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Sublinear operators with rough kernel generated by Calderón-Zygmund operators and their commutators on generalized local Morrey spaces

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Abstract

In this paper, we will study the boundedness of a large class of sublinear operators with rough kernel T_Ω on the generalized local Morrey spaces $LM_{p,\varphi}^{\{x_0\}}$, for $s' \le p$, $p \ne 1$ or $p < s$, where $\Omega \in L_s(S^{n-1})$ with $s > 1$ are homogeneous of degree zero. In the case when $b \in LC_{p,\lambda}^{\{x_0\}}$ is a local Campanato spaces, $1 < p < \infty$, and $T_{\Omega,b}$ be is a sublinear commutator operator, we find the sufficient conditions on the pair (φ_1, φ_2) which ensures the boundedness of the operator $T_{\Omega,b}$ from one generalized local Morrey space $LM_{p,\varphi_1}^{\{x_0\}}$ to another $LM_{p,\varphi_2}^{\{x_0\}}$. In all cases the conditions for the boundedness of T_Ω are given in terms of Zygmund-type integral inequalities on (φ_1, φ_2), which do not make any assumptions on the monotonicity of φ_1, φ_2 in r. Conditions of these theorems are satisfied by many important operators in analysis, in particular pseudo-differential operators, Littlewood-Paley operators, Marcinkiewicz operators, and Bochner-Riesz operators.

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Keywords: sublinear operator; Calderón-Zygmund operator; rough kernel; generalized local Morrey space; commutator; local Campanato space

1 Introduction

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at x of radius r, ${}^{\complement}B(x,r)$ denote its complement and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$. Suppose that S^{n-1} is the unit sphere in $\mathbb{R}^n$ ($n \ge 2$) equipped with the normalized Lebesgue measure $d\sigma$.

Let $\Omega \in L_s(S^{n-1})$ with $1 < s \le \infty$ be homogeneous of degree zero. Suppose that T_Ω represents a linear or a sublinear operator, which satisfies, for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$,

$$\bigl|T_\Omega f(x)\bigr| \le c_0 \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} \bigl|f(y)\bigr|\,dy, \tag{1.1}$$

where c_0 is independent of f and x.

For a function b, suppose that the commutator operator $T_{\Omega,b}$ represents a linear or a sublinear operator, which satisfies, for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$,

$$\left|T_{\Omega,b}f(x)\right| \leq c_0 \int_{\mathbb{R}^n} \left|b(x)-b(y)\right| \frac{|\Omega(x-y)|}{|x-y|^n} \left|f(y)\right| dy, \tag{1.2}$$

where c_0 is independent of f and x.

We point out that the condition (1.1) in the case $\Omega \equiv 1$ was first introduced by Soria and Weiss in [1]. The condition (1.1) is satisfied by many interesting operators in harmonic analysis, such as the Calderón-Zygmund operators, Carleson maximal operators, Hardy-Littlewood maximal operators, C Fefferman singular multipliers, R Fefferman singular integrals, Ricci-Stein oscillatory singular integrals, the Bochner-Riesz means, and so on (see [1, 2] for details).

Let $\Omega \in L_s(S^{n-1})$ with $1 < s \leq \infty$ be homogeneous of degree zero and satisfy the cancelation condition

$$\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0,$$

where $x' = x/|x|$ for any $x \neq 0$. The homogeneous singular integral operator $\overline{T}_\Omega$ defined by

$$\overline{T}_\Omega f(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy$$

satisfies the condition (1.1).

It is obvious that when $\Omega \equiv 1$, $\overline{T}_\Omega$ is the singular integral operator $\overline{T}$.

Theorem A ([3]) *Suppose that $1 \leq p < \infty$, $\Omega \in L_s(S^{n-1})$, $s > 1$, is homogeneous of degree zero and has mean value zero on S^{n-1}. If $s' \leq p$, $p \neq 1$ or $p < s$, then the operator $\overline{T}_\Omega$ is bounded on $L_p(\mathbb{R}^n)$. Also the operator $\overline{T}_\Omega$ is bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$.*

Let b be a locally integrable function on $\mathbb{R}^n$, then we shall define the commutators generated by singular integral operators with rough kernels and b as follows:

$$[b, \overline{T}_\Omega]f(x) \equiv b(x)\overline{T}_\Omega f_1(x) - \overline{T}_\Omega(bf)(x) = p.v. \int_{\mathbb{R}^n} \left[b(x)-b(y)\right] \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy.$$

Theorem B ([3]) *Suppose that $\Omega \in L_s(S^{n-1})$, $s > 1$, is homogeneous of degree zero and has mean value zero on S^{n-1}. Let $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$. If $s' \leq p$ or $p < s$, then the commutator operator $[b, \overline{T}_\Omega]$ is bounded on $L_p(\mathbb{R}^n)$.*

The classical Morrey spaces $M_{p,\lambda}$ were first introduced by Morrey in [4] to study the local behavior of solutions to second order elliptic partial differential equations. For the boundedness of the Hardy-Littlewood maximal operator, the fractional integral operator and the Calderón-Zygmund singular integral operator on these spaces, we refer the readers to [5–7]. For the properties and applications of classical Morrey spaces, see [8–11] and references therein. The generalized Morrey spaces $M_{p,\varphi}$ are obtained by replacing r^λ by a function $\varphi(r)$ in the definition of the Morrey space. During the last decades various classical operators, such as maximal, singular, and potential operators, were widely investigated in both in classical and generalized Morrey spaces.

In this paper, we prove the boundedness of the operators T_Ω from one generalized local Morrey space $LM^{\{x_0\}}_{p,\varphi_1}$ to another $LM^{\{x_0\}}_{p,\varphi_2}$, $1 < p < \infty$, and from the space $LM^{\{x_0\}}_{1,\varphi_1}$ to the weak space $WLM^{\{x_0\}}_{1,\varphi_2}$. In the case $b \in LC^{\{x_0\}}_{p_2,\lambda}$, we find the sufficient conditions on the pair (φ_1, φ_2) which ensure the boundedness of the commutator operators $[b, T_\Omega]$ from $LM^{\{x_0\}}_{p,\varphi_1}$ to $LM^{\{x_0\}}_{p,\varphi_2}$, $1 < p < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Generalized local Morrey spaces

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 2.1 Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and $1 \le p < \infty$. We denote by $\mathcal{M}_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L^{\mathrm{loc}}_p(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{\mathcal{M}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \|f\|_{L_p(B(x,r))}. \tag{2.1}$$

The generalized Morrey spaces $\mathcal{M}_{p,\varphi}(\mathbb{R}^n)$ with norm (2.1) introduced by Mizuhara in [12], which was later extended and studied by many authors (see [13, 14]). Note that the generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ with normalized form

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \left|B(x,r)\right|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))} \tag{2.2}$$

were first defined by Guliyev in [15].

