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The common solution for a generalized equilibrium problem, a variational inequality problem and a hierarchical fixed point problem

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Abstract
The present paper aims to deal with a new iterative method to find a common solution of a generalized equilibrium problem, a variational inequality problem and a hierarchical fixed point problem for a sequence of nearly nonexpansive mappings. It is proved that the proposed method converges strongly to a common solution of above problems under some assumptions. The results here improve and extend some recent corresponding results by many other authors.
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1 Introduction

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively, C be a nonempty, closed, and convex subset of H. It is well known that for any $x \in H$, there exists a unique point $y_0 \in C$ such that

$$\|x - y_0\| = \inf\bigl\{\|x - y\| : y \in C\bigr\}.$$

Here, y_0 is denoted by $P_C x$, where P_C is called the metric projection of H onto C.

Let us recall some kinds of nonlinear mappings as follows, which are needed in the next sections. A mapping $T : C \to H$ is called L-Lipschitzian if there exists a constant $L > 0$ such that $\|Tx - Ty\| \le L\|x - y\|$, $\forall x, y \in C$. In particular, if $L \in [0, 1)$, then T is said to be a contraction; if $L = 1$, then T is called a nonexpansive mapping. Let us fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \to 0$. If the inequality $\|T^n x - T^n y\| \le \|x - y\| + a_n$ holds for all $x, y \in C$ and $n \ge 1$, then T is said to be nearly nonexpansive [1, 2] with respect to $\{a_n\}$. Let $\{T_n\}$ be a sequence of mappings from C into H. Then the sequence $\{T_n\}$ is called a sequence of nearly nonexpansive mappings [3, 4] with respect to a sequence $\{a_n\}$ if

$$\|T_n x - T_n y\| \le \|x - y\| + a_n, \quad \forall x, y \in C, \forall n \ge 1. \tag{1.1}$$

It is obvious that the sequence of nearly nonexpansive mappings is a wider class of sequence of nonexpansive mappings. A mapping $A : C \to H$ is called α-inverse strongly monotone if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C,$$

and a mapping $F : C \to H$ is called η-strongly monotone if there exists a constant $\eta \geq 0$ such that

$$\langle Fx - Fy, x - y\rangle \geq \eta \|x - y\|^2, \quad \forall x, y \in C.$$

In particular, if $\eta = 0$, then F is said to be monotone.

Let $G : C \times C \to \mathbb{R}$ be a bifunction and B be a nonlinear mapping. The generalized equilibrium problem, denoted by GEP, is to find a point $x \in C$ such that

$$G(x, y) + \langle Bx, y - x\rangle \geq 0 \tag{1.2}$$

for all $y \in C$, and the solution of the problem (1.2) is denoted by GEP(G), *i.e.*,

$$\mathrm{GEP}(G) = \bigl\{x \in C : G(x, y) + \langle Bx, y - x\rangle \geq 0, \ \forall y \in C\bigr\}.$$

If $B = 0$, then the GEP is reduced to equilibrium problem, denoted by EP, which is to find a point $x \in C$ such that

$$G(x, y) \geq 0$$

for all $y \in C$. The set of solutions of EP is denoted by EP(G). In the case of $G = 0$, then GEP is equivalent to find a $x \in C$ such that

$$\langle Bx, y - x\rangle \geq 0 \tag{1.3}$$

for all $y \in C$. The problem (1.3) is called variational inequality problem, denoted by $VI(C, B)$, and the solution of $VI(C, B)$ is denoted by Ω, *i.e.*,

$$\Omega = \bigl\{x \in C : \langle Bx, y - x\rangle \geq 0, \ \forall y \in C\bigr\}.$$

The generalized equilibrium problem includes, as special cases, the optimization problem, the variational inequality problem, the fixed point problem, the nonlinear complementarity, the Nash equilibrium problem in noncooperative games, the vector optimization problem, etc. Hence, the existence of solutions of generalized equilibrium problems has been extensively studied by many authors in the literature (see, *e.g.*, [5–9]).

Let $S : C \to H$ be a nonexpansive mapping. The following problem is called a hierarchical fixed point problem: Finding $x^* \in \mathrm{Fix}(T)$ such that

$$\bigl\langle x^* - Sx^*, x - x^*\bigr\rangle \geq 0, \quad x \in \mathrm{Fix}(T), \tag{1.4}$$

where $\mathrm{Fix}(T)$ is the set of fixed points of T, *i.e.*, $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$. The problem (1.4) is equivalent to the following fixed point problem: Finding an $x^* \in C$ that satisfies $x^* = P_{\mathrm{Fix}(T)}Sx^*$. Since $\mathrm{Fix}(T)$ is closed and convex, the metric projection $P_{\mathrm{Fix}(T)}$ is well defined.

