabla u - \frac{uB(x)}{A(x)}\right)\Phi\left(\frac{u}{v}\nabla v\right)\Biggr] dx.
\end{aligned} \tag{3.2}$$

Since $u = 0$ on ∂G and $v \neq 0$ on $\bar{G}$, we observe that u cannot be written in the form $u = c_0 e^{\alpha(x)} v$, and hence $\nabla(\frac{u}{v}) - \frac{B(x)}{A(x)}\frac{u}{v} \neq 0$. Therefore from Lemma 2.1 we see that

$$\int_G A(x)\left[\left|\nabla u - \frac{uB(x)}{A(x)}\right|^{\alpha+1} + \alpha\left|\frac{u}{v}\nabla v\right|^{\alpha+1} - (\alpha+1)\left(\nabla u - \frac{uB(x)}{A(x)}\right)\Phi\left(\frac{u}{v}\nabla v\right)\right] dx > 0,$$

which together with (3.2) implies that $M[u] > 0$. This contradicts the hypothesis $M[u] \leq 0$. The proof of the first statement (1) is complete.

(The second statement) Next we consider the case where $\partial G \in C^1$. Let $v \in D_{P_\alpha}(G)$ be a solution of $P_\alpha(v) = f(x)$ satisfying $vf(x) \leq 0$ in G and $v \neq 0$ on G. Since $\partial G \in C^1$, $u \in C^1(\bar{G}, R)$ and $u = 0$ on ∂G, we find that u belongs to the Sobolev space $W_0^{1,\alpha+1}(G)$, which is the closure in the norm

$$\|w\| := \left(\int_G \left[|w|^{\alpha+1} + |\nabla w|^{\alpha+1}\right] dx\right)^{\frac{1}{\alpha+1}} \tag{3.3}$$

of the class $C_0^\infty(G)$ of infinitely differentiable functions with compact supports in G, [34, 35]. Let u_k be a sequence of functions in $C_0^\infty(G)$ converging to u in the norm (3.3). Integrating (2.7) with $u = u_k$ over G and then applying the divergence theorem, we observe that

$$\begin{aligned} M[u_k] \geq \int_G A(x)\Bigg[&\left|\nabla u_k - \frac{u_k B(x)}{A(x)}\right|^{\alpha+1} + \alpha\left|\frac{u_k}{v}\nabla v\right|^{\alpha+1} \\ &- (\alpha+1)\left(\nabla u_k - \frac{u_k B(x)}{A(x)}\right)\Phi\left(\frac{u_k}{v}\nabla v\right)\Bigg] dx \geq 0. \end{aligned} \tag{3.4}$$

We first claim that $\lim_{k\to\infty} M[u_k] = M[u] = 0$. Since $A(x)$, $C(x)$, $D_i(x)$ ($i = 1, 2, \ldots, \ell$), and $f(x)$ are bounded on $\bar{G}$, there exists a constant $K_1 > 0$ such that

$$A(x) \leq K_1 \quad \text{and} \quad \left|C_1(x)\right| \leq K_1.$$

It is easy to see that

$$\begin{aligned} \left|M[u_k] - M[u]\right| \leq K_1 \int_G &\left|\left|\nabla u_k - \frac{u_k B(x)}{A(x)}\right|^{\alpha+1} - \left|\nabla u - \frac{uB(x)}{A(x)}\right|^{\alpha+1}\right| dx \\ &+ K_1 \int_G \left||u_k|^{\alpha+1} - |u|^{\alpha+1}\right| dx. \end{aligned} \tag{3.5}$$

It follows from the mean value theorem that

$$\begin{aligned} &\left|\left|\nabla u_k - \frac{u_k B(x)}{A(x)}\right|^{\alpha+1} - \left|\nabla u - \frac{uB(x)}{A(x)}\right|^{\alpha+1}\right| \\ &\quad \leq (\alpha+1)\left(\left|\nabla u_k - \frac{u_k B(x)}{A(x)}\right| + \left|\nabla u - \frac{uB(x)}{A(x)}\right|\right)^{\alpha}\left|\nabla(u_k - u) + \frac{B(x)}{A(x)}(u_k - u)\right| \\ &\quad \leq (\alpha+1)\left(|\nabla u_k| + |\nabla u| + \frac{|B(x)|}{A(x)}|u_k| + \frac{|B(x)|}{A(x)}|u|\right)^{\alpha}\left(\left|\nabla(u_k - u)\right| + \frac{|B(x)|}{A(x)}|u_k - u|\right). \end{aligned}$$