Also, in [15], there was defined the weak generalized Morrey space $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ of all functions $f \in WL^{\mathrm{loc}}_p(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \left|B(x,r)\right|^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))} < \infty.$$

According to this definition, we recover the Morrey space $M_{p,\lambda}$ and weak Morrey space $WM_{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}, \qquad WM_{p,\lambda} = WM_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

Recall that in 1994 the doctoral thesis [16] by Guliyev (see also [17–20]) introduced the local Morrey-type space $LM_{p\theta,w}$ given by

$$\|f\|_{LM_{p\theta,w}} = \left\| w(r) \|f\|_{L_p(B(0,r))} \right\|_{L_\theta(0,\infty)} < \infty,$$

where w is a positive measurable function defined on $(0, \infty)$. The main purpose of [16] (also of [17–20]) is to give some sufficient conditions for the boundedness of fractional integral operators and singular integral operators defined on homogeneous Lie groups in the local Morrey-type space $LM_{p\theta,w}$. In a series of papers by Burenkov, H Guliyev and

V Guliyev, *etc.* (see [21–24]), some necessary and sufficient conditions for the boundedness of fractional maximal operators, fractional integral operators, and singular integral operators in local Morrey-type spaces $LM_{p\theta,w}$ were given.

Definition 2.2 Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $LM_{p,\varphi} \equiv LM_{p,\varphi}(\mathbb{R}^n)$ the generalized central (local) Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LM_{p,\varphi}} = \sup_{r>0} \varphi(0,r)^{-1} \left|B(0,r)\right|^{-\frac{1}{p}} \|f\|_{L_p(B(0,r))}.$$

Also by $WLM_{p,\varphi} \equiv WLM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\varphi}} = \sup_{r>0} \varphi(0,r)^{-1} \left|B(0,r)\right|^{-\frac{1}{p}} \|f\|_{WL_p(B(0,r))} < \infty.$$

Particularly, if $\theta = \infty$, $LM_{p\infty,w} = LM_{p,w}$, then the generalized central Morrey spaces $LM_{p,\varphi}$ are the same spaces as the local Morrey spaces $LM_{p\theta,w}$ with $w(r) = \varphi(0,r)^{-1} r^{-n/p}$. Note that $f \in M_{p,\varphi}$ if and only if $f(\cdot - x)_{x \in \mathbb{R}^n}$ forms a bounded set in $LM_{p\varphi}$.

Definition 2.3 Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $LM_{p,\varphi}^{\{x_0\}} \equiv LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ the generalized local Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LM_{p,\varphi}^{\{x_0\}}} = \left\|f(x_0 + \cdot)\right\|_{LM_{p,\varphi}}.$$

Also by $WLM_{p,\varphi}^{\{x_0\}} \equiv WLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ we denote the weak generalized local Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\varphi}^{\{x_0\}}} = \left\|f(x_0 + \cdot)\right\|_{WLM_{p,\varphi}} < \infty.$$

According to this definition, we recover the local Morrey space $LM_{p,\lambda}^{\{x_0\}}$ and weak local Morrey space $WLM_{p,\lambda}^{\{x_0\}}$ under the choice $\varphi(x_0,r) = r^{\frac{\lambda-n}{p}}$:

$$LM_{p,\lambda}^{\{x_0\}} = LM_{p,\varphi}^{\{x_0\}}\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}, \qquad WLM_{p,\lambda}^{\{x_0\}} = WLM_{p,\varphi}^{\{x_0\}}\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}.$$

Wiener [25, 26] looked for a way to describe the behavior of a function at the infinity. The conditions he considered are related to appropriate weighted L_q spaces. Beurling [27] extended this idea and defined a pair of dual Banach spaces A_q and $B_{q'}$, where $1/q + 1/q' = 1$. To be precise, A_q is a Banach algebra with respect to the convolution, expressed as a union of certain weighted L_q spaces; the space $B_{q'}$ is expressed as the intersection of the corresponding weighted $L_{q'}$ spaces. Feichtinger [28] observed that the space B_q can be described by

$$\|f\|_{B_q} = \sup_{k\ge 0} 2^{-\frac{kn}{q}} \|f\chi_k\|_{L_q(\mathbb{R}^n)}, \tag{2.3}$$

where χ_0 is the characteristic function of the unit ball $\{x \in \mathbb{R}^n : |x| \leq 1\}$, χ_k is the characteristic function of the annulus $\{x \in \mathbb{R}^n : 2^{k-1} < |x| \leq 2^k\}$, $k = 1, 2, \ldots$. By duality, the space $A_q(\mathbb{R}^n)$, called the Beurling algebra now, can be described by

$$\|f\|_{A_q} = \sum_{k=0}^{\infty} 2^{-\frac{kn}{q'}} \|f\chi_k\|_{L_q(\mathbb{R}^n)}. \tag{2.4}$$

Let $\dot{B}_q(\mathbb{R}^n)$ and $\dot{A}_q(\mathbb{R}^n)$ be the homogeneous versions of $B_q(\mathbb{R}^n)$ and $A_q(\mathbb{R}^n)$ by taking $k \in \mathbb{Z}$ in (2.3) and (2.4) instead of $k \geq 0$ there.

If $\lambda < 0$ or $\lambda > n$, then $LM_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$. Note that $LM_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ and $LM_{p,n}(\mathbb{R}^n) = \dot{B}_p(\mathbb{R}^n)$;

$$\dot{B}_{p,\mu} = LM_{p,\varphi}\big|_{\varphi(0,r)=r^{\mu n}}, \qquad W\dot{B}_{p,\mu} = WLM_{p,\varphi}\big|_{\varphi(0,r)=r^{\mu n}}.$$

Alvarez *et al.* [29], in order to study the relationship between central BMO spaces and Morrey spaces, introduced λ-central bounded mean oscillation spaces and central Morrey spaces $\dot{B}_{p,\mu}(\mathbb{R}^n) \equiv LM_{p,n+np\mu}(\mathbb{R}^n)$, $\mu \in [-\frac{1}{p}, 0]$. If $\mu < -\frac{1}{p}$ or $\mu > 0$, then $\dot{B}_{p,\mu}(\mathbb{R}^n) = \Theta$. Note that $\dot{B}_{p,-\frac{1}{p}}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ and $\dot{B}_{p,0}(\mathbb{R}^n) = \dot{B}_p(\mathbb{R}^n)$. Also define the weak central Morrey spaces $W\dot{B}_{p,\mu}(\mathbb{R}^n) \equiv WLM_{p,n+np\mu}(\mathbb{R}^n)$.

Inspired by this, we consider the boundedness of singular integral operator with rough kernel on generalized local Morrey spaces and give the central bounded mean oscillation estimates for their commutators.

3 Sublinear operators with rough kernel generated by Calderón-Zygmund operators in the spaces $LM_{p,\varphi}^{\{x_0\}}$

In this section we are going to use the following statement on the boundedness of the weighted Hardy operator:

$$H_w g(t) := \int_t^{\infty} g(s)w(s)\,ds, \quad 0 < t < \infty,$$

where w is a fixed function non-negative and measurable on $(0,\infty)$.

The following theorem was proved in [30, 31].