It is well known that the hierarchical fixed point problem (1.4) links with some monotone variational inequalities and convex programming problems; see [10–15]. Therefore, there exist various methods to solve the hierarchical fixed point problem; see Yao and Liou in [16], Xu in [17], Marino and Xu in [18] and Bnouhachem and Noor in [19].

Now, we give some iteration schemes which are related with the problems (1.2), (1.3), and (1.4). In 2011, Ceng *et al.* [25] investigated the following iterative method:

$$x_{n+1} = P_C\big[\alpha_n \rho V x_n + (1 - \alpha_n \mu F)Tx_n\big], \quad \forall n \geq 0, \tag{1.5}$$

where F is a L-Lipschitzian and η-strongly monotone operator with constants $L, \eta > 0$ and$V$ is a γ-Lipschitzian (possibly non-self-)mapping with constant $\gamma \geq 0$ such that $0 < \mu < \frac{2\eta}{L^2}$ and $0 \leq \rho\gamma < 1 - \sqrt{1 - \mu(2\eta - \mu L^2)}$. They proved that under some approximate assumptions on the operators and parameters, the sequence $\{x_n\}$ generated by (1.5) converges strongly to the unique solution of the variational inequality

$$\big\langle (\rho V - \mu F)x^*, x - x^*\big\rangle \leq 0, \quad \forall x \in \mathrm{Fix}(T). \tag{1.6}$$

Recently, in 2013, Sahu *et al.* [26] introduced the following iterative process for the sequence of nearly nonexpansive mappings $\{T_n\}$ defined by (1.1):

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n S_n x_n, \\ x_{n+1} = P_C[\alpha_n f x_n + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) T_i y_n], \quad \forall n \geq 1, \end{cases} \tag{1.7}$$

where f is a contraction and $\{S_n\}$ is a sequence of nonexpansive mappings from C into itself. They proved that the sequence $\{x_n\}$ generated by (1.7) converges strongly to the unique solution of the following variational inequality:

$$\left\langle \frac{1}{\tau}(I - f)x^* + (I - S)x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_n).$$

In the same year, Bnouhachem and Noor [19] introduced a new iterative scheme to find a common solution of a variational inequality, a generalized equilibrium problem and a hierarchical fixed point problem. Their scheme is as follows:

$$\begin{cases} G(u_n, y) + \langle Bx, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ z_n = P_C(u_n - \lambda_n A u_n), \\ y_n = P_C(\beta_n S x_n + (1 - \beta_n) z_n), \\ x_{n+1} = P_C(\alpha_n f x_n + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) V_i y_n), \quad \forall n \geq 1, \end{cases} \tag{1.8}$$

where $V_i = k_i I + (1 - k_i)T_i$, $0 \leq k_i < 1$, $\{T_i\}_{i=1}^{\infty} : C \to C$ is a countable family of k_i-strict pseudo-contraction mappings, A and B are inverse strongly monotone mappings.

They proved that the sequence $\{x_n\}$ generated by (1.8) converges strongly to a point $z \in P_{\Omega\cap\mathrm{GEP}(G)\cap\mathrm{Fix}(T)}f(z)$ which is the unique solution of the following variational inequality:

$$\bigl\langle (I-f)z, x-z\bigr\rangle \geq 0, \quad \forall x \in \Omega \cap \mathrm{GEP}(G) \cap \mathrm{Fix}(T),$$

where $\mathrm{Fix}(T) = \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i)$.

In 2014, Bnouhachem and Chen [20] introduced the following iterative method:

$$\begin{cases} F_1(u_n, y) + \langle Dx_n, y-u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y-u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C; \\ z_n = P_C(u_n - \lambda_n A u_n); \\ y_n = \beta_n S x_n + (1-\beta_n) z_n; \\ x_{n+1} = P_C[\alpha_n \rho U x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \mu F)(T(y_n))], \quad \forall n \geq 0, \end{cases} \tag{1.9}$$

where $D, A : C \to H$ are inverse strongly monotone mappings, $F_1 : C \times C \to \mathbb{R}$ is a bifunction, $\varphi : C \to \mathbb{R}$ is a proper lower semicontinuous and convex function, $S, T : C \to C$ are nonexpansive mappings, $F : C \to C$ is Lipschitzian and a strongly monotone mapping and $U : C \to C$ is a Lipschitzian mapping. The authors proved the strong convergence of the sequence generated by (1.9) to a common solution of a variational inequality, a generalized mixed equilibrium problem, and a hierarchical fixed point problem.