Since also $B(x)$ is bounded on $\bar{G}$, then there is a constant K_2 such that $\frac{|B(x)|}{A(x)} \leq K_2$ on $\bar{G}$. Let us take $K_3 = \max\{1, K_2\}$. From the above inequality we have

$$\begin{aligned} &\left| \left| \nabla u_k - \frac{u_k B(x)}{A(x)} \right|^{\alpha+1} - \left| \nabla u - \frac{u B(x)}{A(x)} \right|^{\alpha+1} \right| \\ &\quad \leq (\alpha+1) K_3^{\alpha+1} \bigl(|\nabla u_k| + |\nabla u| + |u_k| + |u| \bigr)^{\alpha} \bigl(\left|\nabla (u_k - u)\right| + |u_k - u| \bigr). \end{aligned} \tag{3.6}$$

Using (3.6) and applying Hölder's inequality and then Minkowski's inequality, we get

$$\begin{aligned} &\int_G \left| \left| \nabla u_k - \frac{u_k B(x)}{A(x)} \right|^{\alpha+1} - \left| \nabla u - \frac{u B(x)}{A(x)} \right|^{\alpha+1} \right| dx \\ &\quad \leq (\alpha+1) K_3^{\alpha+1} \left(\int_G \bigl(|\nabla u_k| + |\nabla u| + |u_k| + |u| \bigr)^{\alpha+1} dx \right)^{\frac{\alpha}{\alpha+1}} \\ &\qquad \times \left(\int_G \bigl(\left|\nabla (u_k - u)\right| + |u_k - u| \bigr)^{\alpha+1} dx \right)^{\frac{1}{\alpha+1}} \\ &\quad \leq (\alpha+1) K_3^{\alpha+1} \left[\left(\int_G \bigl(|\nabla u_k| + |u_k| \bigr)^{\alpha+1} dx \right)^{\frac{1}{\alpha+1}} + \left(\int_G \bigl(|\nabla u| + |u| \bigr)^{\alpha+1} dx \right)^{\frac{1}{\alpha+1}} \right]^{\alpha} \\ &\qquad \times \left(\int_G \bigl(\left|\nabla (u_k - u)\right| + |u_k - u| \bigr)^{\alpha+1} dx \right)^{\frac{1}{\alpha+1}} \\ &\quad \leq (\alpha+1) K_3^{\alpha+1} \bigl(\|u_k\| + \|u\| \bigr)^{\alpha} \|u_k - u\|. \end{aligned} \tag{3.7}$$

Similarly we obtain

$$\int_G \left| |u_k|^{\alpha+1} - |u|^{\alpha+1} \right| dx \leq (\alpha+1) \bigl(\|u_k\| + \|u\| \bigr)^{\alpha} \|u_k - u\|. \tag{3.8}$$

Combining (3.5), (3.7), and (3.8), we have

$$\left| M[u_k] - M[u] \right| \leq K_4 \bigl(\|u_k\| + \|u\| \bigr)^{\alpha} \|u_k - u\|$$

for some positive constant K_4 depending only on K_1, K_2, K_3, and α, from which it follows that $\lim_{k\to\infty} M[u_k] = M[u]$. We see from (3.4) that $M[u] \geq 0$, which together with (3.2) implies $M[u] = 0$.