Theorem 3.1 *Let v_1, v_2, and w be positive almost everywhere and measurable functions on $(0,\infty)$. The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t)H_w g(t) \leq C \operatorname*{ess\,sup}_{t>0} v_1(t)g(t) \tag{3.1}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^{\infty} \frac{w(s)\,ds}{\operatorname{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Moreover, the value $C = B$ is the best constant for (3.1).

The following statement, containing the results obtained in [12, 13] was proved in [3].

Theorem 3.2 *Suppose that $\Omega \in L_s(S^{n-1})$, $s > 1$, is homogeneous of degree zero and has mean value zero on S^{n-1}. Let $1 \le s' < p < \infty$ and $\varphi(x,r)$ satisfy the conditions*

$$c^{-1}\varphi(x,r) \le \varphi(x,t) \le c\varphi(x,r) \tag{3.2}$$

whenever $r \le t \le 2r$, where c (≥ 1) does not depend on $t, r, x \in \mathbb{R}^n$, and

$$\int_r^\infty \varphi(x,t)^p \frac{dt}{t} \le C\varphi(x,r)^p, \tag{3.3}$$

where C does not depend on x and r. Then the operator $\overline{T}_\Omega$ is bounded on $M_{p,\varphi}$.

The following statement, containing the results obtained in [12, 13] was proved in [15, 16] (see also [17, 21–23, 32]).

Theorem 3.3 *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \varphi_1(0,t) \frac{dt}{t} \le C\varphi_2(0,r), \tag{3.4}$$

where C does not depend on r. Then the operator $\overline{T}$ is bounded from LM_{p,φ_1} to LM_{p,φ_2} for $p > 1$ and from LM_{1,φ_1} to WLM_{1,φ_2} for $p = 1$.

Corollary 3.4 *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \varphi_1(x,t) \frac{dt}{t} \le C\varphi_2(x,r), \tag{3.5}$$

where C does not depend on x and r. Then the operator $\overline{T}$ is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2} for $p = 1$.

The following statement, containing results obtained in [15, 16], was proved in [30].

Theorem 3.5 *Let $x_0 \in \mathbb{R}^n$, $1 \le p < \infty$, and $\Omega \in L_s(S^{n-1})$, $s > 1$, be a homogeneous of degree zero. Let also, for $s' \le p$ or $p < s$, the pair (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}\, dt \le C\varphi_2(x_0,r), \tag{3.6}$$

where C does not depend on r. Then the operator $\overline{T}_\Omega$ is bounded from $LM^{\{x_0\}}_{p,\varphi_1}$ to $LM^{\{x_0\}}_{p,\varphi_2}$ for $p > 1$ and from $LM^{\{x_0\}}_{1,\varphi_1}$ to $WLM^{\{x_0\}}_{1,\varphi_2}$ for $p = 1$.

Corollary 3.6 *Let $1 \le p < \infty$, $\Omega \in L_s(S^{n-1})$, $s > 1$, be a homogeneous of degree zero. Let also, for $s' \le p$ or $p < s$, the pair (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}\, dt \le C\varphi_2(x,r), \tag{3.7}$$

where C does not depend on x and r. Then the operator $\overline{T}_\Omega$ is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2} for $p = 1$.

Lemma 3.7 *Let $x_0 \in \mathbb{R}^n$, $1 \le p < \infty$, T_Ω be a sublinear operator satisfying condition* (1.1) *with $\Omega \in L_s(S^{n-1})$, $s > 1$, be a homogeneous of degree zero, bounded on $L_p(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$.*

If $p > 1$ and $s' \le p$, then the inequality

$$\|T_\Omega f\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} t^{-\frac{n}{p}-1}\,dt$$

holds for any ball $B(x_0,r)$ and for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

If $p > 1$ and $p < s$, then the inequality

$$\|T_\Omega f\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}-\frac{n}{s}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} t^{\frac{n}{s}-\frac{n}{p}-1}\,dt$$

holds for any ball $B(x_0,r)$ and for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

Moreover, for $s > 1$ the inequality

$$\|T_\Omega f\|_{WL_1(B(x_0,r))} \lesssim r^n \int_{2r}^{\infty} t^{-n-1} \|f\|_{L_1(B(x_0,t))}\,dt \tag{3.8}$$

holds for any ball $B(x_0,r)$ and for all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof Let $1 < p < \infty$ and $s' \le p$. Set $B = B(x_0,r)$ for the ball centered at x_0 and of radius r. We represent f as

$$f = f_1 + f_2, \qquad f_1(y) = f(y)\chi_{2B}(y), \qquad f_2(y) = f(y)\chi_{\complement_{(2B)}}(y), \quad r > 0, \tag{3.9}$$

and have

$$\|T_\Omega f\|_{L(B)} \le \|T_\Omega f_1\|_{L_p(B)} + \|T_\Omega f_2\|_{L_p(B)}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$, $T_\Omega f_1 \in L_p(\mathbb{R}^n)$ and from the boundedness of T_Ω on $L_p(\mathbb{R}^n)$ it follows that

$$\|T_\Omega f_1\|_{L_p(B)} \le \|T_\Omega f_1\|_{L_p(\mathbb{R}^n)} \le C\|f_1\|_{L_p(\mathbb{R}^n)} = C\|f\|_{L_p(2B)},$$

where constant $C > 0$ is independent of f.

Note that

$$\begin{aligned}
\big\|\Omega(x-\cdot)\big\|_{L_s(B(x_0,t))} &= \left(\int_{B(x-x_0,t)} \big|\Omega(y)\big|^s\,dy\right)^{\frac{1}{s}} \\
&\le \left(\int_{B(0,t+|x-x_0|)} \big|\Omega(y)\big|^s\,dy\right)^{\frac{1}{s}} \\
&= \left(\int_0^{t+|x-x_0|} r^{n-1}\,dr \int_{S^{n-1}} \big|\Omega(y')\big|^s d\sigma(y')\right)^{\frac{1}{s}} \\
&= c_0 \|\Omega\|_{L_s(S^{n-1})} \big|B\big(0, t+|x-x_0|\big)\big|^{\frac{1}{s}},
\end{aligned} \tag{3.10}$$

where $c_0 = (n\nu_n)^{-1/s}$ and $\nu_n = |B(0,1)|$.