In addition to all these papers, similar problems are considered in several papers; see, *e.g.*, [21–24].

In this paper, motivated by the above works and by the recent work going in this direction, we introduce an iterative projection method and prove a strong convergence theorem based on this method for computing an approximate element of the common set of solution of a generalized equilibrium problem, a variational inequality problem and a fixed point problem for a sequence of nearly nonexpansive mappings defined by (1.1). The proposed method improves and extends many known results; see, *e.g.*, [4, 11, 25, 27, 28] and the references therein.

2 Preliminaries

Let $\{x_n\}$ be a sequence in a Hilbert space H and $x \in H$. Throughout this paper, $x_n \to x$ denotes the strong convergence of $\{x_n\}$ to x and $x_n \rightharpoonup x$ denotes the weak convergence. Let C be a nonempty subset of a real Hilbert space H. For solving an equilibrium problem for a bifunction $G : C \times C \to \mathbb{R}$, let us assume that G satisfies the following conditions:

(A1) $G(x,x) = 0$, $\forall x \in C$,

(A2) G is monotone, *i.e.* $G(x,y) + G(y,x) \leq 0$, $\forall x, y \in C$,

(A3) $\forall x, y, z \in C$,

$$\lim_{t\to 0^+} G\bigl(tz + (1-t)x, y\bigr) \leq G(x,y),$$

(A4) $\forall x \in C$, $y \longmapsto G(x,y)$ is convex and lower semicontinuous.

Lemma 1 [29] *Let C be a nonempty, closed, and convex subset of H, and let G be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4). *Let $r > 0$ and $x \in H$. Then there exists $z \in C$*

such that

$$G(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \tag{2.1}$$

for all $x \in C$.

Lemma 2 [30] *Suppose that* $G : C \times C \to \mathbb{R}$ *satisfies* (A1)-(A4). *For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r : H \to C$ *as follows*:

$$T_r(x) = \left\{ z \in C : G(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\}$$

for all $z \in H$. *Then the following hold*:

(1) T_r *is single valued*,

(2) T_r *is firmly nonexpansive i.e.*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \quad \forall x, y \in H,$$

(3) $\mathrm{Fix}(T_r) = \mathrm{EP}(G)$,

(4) $\mathrm{EP}(G)$ *is closed and convex*.

Let $T_1, T_2 : C \to H$ be two mappings. We denote $\mathcal{B}(C)$, the collection of all bounded subsets of C. The deviation between T_1 and T_2 on $B \in \mathcal{B}(C)$, denoted by $\mathfrak{D}_B(T_1, T_2)$, is defined by

$$\mathfrak{D}_B(T_1, T_2) = \sup\left\{ \|T_1 x - T_2 x\| : x \in B \right\}.$$

The following lemmas will be used in the next section.

Lemma 3 [3] *Let* C *be a nonempty, closed, and bounded subset of a Banach space* X *and* $\{T_n\}$ *be a sequence of nearly nonexpansive self-mappings on* C *with a sequence* $\{a_n\}$ *such that* $\mathfrak{D}_C(T_n, T_{n+1}) < \infty$. *Then, for each* $x \in C$, $\{T_n x\}$ *converges strongly to some point of* C. *Moreover, if* T *is a mapping from* C *into itself defined by* $Tz = \lim_{n\to\infty} T_n z$ *for all* $z \in C$, *then* T *is nonexpansive and* $\lim_{n\to\infty} \mathfrak{D}_C(T_n, T) = 0$.

Lemma 4 [25] *Let* $V : C \to H$ *be a* γ*-Lipschitzian mapping with a constant* $\gamma \geq 0$ *and let* $F : C \to H$ *be a L-Lipschitzian and* η*-strongly monotone operator with constants* $L, \eta > 0$. *Then for* $0 \leq \rho\gamma < \mu\eta$,

$$\left\langle (\mu F - \rho V)x - (\mu F - \rho V)y, x - y \right\rangle \geq (\mu\eta - \rho\gamma)\|x - y\|^2, \quad \forall x, y \in C.$$

That is, $\mu F - \rho V$ *is strongly monotone with coefficient* $\mu\eta - \rho\gamma$.

Lemma 5 [31] *Let* C *be a nonempty subset of a real Hilbert space* H. *Suppose that* $\lambda \in (0,1)$ *and* $\mu > 0$. *Let* $F : C \to H$ *be a L-Lipschitzian and* η*-strongly monotone operator on* C. *Define the mapping* $G : C \to H$ *by*

$$Gx = x - \lambda\mu F x, \quad \forall x \in C.$$

Then G is a contraction that provided $\mu < \frac{2\eta}{L^2}$. *More precisely, for* $\mu \in (0, \frac{2\eta}{L^2})$,

$$\|Gx - Gy\| \leq (1 - \lambda\nu)\|x - y\|, \quad \forall x, y \in C,$$

where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu L^2)}$.