Let $\mathcal{B}$ be an arbitrary ball with $\bar{\mathcal{B}} \subset G$ and define

$$\begin{aligned} Q_{\mathcal{B}}[w] := \int_{\mathcal{B}} A(x) \Biggl[&\left| \nabla w - \frac{w B(x)}{A(x)} \right|^{\alpha+1} + \alpha \left| \frac{w}{v} \nabla v \right|^{\alpha+1} \\ &- (\alpha+1) \left(\nabla w - \frac{w B(x)}{A(x)} \right) \Phi \left(\frac{w}{v} \nabla v \right) \Biggr] \end{aligned} \tag{3.9}$$

for $w \in C^1(G; R)$. It is easily verified that

$$0 \leq Q_{\mathcal{B}}[u_k] \leq Q_G[u_k] \leq M[u_k], \tag{3.10}$$

where $Q_G[u_k]$ denotes the right-hand side of (3.4) with $w = u_k$ and with $\mathcal{B}$ replaced by G.

A simple calculation yields

$$\begin{aligned}\left|Q_{\mathcal{B}}[u_k]-Q_{\mathcal{B}}[u]\right| &\leq K_5\left(\|u_k\|_{\mathcal{B}}+\|u\|_{\mathcal{B}}\right)^{\alpha}\|u_k-u\|_{\mathcal{B}}+K_6\left(\|u_k\|_{\mathcal{B}}\right)^{\alpha}\|u_k-u\|_{\mathcal{B}}\\ &\quad+K_7\left\|\varphi[u_k]-\varphi[u]\right\|_{L^q_{(\mathcal{B})}}\|u\|_{\mathcal{B}},\end{aligned} \tag{3.11}$$

where $q=\frac{\alpha+1}{\alpha}$, the constants K_5, K_6, and K_7 are independent of k, and the subscript $\mathcal{B}$ indicates the integrals involved in the norm (3.3) are to be taken over $\mathcal{B}$ instead of G. It is well known that the Nemitski operator $\varphi: L^{\alpha+1}(G)\to L^q(G)$ is continuous [36] and it is clear that $\|u_k-u\|_{\mathcal{B}}\to 0$ as $\|u_k-u\|_G\to 0$.

Therefore, letting $k\to\infty$ in (3.11), we find that $Q_{\mathcal{B}}(u)=0$. Since $A(x)>0$ in $\mathcal{B}$, it follows that

$$\left[\left|\nabla u-\frac{uB(x)}{A(x)}\right|^{\alpha+1}+\alpha\left|\frac{u}{v}\nabla v\right|^{\alpha+1}-(\alpha+1)\left(\nabla u-\frac{uB(x)}{A(x)}\right)\cdot\Phi\left(\frac{u}{v}\nabla v\right)\right]\equiv 0 \quad \text{in } \mathcal{B}. \tag{3.12}$$

From this, Lemma 2.1 implies that

$$\nabla u-\frac{uB(x)}{A(x)}\equiv\frac{u}{v}\nabla v \quad \textit{i.e. } \nabla\left(\frac{u}{v}\right)-\frac{B(x)}{A(x)}\frac{u}{v}\equiv 0 \text{ in } \mathcal{B}.$$

Hence we observe that $\frac{u}{v}=c_0e^{\alpha(x)}$ in $\mathcal{B}$ for some constant c_0 and some continuous function $\alpha(x)$. Since $\mathcal{B}$ an arbitrary ball with $\bar{\mathcal{B}}\subset G$, we conclude that $\frac{u}{v}=c_0e^{\alpha(x)}$ in G where $c_0\neq 0$. □

Corollary 3.1 *Assume that $f(x)\geq 0$ (or $f(x)\leq 0$) in G. If there is a nontrivial function $u\in C^1(G,R)$ such that $u=0$ on ∂G and $M[u]\leq 0$, then $P_\alpha(v)=f(x)$ has no negative (or positive) solution on $\bar{G}$.*

Proof Suppose that $P_\alpha(v)=f(x)$ has a negative (or positive) solution v on $\bar{G}$. It is easy to see that $vf(x)\leq 0$ in G, and therefore it follows from Theorem 3.1 that v must vanish at some point of $\bar{G}$. This is a contradiction and the proof is complete. □