It is clear that $x \in B$, $y \in {}^{\complement}(2B)$ implies $\frac{1}{2}|x_0 - y| \le |x - y| \le \frac{3}{2}|x_0 - y|$. We get

$$\left|T_\Omega f_2(x)\right| \le 2^n c_1 \int_{{}^{\complement}(2B)} \frac{|f(y)||\Omega(x-y)|}{|x_0 - y|^n}\, dy.$$

By the Fubini theorem we have

$$\begin{aligned}\int_{{}^{\complement}(2B)} \frac{|f(y)||\Omega(x-y)|}{|x_0 - y|^n}\, dy &\approx \int_{{}^{\complement}(2B)} \left|f(y)\right|\left|\Omega(x-y)\right| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}}\, dy \\ &\approx \int_{2r}^{\infty} \int_{2r \le |x_0 - y| \le t} \left|f(y)\right|\left|\Omega(x-y)\right| dy \frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} \left|f(y)\right|\left|\Omega(x-y)\right| dy \frac{dt}{t^{n+1}}.\end{aligned}$$

Applying the Hölder inequality, we get

$$\begin{aligned}\int_{{}^{\complement}(2B)} \frac{|f(y)||\Omega(x-y)|}{|x_0 - y|^n}\, dy &\lesssim \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \left\|\Omega(x-\cdot)\right\|_{L_s(B(x_0,t))} \left|B(x_0,t)\right|^{1-\frac{1}{p}-\frac{1}{s}} \frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \left|B\left(x_0, t + |x - x_0|\right)\right|^{\frac{1}{s}} \left|B(x_0,t)\right|^{1-\frac{1}{p}-\frac{1}{s}} \frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \left|B(x_0,t)\right|^{1-\frac{1}{p}} \frac{dt}{t^{n+1}} \\ &\approx \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}. \end{aligned} \tag{3.11}$$

Moreover, for all $p \in [1,\infty)$, the inequality

$$\|T_\Omega f_2\|_{L_p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}} \tag{3.12}$$

is valid. Thus

$$\|T_\Omega f\|_{L_p(B)} \lesssim \|f\|_{L_p(2B)} + r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.$$

On the other hand,

$$\|f\|_{L_p(2B)} \approx r^{\frac{n}{p}} \|f\|_{L_p(2B)} \int_{2r}^{\infty} \frac{dt}{t^{\frac{n}{p}+1}} \le r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}. \tag{3.13}$$

Thus

$$\|T_\Omega f\|_{L_p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.$$

When $1 < p < s$, by the Fubini theorem, the Minkowski inequality and (3.10), we get

$$\begin{aligned}\|T_\Omega f_2\|_{L_p(B)} &\le \left(\int_B \left(\int_{2r}^{\infty} \int_{B(x_0,t)} \left|f(y)\right|\left|\Omega(x-y)\right| dy \frac{dt}{t^{n+1}}\right)^p dx\right)^{\frac{1}{p}} \\ &\le \int_{2r}^{\infty} \int_{B(x_0,t)} \left|f(y)\right| \left\|\Omega(\cdot - y)\right\|_{L_p(B)} dy \frac{dt}{t^{n+1}}\end{aligned}$$

$$\begin{aligned}
&\le \left|B(x_0,r)\right|^{\frac{1}{p}-\frac{1}{s}} \int_{2r}^{\infty} \int_{B(x_0,t)} \left|f(y)\right| \left\|\Omega(\cdot-y)\right\|_{L_s(B)}\, dy \frac{dt}{t^{n+1}} \\
&\le \left|B(x_0,r)\right|^{\frac{1}{p}-\frac{1}{s}} \int_{2r}^{\infty} \int_{B(x_0,t)} \left|f(y)\right| \left|B\bigl(x_0, r+|x_0-y|\bigr)\right|^{\frac{1}{s}}\, dy \frac{dt}{t^{n+1}} \\
&\lesssim r^{\frac{n}{p}-\frac{n}{s}} \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \left|B(x_0,t)\right|^{\frac{1}{s}} \frac{dt}{t^{n+1}} \\
&\approx r^{\frac{n}{p}-\frac{n}{s}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} t^{\frac{n}{s}-\frac{n}{p}-1}\, dt.
\end{aligned} \tag{3.14}$$

Let $p = 1 < s \le \infty$. From the weak (1,1) boundedness of T_Ω and (3.13) it follows that

$$\begin{aligned}
\|T_\Omega f_1\|_{WL_1(B)} &\le \|T_\Omega f_1\|_{WL_1(\mathbb{R}^n)} \lesssim \|f_1\|_{L_1(\mathbb{R}^n)} \\
&= \|f\|_{L_1(2B)} \lesssim r^n \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \frac{dt}{t^{n+1}}.
\end{aligned} \tag{3.15}$$

Then from (3.12) and (3.15) we get the inequality (3.8). □

Theorem 3.8 *Let $x_0 \in \mathbb{R}^n$, $1 \le p < \infty$, T_Ω be a sublinear operator satisfying condition* (1.1) *with $\Omega \in L_s(S^{n-1})$, $s > 1$, be a homogeneous of degree zero. Suppose that the operator T_Ω is bounded on $L_p(\mathbb{R}^n)$ for $p > 1$ and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. Let also, for $s' \le p$, $p \ne 1$, the pair (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}\, dt \le C\varphi_2(x_0,r), \tag{3.16}$$

and for $1 < p < s$ the pair (φ_1, φ_2) satisfy the condition

$$\int_r^{\infty} \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}-\frac{n}{s}+1}}\, dt \le C\varphi_2(x_0,r) r^{\frac{n}{s}}, \tag{3.17}$$

where C does not depend on r.

Then the operator T_Ω is bounded from $LM_{p,\varphi_1}^{\{x_0\}}$ to $LM_{p,\varphi_2}^{\{x_0\}}$. Moreover,

$$\|T_\Omega f\|_{LM_{p,\varphi_2}^{\{x_0\}}} \lesssim \|f\|_{LM_{p,\varphi_1}^{\{x_0\}}}.$$

Also the operator T_Ω is bounded from $LM_{1,\varphi_1}^{\{x_0\}}$ to $WLM_{1,\varphi_2}^{\{x_0\}}$ and

$$\|T_\Omega f\|_{WLM_{1,\varphi_2}^{\{x_0\}}} \lesssim \|f\|_{LM_{1,\varphi_1}^{\{x_0\}}}.$$

Proof Let $1 < p < \infty$ and $s' \le p$. By Lemma 3.7 and Theorem 3.1 with $\nu_2(r) = \varphi_2(x_0,r)^{-1}$, $\nu_1(r) = \varphi_1(x_0,r)^{-1} r^{-\frac{n}{p}}$, $g(r) = \|f\|_{L_p(B(x_0,r))}$, and $w(r) = r^{-\frac{n}{p}-1}$ we have

$$\begin{aligned}
\|T_\Omega f\|_{LM_{p,\varphi_2}^{\{x_0\}}} &\lesssim \sup_{r>0} \varphi_2(x_0,r)^{-1} \int_r^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}} \\
&\lesssim \sup_{r>0} \varphi_1(x_0,r)^{-1} r^{-\frac{n}{p}} \|f\|_{L_p(B(x_0,r))} = \|f\|_{LM_{p,\varphi_1}^{\{x_0\}}}.
\end{aligned}$$