Theorem 3.2 (Sturmian comparison theorem) *If there is a nontrivial solution $u\in D_\ell(G)$ of $\ell(u)=0$ such that $u=0$ on ∂G and*

$$\begin{aligned}V[u]:=\int_G(\nabla u)^T\left(a(x)-\left|b(x)\right|\right)(\nabla u)-A(x)\left|\nabla u-\frac{uB(x)}{A(x)}\right|^{\alpha+1}\\ +C_1(x)|u|^{\alpha+1}-\left(\left|b(x)\right|+c(x)\right)u^2\geq 0.\end{aligned} \tag{3.13}$$

Then every solution $v\in D_{P_\alpha}(G)$ of $P_\alpha(v)=f(x)$ satisfying $vf(x)\leq 0$ in G must vanish at some point of $\bar{G}$. Furthermore, if $\partial G\in C^1$, then every solution $v\in D_{P_\alpha}(G)$ of $P_\alpha(v)=f(x)$ satisfying $vf(x)\leq 0$ in G has one of the following properties:

(1) *v has a zero in G,*
(2) *$u=c_0e^{\alpha(x)}v$, where $c_0\neq 0$ is a constant and $\nabla\alpha(x)=\frac{B(x)}{A(x)}$.*

Proof The theorem can be proof via the inequality (2.1) by applying the same argument as that use in the proof of Theorem 3.1. But here we will give an alternative proof. By using the definition of $M[u]$ and $V[u]$, we have the following:

$$M[u] = -V[u] + \int_G \{(\nabla u)^T (a(x) - |b(x)|)(\nabla u) - (c(x) + |b(x)|)u^2\} \, dx.$$

For the last integral over G, considering the integral of the inequality (2.4) by using the divergence theorem and in view of (3.13) implies that $M[u] \leq 0$. Then the conclusion of the theorem follows from Theorem 3.1. □

Since the hypothesis $V[u] \geq 0$ contains the nontrivial solution u of $\ell(u) = 0$, Theorem 3.2 is not efficient enough. But when we take $\alpha = 1$ and $B(x) \equiv 0$ in $P_\alpha(v) = f(x)$, that is,

$$P_1(v) = \nabla \cdot (A(x)\nabla v) + C(x)v + \sum_{i=1}^{\ell} D_i(x)|v|^{\beta_i - 1}v + \sum_{j=1}^{m} E_j(x)|v|^{\gamma_j - 1}v = f(x), \tag{3.14}$$

where β_i, γ_j are real positive constants such that $\gamma_j < 1 < \beta_i$ $(i = 1, 2, \dots, \ell; j = 1, 2, \dots, m)$, we observe some interesting results for the pair of $\ell(u) = 0$ and $P_1(v) = f(x)$. Now, we will consider the equations $\ell(u) = 0$ and $P_1(v) = f(x)$.

Corollary 3.2 *If $u \in D_\ell(G)$ of $\ell(u) = 0$, $v \in D_{P_1}(G)$, and $v \neq 0$ in G and $vf(x) \leq 0$ in G, then for any $u \in C^1(G, R)$ the following Picone-type inequality holds*:

$$\begin{aligned}
&\nabla \cdot \left(\frac{u}{\varphi(v)} [\varphi(v)a(x)\nabla u - \varphi(u)A(x)\nabla v] \right) \\
&\quad \geq (\nabla u)^T (a(x) - |b(x)| - A(x))(\nabla u) - \frac{u\varphi(u)}{\varphi(v)} [P(v) - f(x)] \\
&\qquad + \left(C(x) + \sum_{i=1}^{N} \mathfrak{G}(\beta_i, 1, D_i(x), f(x)) - |b(x)| - c(x) \right)(u)^2 \\
&\qquad + A(x)\left(\nabla u - \frac{u}{v}\nabla v \right)^2,
\end{aligned} \tag{3.15}$$

where β_i, γ_j are real positive constants such that $\gamma_j < 1 < \beta_i$ $(i = 1, 2, \dots, \ell; j = 1, 2, \dots, m)$.