Let $1 < p < s$. By Lemma 3.7 and Theorem 3.1 with $v_2(r) = \varphi_2(x_0,r)^{-1}$, $v_1(r) = \varphi_1(x_0,r)^{-1}r^{-\frac{n}{p}+\frac{n}{s}}$, $g(r) = \|f\|_{L_p(B(x_0,r))}$, and $w(r) = r^{-\frac{n}{p}+\frac{n}{s}-1}$ we have

$$\begin{aligned}\|T_\Omega f\|_{LM^{\{x_0\}}_{p,\varphi_2}} &\lesssim \sup_{r>0}\varphi_2(x_0,r)^{-1}r^{-\frac{n}{s}}\int_r^\infty \|f\|_{L_p(B(x_0,t))}\frac{dt}{t^{\frac{n}{p}-\frac{n}{s}+1}} \\ &\lesssim \sup_{r>0}\varphi_1(x_0,r)^{-1}r^{-\frac{n}{p}}\|f\|_{L_p(B(x_0,r))} = \|f\|_{LM^{\{x_0\}}_{p,\varphi_1}}.\end{aligned}$$

Also for $p = 1$

$$\begin{aligned}\|T_\Omega f\|_{WLM^{\{x_0\}}_{1,\varphi_2}} &\lesssim \sup_{r>0}\varphi_2(x_0,r)^{-1}\int_r^\infty \|f\|_{L_1(B(x_0,t))}\frac{dt}{t^{n+1}} \\ &\lesssim \sup_{r>0}\varphi_1(x_0,r)^{-1}r^{-n}\|f\|_{L_p(B(x_0,r))} = \|f\|_{LM^{\{x_0\}}_{1,\varphi_1}}.\end{aligned}$$ □

Corollary 3.9 *Let $x_0 \in \mathbb{R}^n$, $1 \le p < \infty$, T_Ω be a sublinear operator satisfying condition* (1.1), *with $\Omega \in L_s(S^{n-1})$, $s > 1$, being a homogeneous of degree zero and bounded on $L_p(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$.*

Let also, for $s' \le p$, $p \ne 1$, the pair (φ_1,φ_2) satisfy the condition

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<\tau<\infty}\varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}\,dt \le C\varphi_2(x,r), \tag{3.18}$$

and, for $1 < p < s$, the pair (φ_1,φ_2) satisfy the condition

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<\tau<\infty}\varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}-\frac{n}{s}+1}}\,dt \le C\varphi_2(x,r)r^{\frac{n}{s}}, \tag{3.19}$$

where C does not depend on x and r.

Then the operator T_Ω is bounded from M_{p,φ_1} to M_{p,φ_2}. Moreover,

$$\|T_\Omega f\|_{M_{p,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}}.$$

Also the operator T_Ω is bounded from M_{1,φ_1} to WM_{1,φ_2} and

$$\|T_\Omega f\|_{WM_{1,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}.$$

Corollary 3.10 *Let $1 \le p < \infty$ and (φ_1,φ_2) satisfy condition* (3.16). *Then the operator $\overline{T}$ is bounded from $LM^{\{x_0\}}_{p,\varphi_1}$ to $LM^{\{x_0\}}_{p,\varphi_2}$ for $p > 1$ and from $LM^{\{x_0\}}_{1,\varphi_1}$ to $WLM^{\{x_0\}}_{1,\varphi_2}$.*

Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. The rough Hardy-Littlewood maximal function M_Ω is defined by

$$M_\Omega f(x) = \sup_{t>0}\frac{1}{|B(x,t)|}\int_{B(x,t)}\left|\Omega(x-y)\right|\left|f(y)\right|dy.$$

Then we can give the following corollary.

Corollary 3.11 *Let $1 \le p < \infty$, $\Omega \in L_s(S^{n-1})$. For $s' \le p$, $p \ne 1$, the pair (φ_1,φ_2) satisfies condition* (3.16) *and, for $1 < p < s$, the pair (φ_1,φ_2) satisfies condition* (3.17). *Then the operators M_Ω and $\overline{T}_\Omega$ are bounded from $LM^{\{x_0\}}_{p,\varphi_1}$ to $LM^{\{x_0\}}_{p,\varphi_2}$, for $p > 1$, and from $LM^{\{x_0\}}_{1,\varphi_1}$ to $WLM^{\{x_0\}}_{1,\varphi_2}$.*

Corollary 3.12 *Let $1 \le p < \infty$, $\Omega \in L_s(S^{n-1})$. For $s' \le p$, $p \ne 1$, the pair (φ_1, φ_2) satisfies condition* (3.18) *and, for $1 < p < s$, the pair (φ_1, φ_2) satisfies condition* (3.19). *Then the operators M_Ω and $\overline{T}_\Omega$ are bounded from M_{p,φ_1} to M_{p,φ_2} and from M_{1,φ_1} to WM_{1,φ_2}.*

Remark 3.13 Note that, in the case $s = \infty$, Corollary 3.9 was proved in [33]. The condition (3.16) in Theorem 3.8 is weaker than condition (3.4) in Theorem 3.3 (see [33]).

4 Commutators of linear operators with rough kernel generated by Calderón-Zygmund operators in the spaces $LM^{\{x_0\}}_{p,\varphi}$

Let T be a linear operator; for a function b, we define the commutator $[b, T]$ by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x)$$

for any suitable function f. Let $\widetilde{T}$ be a Calderón-Zygmund singular integral operator. A well-known result of Coifman *et al.* [34] states that the commutator $[b, \widetilde{T}]f = b\widetilde{T}f - \widetilde{T}(bf)$ is bounded on $L_p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $b \in BMO(\mathbb{R}^n)$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [8–10, 35]).

The definition of a local Campanato space is as follows.

Definition 4.1 Let $1 \le q < \infty$ and $0 \le \lambda < \frac{1}{n}$. A function $f \in L^{\mathrm{loc}}_q(\mathbb{R}^n)$ is said to belong to the $LC^{\{x_0\}}_{q,\lambda}(\mathbb{R}^n)$ (local Campanato space), if

$$\|f\|_{LC^{\{x_0\}}_{q,\lambda}} = \sup_{r>0} \left(\frac{1}{|B(x_0,r)|^{1+\lambda q}} \int_{B(x_0,r)} \left| f(y) - f_{B(x_0,r)} \right|^q dy \right)^{1/q} < \infty,$$

where

$$f_{B(x_0,r)} = \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} f(y)\, dy.$$

Define

$$LC^{\{x_0\}}_{q,\lambda}(\mathbb{R}^n) = \left\{ f \in L^{\mathrm{loc}}_q(\mathbb{R}^n) : \|f\|_{LC^{\{x_0\}}_{q,\lambda}} < \infty \right\}.$$

In [36], Lu and Yang introduced the central BMO space $CBMO_q(\mathbb{R}^n) = LC^{\{0\}}_{q,0}(\mathbb{R}^n)$. Note that $BMO(\mathbb{R}^n) \subset \bigcap_{q>1} CBMO^{\{x_0\}}_q(\mathbb{R}^n)$, $1 \le q < \infty$. The space $CBMO^{\{x_0\}}_q(\mathbb{R}^n)$ can be regarded as a local version of $BMO(\mathbb{R}^n)$, the space of bounded mean oscillation, at the origin. But they have quite different properties. The classical John-Nirenberg inequality shows that functions in $BMO(\mathbb{R}^n)$ are locally exponentially integrable. This implies that, for any $1 \le q < \infty$, the functions in $BMO(\mathbb{R}^n)$ can be described by means of the condition:

$$\sup_{B \subset \mathbb{R}^n} \left(\frac{1}{|B|} \int_B \left| f(y) - f_B \right|^q dy \right)^{1/q} < \infty,$$

where B denotes an arbitrary ball in $\mathbb{R}^n$. However, the space $CBMO^{\{x_0\}}_q(\mathbb{R}^n)$ depends on q. If $q_1 < q_2$, then $CBMO^{\{x_0\}}_{q_2}(\mathbb{R}^n) \subsetneqq CBMO^{\{x_0\}}_{q_1}(\mathbb{R}^n)$. Therefore, there is no analogy of the famous

John-Nirenberg inequality of $BMO(\mathbb{R}^n)$ for the space $CBMO_q^{\{x_0\}}(\mathbb{R}^n)$. One can imagine that the behavior of $CBMO_q^{\{x_0\}}(\mathbb{R}^n)$ may be quite different from that of $BMO(\mathbb{R}^n)$.