Corollary 3.3 *If there is a nontrivial function $u \in C^1(\bar{G}, R)$ such that $u = 0$ on ∂G and*

$$M_G[u] := \int_G \left\{ (\nabla u)^T A(x)(\nabla u) - \left(C(x) + \sum_{i=1}^{N} \mathfrak{G}(\beta_i, 1, D_i(x), f(x)) \right) u^2 \right\} dx \leq 0, \tag{3.16}$$

then every solution $v \in D_{P_1}(G)$ of (3.14) *satisfying $vf(x) \leq 0$ in G vanishes at some point of $\bar{G}$. Furthermore, if $\partial G \in C^1$, then every solution $v \in D_{P_1}(G)$ of* (3.14) *satisfying $vf(x) \leq 0$ in G has one of the following properties*:

(1) *v has a zero in G,*

(2) *$u = c_0 v$, where $c_0 \neq 0$ is a constant.*

The proof can be given by using the same process as in the proof of Theorem 3.1.

Theorem 3.3 *If there is a nontrivial solution $u \in D_\ell(G)$ of $\ell(u) = 0$ such that $u = 0$ on ∂G and*

$$V_G[u] := \int_G \Biggl\{ (\nabla u)^T \bigl(a(x) - |b(x)| - A(x)\bigr)(\nabla u) + \left(C(x) + \sum_{i=1}^{N} \mathfrak{G}\bigl(\beta_i, 1, D_i(x), f(x)\bigr) - |b(x)| - c(x) \right)(x)(u)^2 \Biggr\}\, dx \geq 0. \tag{3.17}$$

Then every solution $v \in D_{P_1}(G)$ of (3.14) *satisfying $vf(x) \leq 0$ in G must vanish at some point of $\bar{G}$.*

Proof Suppose that, contrary to our claim, there exists a solution $v \in D_{P_1}(G)$ of (3.14) satisfying $vf(x) \leq 0$ and $v \neq 0$ on $\bar{G}$. We integrate (3.15) over G and then apply the divergence theorem to obtain

$$0 \geq V_G[u] + \int_G A(x)\left(\nabla u - \frac{u}{v}\nabla v\right)^2 dx \geq 0,$$

and therefore

$$\nabla u - \frac{u}{v}\nabla v \equiv 0 \quad \textit{i.e.}\ \nabla\left(\frac{u}{v}\right) \equiv 0 \text{ in } G,$$

that is, $u/v = c_0$ on $\bar{G}$ for some constant c_0. Since $u = 0$ on ∂G we see that $c_0 = 0$, which contradicts the fact that u is nontrivial. The proof is complete. □

Corollary 3.4 *Assume that*

$$a(x) \geq |b(x)| + A(x) \tag{3.18}$$

and

$$C(x) + \sum_{i=1}^{N} \mathfrak{G}\bigl(\beta_i, 1, D_i(x), f(x)\bigr) \geq |b(x)| + c(x) \tag{3.19}$$

in G. If there exists a nontrivial solution $u \in D_\ell(G)$ of $\ell(u) = 0$ such that $u = 0$ on ∂G, then every solution $v \in D_{P_1}(G)$ of (3.14) *satisfying $vf(x) \leq 0$ must vanish at some point of $\bar{G}$.*

In the special case $b(x) \equiv 0$ and $f(x) \equiv 0$ we consider the following equations:

$$\nabla \cdot \bigl(a(x)\nabla u\bigr) + c(x)u = 0 \tag{3.20}$$

and

$$\nabla \cdot \bigl(A(x)\nabla v\bigr) + C(x)v + \sum_{i=1}^{\ell} D_i(x)|v|^{\beta_i - 1}v + \sum_{j}^{m} E_j(x)|v|^{\gamma_j - 1}v = 0. \tag{3.21}$$

For (3.20) and (3.21) the following corollary can be given as a result of a special case of Theorem 3.3.

Corollary 3.5 *Assume that*

$$a(x) \geq A(x)$$

and

$$C(x) \geq c(x) \quad \text{in } G.$$

If there is a nontrivial solution u of (3.20) *such that $u = 0$ on ∂G, then every solution v of* (3.21) *must vanish at some point of $\bar{G}$.*

Note that when we take $\alpha = 1$ and $b(x) \equiv B(x) \equiv 0$, we obtain interesting results for the considered the pair of linear and nonlinear equations, that is, our results are reduced to the well-known results in the literature. For example, if we omit the damped terms, that is, $b(x) \equiv B(x) \equiv 0$ and if we substitute $C(x) \equiv 0$, $D_1(x) \equiv C(x)$, $\beta_1 = \beta$, $E_1(x) \equiv D(x)$, $\gamma_1 = \gamma$, and $D_i(x) \equiv E_j(x) \equiv 0$ ($i = 2, 3, \ldots, \ell$; $j = 2, 3, \ldots, m$), then it is seen that our results are reduced to the results which are given in [25] in the case $a_{ij}(x) \equiv a(x)$ and $A_{ij}(x) \equiv A(x)$. In special cases these results are also reduced to the results in [23, 37].

Theorem 3.4 *Suppose that G is divided into two subdomains G_1 and G_2 by $(n-1)$-dimensional piecewise smooth hypersurface in such a way that*

$$f(x) \geq 0 \quad \text{in } G_1 \quad \text{and} \quad f(x) \leq 0 \quad \text{in } G_2. \tag{3.22}$$

If there are nontrivial functions $u_k \in C^1(\bar{G}_k, R)$ such that $u_k = 0$ on ∂G_k and

$$M_{G_k}[u_k] := \int_{G_k} \left\{ (\nabla u_k)^T A(x)(\nabla u_k) - \left(C(x) + \sum_{i=1}^{N} \mathfrak{G}\big(\beta_i, 1D_i(x), f(x)\big) \right) u_k^2 \right\} dx \leq 0, \quad k = 1, 2, \tag{3.23}$$

then every solution $v \in D_{P_1}(G)$ of (3.14) *has a zero on $\bar{G}$.*

Proof Suppose that (3.14) has a solution $v \in D_{P_1}(G)$ with no zero on $\bar{G}$. Then either $v < 0$ on $\bar{G}$ or $v > 0$ on $\bar{G}$. If $v < 0$ on $\bar{G}$, then $v < 0$ on $\bar{G}_1$, so that $vf(x) \leq 0$ in $\bar{G}_1$. Using Corollary 3.1, we see that no solution of (3.14) can be negative on $\bar{G}_1$. This contradiction shows that it is impossible that $v < 0$ on $\bar{G}$. In the case where $v > 0$ on $\bar{G}$, a similar argument leads us to a contradiction and the proof is complete. □

In [33], Yoshida also studied a similar problem. Inspired by his results we establish the following theorems by using Picone-type inequalities given in Theorems 2.3 and 2.4. Since the proofs of the theorems can be given by similar lines of thought to the proofs of Theorems 3.1 and 3.2 and the proof of Theorem 1 in [33], the proofs are omitted.

Theorem 3.5 *Let $k_0 > 0$ be a constant. Assume that there exists a nontrivial function $u \in C^1(\bar{G}; R)$ such that $u = 0$ on ∂G and*

$$\tilde{M}[u] := \int_G \left\{ A(x) \left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} - \left(C_2(x) - k_0^{-\alpha} \left|f(x)\right|\right) |u|^{\alpha+1} \right\} dx \leq 0. \tag{3.24}$$

Then for every solution $v \in D_{P_\alpha(G)}$ of (2.9), either v has a zero on $\bar{G}$ or $|v(x_0)| < k_0$ for some $x_0 \in G$.

Theorem 3.6 *If there is a nontrivial solution $u \in D_\ell(G)$ of $\ell(u) = 0$ such that $u = 0$ on ∂G and*

$$\begin{aligned} \tilde{V}[u] := \int_G &(\nabla u)^T \left(a(x) - \left|b(x)\right|\right)(\nabla u) - A(x) \left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} \\ &+ \left(C_2(x) - k_0^{-\alpha} \left|f(x)\right|\right) |u|^{\alpha+1} - \left(\left|b(x)\right| + c(x)\right) u^2 \geq 0. \end{aligned} \tag{3.25}$$

Then every solution $v \in D_{P_\alpha}(G)$ of $P_\alpha(v) = f(x)$ in G must vanish at some point of $\bar{G}$ or $|v(x_0)| < k_0$ for $x_0 \in G$.