We will use the following statement on the boundedness of the weighted Hardy operator:

$$H_w^* g(r) := \int_r^\infty \left(1 + \ln \frac{t}{r}\right) g(t) w(t)\, dt, \quad r \in (0, \infty),$$

where w is a weight.

The following theorem was proved in [37].

Theorem 4.2 ([37]) *Let v_1, v_2, and w be positive almost everywhere and measurable functions on $(0,\infty)$. The inequality*

$$\operatorname*{ess\,sup}_{r>0} v_2(r) H_w^* g(r) \le C \operatorname*{ess\,sup}_{r>0} v_1(r) g(r) \tag{4.1}$$

holds, for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$, if and only if

$$B := \sup_{r>0} v_2(r) \int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{w(t)\, dt}{\sup_{t<s<\infty} v_1(s)} < \infty. \tag{4.2}$$

Moreover, the value $C = B$ is the best constant for (4.1).

Remark 4.3 In (4.1)-(4.2) it is assumed that $0 \cdot \infty = 0$.

Lemma 4.4 *Let b be a function in $LC_{q,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, $1 \le q < \infty$, $0 \le \lambda < \frac{1}{n}$, and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{|B(x_0,r_1)|^{1+\lambda q}} \int_{B(x_0,r_1)} \left|b(y) - b_{B(x_0,r_2)}\right|^q dy\right)^{\frac{1}{q}} \le C\left(1 + \left|\ln \frac{r_1}{r_2}\right|\right) \|b\|_{LC_{q,\lambda}^{\{x_0\}}},$$

where $C > 0$ is independent of b, r_1, and r_2.

In [3] the following statement was proved for the commutators of singular integral operators with rough kernels, containing the result in [12, 13].

Theorem 4.5 *Suppose that $\Omega \in L_s(S^{n-1})$, $s > 1$, is homogeneous of degree zero and $b \in BMO(\mathbb{R}^n)$. Let $1 \le s' < p < \infty$, $\varphi(x,r)$ satisfy the conditions* (3.2) *and* (3.3). *Then the operator $[b, \overline{T}_\Omega]$ is bounded on $M_{p,\varphi}$.*

Lemma 4.6 *Let $x_0 \in \mathbb{R}^n$, $1 < p < \infty$, $b \in LC_{p_2,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and $0 \le \lambda < \frac{1}{n}$. Let also T_Ω be a linear operator satisfying condition* (1.1) *with $\Omega \in L_s(S^{n-1})$, $s > 1$, be a homogeneous of degree zero and bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$.*

Then, for $s' \le p_1$, the inequality

$$\left\|[b, T_\Omega] f\right\|_{L_p(B(x_0,r))} \lesssim \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} r^{\frac{n}{p}} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right) t^{n\lambda - \frac{n}{p_1} - 1} \|f\|_{L_{p_1}(B(x_0,t))}\, dt$$

holds, for any ball $B(x_0,r)$ and for all $f \in L_{p_1}^{\mathrm{loc}}(\mathbb{R}^n)$.

Also, for $p_1 < s$, the inequality

$$\big\|[b,T_\Omega]f\big\|_{L_p(B(x_0,r))} \lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p}-\frac{n}{s}} \int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)t^{n\lambda-\frac{n}{p_1}+\frac{n}{s}-1}\|f\|_{L_{p_1}(B(x_0,t))}\,dt$$

holds, for any ball $B(x_0,r)$ and for all $f \in L^{\mathrm{loc}}_{p_1}(\mathbb{R}^n)$.

Proof Let $1 < p < \infty$, $b \in LC^{\{x_0\}}_{p_2,\lambda}(\mathbb{R}^n)$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. As in the proof of Lemma 3.7, we represent the function f in the form (3.9) and have

$$\begin{aligned}[b,T_\Omega]f(x) &\equiv J_1+J_2+J_3+J_4 = \big(b(x)-b_B\big)T_\Omega f_1(x)\\ &\quad - T_\Omega\big(\big(b(\cdot)-b_B\big)f_1\big)(x) + \big(b(x)-b_B\big)T_\Omega f_2(x) - T_\Omega\big(\big(b(\cdot)-b_B\big)f_2\big)(x).\end{aligned}$$

Hence we get

$$\big\|[b,T_\Omega]f\big\|_{L_p(B)} \le \|J_1\|_{L_p(B)} + \|J_2\|_{L_p(B)} + \|J_3\|_{L_p(B)} + \|J_4\|_{L_p(B)}.$$

From the boundedness of T_Ω on $L_p(\mathbb{R}^n)$ and Lemma 4.4 it follows that

$$\begin{aligned}\|J_1\|_{L_p(B)} &\le \big\|\big(b(\cdot)-b_B\big)T_\Omega f_1(\cdot)\big\|_{L_p(\mathbb{R}^n)}\\ &\le \big\|\big(b(\cdot)-b_B\big)\big\|_{L_{p_2}(\mathbb{R}^n)}\big\|T_\Omega f_1(\cdot)\big\|_{L_{p_1}(\mathbb{R}^n)}\\ &\le C\|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p_2}+n\lambda}\|f_1\|_{L_{p_1}(\mathbb{R}^n)}\\ &= C\|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p_2}+\frac{n}{p_1}+n\lambda}\|f\|_{L_{p_1}(2B)}\int_{2r}^{\infty}t^{-1-\frac{n}{p_1}}\,dt\\ &\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)t^{n\lambda-\frac{n}{p_1}-1}\|f\|_{L_{p_1}(B(x_0,t))}\,dt.\end{aligned}$$

From Lemma 4.4 for J_2 we have

$$\begin{aligned}\|J_2\|_{L_p(B)} &\le \big\|T_\Omega\big(b(\cdot)-b_B\big)f_1\big\|_{L_p(\mathbb{R}^n)}\\ &\lesssim \big\|\big(b(\cdot)-b_B\big)f_1\big\|_{L_p(\mathbb{R}^n)}\\ &\lesssim \big\|b(\cdot)-b_B\big\|_{L_{p_2}(\mathbb{R}^n)}\|f_1\|_{L_{p_1}(\mathbb{R}^n)}\\ &\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p_2}+\frac{n}{p_1}+n\lambda}\|f\|_{L_{p_1}(2B)}\int_{2r}^{\infty}t^{-1-\frac{n}{p_1}}\,dt\\ &\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)t^{n\lambda-\frac{n}{p_1}-1}\|f\|_{L_{p_1}(B(x_0,t))}\,dt.\end{aligned}$$

For J_3, it is known that $x \in B$, $y \in \complement(2B)$, which implies $\frac{1}{2}|x_0-y| \le |x-y| \le \frac{3}{2}|x_0-y|$.

When $s' \le p_1$, by the Fubini theorem and (3.10), and applying the Hölder inequality, we have

$$\begin{aligned}\big|T_\Omega f_2(x)\big| &\le c_0\int_{\complement(2B)}\big|\Omega(x-y)\big|\frac{|f(y)|}{|x_0-y|^n}\,dy\\ &\approx \int_{2r}^{\infty}\int_{2r<|x_0-y|<t}\big|\Omega(x-y)\big|\big|f(y)\big|\,dy\,t^{-1-n}\,dt\end{aligned}$$

$$\begin{aligned}
&\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} \left|\Omega(x-y)\right| \left|f(y)\right| dy t^{-1-n} \, dt \\
&\lesssim \int_{2r}^{\infty} \|f\|_{L_{p_1}(B(x_0,t))} \left\|\Omega(x-\cdot)\right\|_{L_s(B(x_0,t))} \left|B(x_0,t)\right|^{1-\frac{1}{p_1}-\frac{1}{s}} t^{-1-n} \, dt \\
&\lesssim \int_{2r}^{\infty} \|f\|_{L_{p_1}(B(x_0,t))} \left|B\bigl(x_0, t+|x-x_0|\bigr)\right|^{\frac{1}{s}} \left|B(x_0,t)\right|^{1-\frac{1}{p_1}-\frac{1}{s}} \frac{dt}{t^{n+1}} \\
&\lesssim \int_{2r}^{\infty} t^{-\frac{n}{p_1}-1} \|f\|_{L_{p_1}(B(x_0,t))} \, dt.
\end{aligned}$$

Hence, from Lemma 4.4 we get

$$\begin{aligned}
\|J_3\|_{L_p(B)} &= \left\| \bigl(b(\cdot)-b_B\bigr) T_\Omega f_2(\cdot) \right\|_{L_p(\mathbb{R}^n)} \\
&\le \left\| \bigl(b(\cdot)-b_B\bigr) \right\|_{L_p(\mathbb{R}^n)} \int_{2r}^{\infty} t^{-\frac{n}{p_1}-1} \|f\|_{L_{p_1}(B(x_0,t))} \, dt \\
&\le \left\| \bigl(b(\cdot)-b_B\bigr) \right\|_{L_{p_2}(\mathbb{R}^n)} r^{\frac{n}{p_1}} \int_{2r}^{\infty} t^{-\frac{n}{p_1}-1} \|f\|_{L_{p_1}(B(x_0,t))} \, dt \\
&\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p}+n\lambda} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) t^{-\frac{n}{p_1}-1} \|f\|_{L_{p_1}(B(x_0,t))} \, dt \\
&\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) t^{n\lambda-\frac{n}{p_1}-1} \|f\|_{L_{p_1}(B(x_0,t))} \, dt.
\end{aligned}$$

When $p_1 < s$, by the Fubini theorem, the Minkowski inequality, (3.10) and from Lemma 4.4, we get

$$\begin{aligned}
\|J_3\|_{L_p(B)} &\le \left(\int_B \left(\int_{2r}^{\infty} \int_{B(x_0,t)} \left|f(y)\right| \left|b(x)-b_B\right| \left|\Omega(x-y)\right| dy \frac{dt}{t^{n+1}} \right)^p dx \right)^{\frac{1}{p}} \\
&\le \int_{2r}^{\infty} \int_{B(x_0,t)} \left|f(y)\right| \left\| \bigl(b(\cdot)-b_B\bigr)\Omega(\cdot-y) \right\|_{L_p(B)} dy \frac{dt}{t^{n+1}} \\
&\le \int_{2r}^{\infty} \int_{B(x_0,t)} \left|f(y)\right| \left\| b(\cdot)-b_B \right\|_{L_{p_2}(B)} \left\| \Omega(\cdot-y) \right\|_{L_{p_1}(B)} dy \frac{dt}{t^{n+1}} \\
&\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p_2}+n\lambda} |B|^{\frac{1}{p_1}-\frac{1}{s}} \int_{2r}^{\infty} \int_{B(x_0,t)} \left|f(y)\right| \left\| \Omega(\cdot-y) \right\|_{L_s(B)} dy \frac{dt}{t^{n+1}} \\
&\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p}-\frac{n}{s}+n\lambda} \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \left|B\bigl(x_0, t+|x_0-y|\bigr)\right|^{\frac{1}{s}} \frac{dt}{t^{n+1}} \\
&\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p}-\frac{n}{s}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) t^{n\lambda+\frac{n}{s}-\frac{n}{p_1}-1} \|f\|_{L_{p_1}(B(x_0,t))} \, dt.
\end{aligned} \tag{4.3}$$

For $x \in B$, by the Fubini theorem, applying the Hölder inequality, and from Lemma 4.4 we have

$$\begin{aligned}
&\left| T_\Omega\bigl(\bigl(b(\cdot)-b_B\bigr)f_2\bigr)(x) \right| \\
&\quad \lesssim \int_{\complement(2B)} \left|b(y)-b_B\right| \left|\Omega(x-y)\right| \frac{|f(y)|}{|x-y|^n} \, dy \\
&\quad \lesssim \int_{\complement(2B)} \left|b(y)-b_B\right| \left|\Omega(x-y)\right| \frac{|f(y)|}{|x_0-y|^n} \, dy
\end{aligned}$$

$$
\begin{aligned}
&\approx \int_{2r}^{\infty} \int_{2r<|x_0-y|<t} \left|b(y)-b_B\right|\left|\Omega(x-y)\right|\left|f(y)\right| dy \frac{dt}{t^{n+1}} \\
&\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} \left|b(y)-b_{B(x_0,t)}\right|\left|\Omega(x-y)\right|\left|f(y)\right| dy \frac{dt}{t^{n+1}} \\
&\quad + \int_{2r}^{\infty} |b_{B(x_0,r)}-b_{B(x_0,t)}| \int_{B(x_0,t)} \left|\Omega(x-y)\right|\left|f(y)\right| dy \frac{dt}{t^{n+1}} \\
&\lesssim \int_{2r}^{\infty} \left\|\big(b(\cdot)-b_{B(x_0,t)}\big)f\right\|_{L_p(B(x_0,t))} \left\|\Omega(\cdot-y)\right\|_{L_s(B(x_0,t))} \left|B(x_0,t)\right|^{1-\frac{1}{p}-\frac{1}{s}} \frac{dt}{t^{n+1}} \\
&\quad + \int_{2r}^{\infty} |b_{B(x_0,r)}-b_{B(x_0,t)}| \|f\|_{L_{p_1}(B(x_0,t))} \left\|\Omega(\cdot-y)\right\|_{L_s(B(x_0,t))} \left|B(x_0,t)\right|^{1-\frac{1}{p_1}-\frac{1}{s}} t^{-n-1}\, dt \\
&\lesssim \int_{2r}^{\infty} \left\|b(\cdot)-b_{B(x_0,t)}\right\|_{L_{p_2}(B(x_0,t))} \|f\|_{L_{p_1}(B(x_0,t))} t^{-1-\frac{n}{p_1}}\, dt \\
&\quad + \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) t^{n\lambda-\frac{n}{p_1}-1} \|f\|_{L_{p_1}(B(x_0,t))}\, dt \\
&\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) t^{n\lambda-\frac{n}{p_1}-1} \|f\|_{L_{p_1}(B(x_0,t))}\, dt.
\end{aligned}
$$