Here we point out that, when we compare Theorems 3.5 and 3.6 with Theorems 3.1 and 3.2, respectively, we see that the condition on $f(x)$ is removed, but, while Theorems 3.5 and 3.6 cannot guarantee a zero in $\bar{G}$, Theorems 3.1 and 3.2 guarantee a zero in $\bar{G}$.

4 An application

Let Ω be an exterior domain in R^n, that is, $\Omega \supset \{x \in R : |x| \geq r_0\}$ for some $r_0 > 0$. We consider the following equations:

$$\ell(u) = 0 \quad \text{in } \Omega \tag{4.1}$$

and

$$P_1(v) = f(x) \quad \text{in } \Omega, \tag{4.2}$$

where the operators ℓ and P_1 are defined before $a, A \in C(\Omega, R^+)$, $b, B \in C(\Omega, R^n)$, $c, C \in C(\Omega, R)$, $D_i, E_j \in C(\Omega, R^+ \cup \{0\})$ ($i = 1, 2, \ldots, \ell$; $j = 1, 2, \ldots, m$) and $f \in C(\Omega, R)$, β_i, γ_j are real positive constants such that $\gamma_j < 1 < \beta_i$.

The domain $D_\ell(\Omega)$ of ℓ is defined to be set of all functions u of class $C^1(\Omega, R)$ with the property that $a(x)\nabla u \in C^1(\Omega, R^n)$. The domain $D_{P_1}(\Omega)$of is defined similarly. A solution of (4.1) (or (4.2)) is said to be the oscillatory in Ω if it has a zero in Ω_r for any $r > 0$, where

$$\Omega_r = \Omega \cap \left\{x \in R^n : |x| > r\right\}.$$

Now we will give an oscillation result for (4.2) in an exterior domain Ω in R^n which contains $\{x \in R^n : |x| \geq r_0\}$ for some $r_0 > 0$.

Theorem 4.1 *Assume that for any $r > 0$ there exists a bounded domain G in Ω_r with piecewise smooth boundary which can be divided into subdomains G_1 and G_2 by an $(n-1)$-dimensional hypersurface in such a way that $f(x) \geq 0$ in G_1 and $f(x) \leq 0$ in G_2. Assume* *furthermore that $D_i(x) \geq 0$ $(i = 1, 2, \dots, \ell)$ and $E_j(x) \geq 0$ $(j = 1, 2, \dots, m)$ in G and that there are nontrivial functions $u_k \in C^1(\bar{G}_k, R)$ such that $u_k = 0$ on ∂G_k and $M_{G_k}[u_k] \leq 0$ $(k = 1, 2)$ where M_{G_k} are defined by* (3.23). *Then every solution $v \in D_{P_1}(\Omega)$ of* (4.2) *is oscillatory in Ω.*

Proof We need to apply Theorem 3.4 to make sure that v has a zero in any domain $\bar{G} \subset \Omega_r$ as mentioned in the hypotheses of Theorem 4.1. For any $r > 0$, there exists a bounded domain G as defined in Theorem 4.1. this leads v is oscillatory in Ω. □

Example Let consider the forced elliptic equation:

$$\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} + K_1(\sin x_1 \sin x_2)|v|^{\beta-1}v + K_2(\sin x_1 \sin x_2)|v|^{2\beta-1}v = \cos x_1 \sin x_2, \tag{4.3}$$

where β, K_1, and K_2 are constants such that $\beta > 1$, $K_1 \geq 0$, and $K_2 \geq 0$, but they are not equal zero at the same time, and Ω is unbounded domain in R^2 containing a horizontal strip such that

$$[2\pi, \infty) \times [0, \pi] \subset \Omega.$$

Here $\ell = 2$, $\beta_1 = \beta$, $\beta_2 = 2\beta$, $A(x) \equiv 1$, $C(x) \equiv 0$, $D_1(x) = K_1(\sin x_1 \sin x_2)$, $D_2(x) = K_2(\sin x_1 \sin x_2)$, $D_i \equiv 0$ $(i = 3, 4, \dots, \ell)$, $E_j(x) \equiv 0$ $(j = 1, 2, \dots, m)$ and $f(x) = \cos x_1 \sin x_2$. For any fixed j, we consider the rectangle

$$G^j = \bigl(2j\pi, (2j+1)\pi\bigr) \times (0, \pi),$$

which is divided into two subdomains,

$$G_1^j = \bigl(2j\pi, \bigl(2j + (1/2)\bigr)\pi\bigr) \times (0, \pi), \qquad G_2^j = \bigl(\bigl(2j + (1/2)\bigr)\pi, (2j+1)\pi\bigr) \times (0, \pi),$$

by the vertical line $x_1 = (2j + (1/2))\pi$. It is easy to see that $f(x) \geq 0$ in G_1^j and $f(x) \leq 0$ in G_2^j. Let $u_k = \sin 2x_1 \sin x_2$ $(k = 1, 2)$, $u_k = 0$ on ∂G_k, and by simple calculations we have

$$\begin{aligned}
&M_{G_k^j}[u_k] \\
&\quad = \int_{G_k^j} \biggl\{ \biggl(\frac{\partial u_k}{\partial x_1}\biggr)^2 + \biggl(\frac{\partial u_k}{\partial x_2}\biggr)^2 - \bigl[\beta(\beta-1)^{\frac{1-\beta}{\beta}} (K_1 \sin x_1 \sin x_2)^{\frac{1}{\beta}} |\cos x_1 \sin x_2|^{\frac{\beta-1}{\beta}} \\
&\qquad + 2\beta(2\beta-1)^{\frac{1-2\beta}{2\beta}} (K_2 \sin x_1 \sin x_2)^{\frac{1}{2\beta}} |\cos x_1 \sin x_2|^{\frac{2\beta-1}{2\beta}} \bigr] u_k^2 \biggr\}\, dx \\
&\quad = \frac{5\pi^2}{8} - \frac{8}{3} K_1^{1/\beta} \beta(\beta-1)^{\frac{1-\beta}{\beta}} B\biggl(\frac{3}{2} + \frac{1}{2\beta}, 2 - \frac{1}{2\beta}\biggr) \\
&\qquad - \frac{16}{3} \beta(2\beta-1)^{\frac{1-2\beta}{2\beta}} K_2^{1/2\beta} B\biggl(\frac{3}{2} + \frac{1}{4\beta}, 2 - \frac{1}{4\beta}\biggr),
\end{aligned} \tag{4.4}$$

where $B(s, t)$ is the beta function. If K_1 and K_2 are chosen satisfying the following condition:

$$\begin{aligned}
\frac{15\pi^2}{64} \leq{}& K_1^{1/\beta} \beta(\beta-1)^{\frac{1-\beta}{\beta}} B\biggl(\frac{3}{2} + \frac{1}{2\beta}, 2 - \frac{1}{2\beta}\biggr) \\
&+ 2\beta(2\beta-1)^{\frac{1-2\beta}{2\beta}} K_2^{1/2\beta} B\biggl(\frac{3}{2} + \frac{1}{4\beta}, 2 - \frac{1}{4\beta}\biggr),
\end{aligned}$$

then $M_{G_k^j}[u_k] \leq 0$ holds for $k = 1, 2$ and for any fixed $j \in N$. Therefore, Theorem 4.1 implies that every solution v of (4.3) is oscillatory in Ω for all sufficiently large β, K_1, and K_2. For example, if we choose $\beta = 2$, $K_1 = 1$, and $K_2 = 95$, then the above inequality holds.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to all the steps in writing this paper. All authors read and approved the final manuscript.

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Acknowledgements
The authors would like to thank the anonymous referees for their valuable comments and useful suggestions, which have contributed to the final version of the paper.

Received: 25 July 2014 Accepted: 11 February 2015 Published online: 03 March 2015

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