Then for J_4 we have

$$
\begin{aligned}
\|J_4\|_{L_p(B)} &\le \left\|T_\Omega\big(b(\cdot)-b_B\big)f_2\right\|_{L_p(\mathbb{R}^n)} \\
&\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) t^{n\lambda-\frac{n}{p_1}-1} \|f\|_{L_{p_1}(B(x_0,t))}\, dt.
\end{aligned}
$$

When $p_1 < s$, by the Fubini theorem, (3.10), and the Minkowski inequality, we get

$$
\begin{aligned}
\|T_\Omega f_2\|_{L_p(B)} &\le \left(\int_B \left(\int_{2r}^{\infty} \int_{B(x_0,t)} \left|f(y)\right|\left|\Omega(x-y)\right| dy \frac{dt}{t^{n+1}}\right)^p dx\right)^{\frac{1}{p}} \\
&\le \int_{2r}^{\infty} \int_{B(x_0,t)} \left|f(y)\right| \left\|\Omega(\cdot-y)\right\|_{L_p(B)} dy \frac{dt}{t^{n+1}} \\
&\le |B|^{\frac{1}{p}-\frac{1}{s}} \int_{2r}^{\infty} \int_{B(x_0,t)} \left|f(y)\right| \left\|\Omega(\cdot-y)\right\|_{L_s(B)} dy \frac{dt}{t^{n+1}} \\
&\lesssim r^{\frac{n}{p}-\frac{n}{s}} \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \left|B\big(x_0, t+|x_0-y|\big)\right|^{\frac{1}{s}} \frac{dt}{t^{n+1}} \\
&\lesssim r^{\frac{n}{p}-\frac{n}{s}} \int_{2r}^{\infty} t^{\frac{n}{s}-\frac{n}{p_1}-1} \|f\|_{L_{p_1}(B(x_0,t))}\, dt. \qquad (4.4)
\end{aligned}
$$

Now combining all the above estimates, we end the proof of Lemma 4.6. □

The following theorem is true.

Theorem 4.7 *Suppose that $x_0 \in \mathbb{R}^n$, $1 < p < \infty$, T_Ω is a linear operator satisfying condition* (1.1) *with $\Omega \in L_s(S^{n-1})$, $s > 1$, is homogeneous of degree zero and bounded on $L_p(\mathbb{R}^n)$. Let $b \in LC^{\{x_0\}}_{p_2,\lambda}(\mathbb{R}^n)$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $0 \le \lambda < \frac{1}{n}$.*

Let also, for $s' \le p_1$, the pair (φ_1, φ_2) satisfy the condition

$$
\int_r^{\infty} \left(1+\ln\frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1-n\lambda}}\, dt \le C\varphi_2(x_0,r), \qquad (4.5)
$$

and, for $p_1 < s$, the pair (φ_1, φ_2) satisfy the condition

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}-\frac{n}{s}+1}}\, dt \le C\varphi_2(x,r) r^{\frac{n}{s}}, \tag{4.6}$$

where C does not depend on r.

Then the operator $[b, T_\Omega]$ is bounded from $LM^{\{x_0\}}_{p,\varphi_1}$ to $LM^{\{x_0\}}_{p,\varphi_2}$. Moreover,

$$\big\| [b, T_\Omega] f \big\|_{LM^{\{x_0\}}_{p,\varphi_2}} \lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda}} \|f\|_{LM^{\{x_0\}}_{p,\varphi_1}}.$$

Proof The statement of Theorem 4.7 follows by Lemma 4.6 and Theorem 4.2 in the same manner as in the proof of Theorem 3.8. □

Corollary 4.8 *Suppose that $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$ with $s > 1$, is homogeneous of degree zero. Let $1 < p < \infty$, $b \in LC^{\{x_0\}}_{p_2,\lambda}(\mathbb{R}^n)$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and $0 \le \lambda < \frac{1}{n}$. Let also, for $s' \le p_1$, the pair (φ_1, φ_2) satisfy the condition* (4.5), *and, for $p < s$, the pair (φ_1, φ_2) satisfy the condition* (4.6). *Then the operator $[b, \overline{T}_\Omega]$ is bounded from $LM^{\{x_0\}}_{p,\varphi_1}$ to $LM^{\{x_0\}}_{p,\varphi_2}$.*

Corollary 4.9 *Let T_Ω be a linear operator satisfying condition* (1.1) *with $\Omega \in L_s(S^{n-1})$, $s > 1$, being homogeneous of degree zero and bounded on $L_p(\mathbb{R}^n)$. Suppose $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$. Let also, for $s' \le p$, the pair (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}\, dt \le C\varphi_2(x,r), \tag{4.7}$$

and, for $p < s$, the pair (φ_1, φ_2) satisfy the condition

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}-\frac{n}{s}+1}}\, dt \le C\varphi_2(x,r) r^{\frac{n}{s}}, \tag{4.8}$$

where C does not depend on x and r.

Then the operator $[b, T_\Omega]$ is bounded from M_{p,φ_1} to M_{p,φ_2}. Moreover,

$$\big\| [b, T_\Omega] f \big\|_{M_{p,\varphi_2}} \lesssim \|b\|_{BMO} \|f\|_{M_{p,\varphi_1}}.$$

Corollary 4.10 *Suppose that $\Omega \in L_s(S^{n-1})$ with $s > 1$, is homogeneous of degree zero. Let $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$. Let also, for $s' \le p$, the pair (φ_1, φ_2) satisfy the condition* (4.7) *and, for $p < s$, the pair (φ_1, φ_2) satisfy the condition* (4.8). *Then the operator $[b, \overline{T}_\Omega]$ is bounded from M_{p,φ_1} to M_{p,φ_2}.*

Remark 4.11 Note that the boundedness of sublinear operators with rough kernel and its commutator on the generalized central (local) Morrey spaces $LM_{p,\varphi}$ were studied in [38]. Also, in the case $s = \infty$ Corollary 4.8 was proved in [30] and Corollary 4.10 in [33].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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