Lemma 6 [32] *Let C be a nonempty, closed, and convex subset of a real Hilbert space H, and T be a nonexpansive self-mapping on C. If* $\mathrm{Fix}(T) \neq \emptyset$, *then* $I - T$ *is demiclosed; that is whenever* $\{x_n\}$ *is a sequence in C weakly converging to some* $x \in C$ *and the sequence* $\{(I - T)x_n\}$ *strongly converges to some y, it follows that* $(I - T)x = y$. *Here I is the identity operator of H.*

Lemma 7 [33] *Assume that* $\{x_n\}$ *is a sequence of nonnegative real numbers satisfying the conditions*

$$x_{n+1} \leq (1 - \alpha_n)x_n + \alpha_n\beta_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences of real numbers such that*

(i) $\{\alpha_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$.

Then $\lim_{n\to\infty} x_n = 0$.

3 Main results

Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $A, B : C \to H$ be α, θ-inverse strongly monotone mappings, respectively. Let $G : C \times C \to \mathbb{R}$ be a bifunction satisfying assumptions (A1)-(A4), $S : C \to H$ be a nonexpansive mapping and $\{T_n\}$ be a sequence of nearly nonexpansive mappings with the sequence $\{a_n\}$ such that $\mathcal{F} := \mathrm{Fix}(T) \cap \Omega \cap \mathrm{GEP}(G) \neq \emptyset$ where $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$ and $\mathrm{Fix}(T) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. It is clear that the mapping T is nonexpansive. Let $V : C \to H$ be a γ-Lipschitzian mapping, $F : C \to H$ be a L-Lipschitzian and η-strongly monotone operator such that these coefficients satisfy $0 < \mu < \frac{2\eta}{L^2}$, $0 \leq \rho\gamma < \nu$, where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu L^2)}$. For an arbitrarily initial value x_1, define the sequence $\{x_n\}$ in C generated by

$$\begin{cases} G(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ z_n = P_C(u_n - \lambda_n A u_n), \\ y_n = P_C[\beta_n S x_n + (1 - \beta_n)z_n], \\ x_{n+1} = P_C[\alpha_n \rho V x_n + (I - \alpha_n \mu F)T_n y_n], \quad n \geq 1, \end{cases} \tag{3.1}$$

where $\{\lambda_n\} \subset (0, 2\alpha)$, $\{r_n\} \subset (0, 2\theta)$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$.

As can be seen, the convergence of the sequence $\{x_n\}$ generated by (3.1) depends on the choice of the control sequences and mappings. We list the following hypotheses on them:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\lim_{n\to\infty} \frac{a_n}{\alpha_n} = 0$, $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = 0$, $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$, $\lim_{n\to\infty} \frac{|\lambda_n - \lambda_{n-1}|}{\alpha_n} = 0$;
$\lim_{n\to\infty} \frac{|\beta_n - \beta_{n-1}|}{\alpha_n} = 0$, and $\lim_{n\to\infty} \frac{|r_n - r_{n-1}|}{\alpha_n} = 0$;

(C3) $\lim_{n\to\infty} \mathfrak{D}_B(T_n, T_{n+1}) = 0$ and $\lim_{n\to\infty} \frac{\mathfrak{D}_B(T_n, T_{n+1})}{\alpha_n} = 0$ for each $B \in \mathcal{B}(C)$.

Now, we need the following lemmas to prove our main theorem.

Lemma 8 *Assume that the conditions* (C1), (C2) *hold and* $p \in \mathcal{F}$. *Then the sequences* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, *and* $\{u_n\}$ *generated by* (3.1) *are bounded*.

Proof It is easy to see that the mapping $I - r_nB$ is nonexpansive, so the mapping $I - \lambda_n A$ is also nonexpansive. From Lemma 2, we have $u_n = T_{r_n}(x_n - r_nBx_n)$. Let $p \in \mathcal{F}$. So, we get $p = T_{r_n}(p - r_nBp)$. Then we obtain

$$\begin{aligned} \|u_n - p\|^2 &= \left\| T_{r_n}(x_n - r_nBx_n) - T_{r_n}(p - r_nBp)\right\|^2 \\ &\le \left\|(x_n - r_nBx_n) - (p - r_nBp)\right\|^2 \\ &= \|x_n - p\|^2 - 2r_n\langle x_n - p, Bx_n - Bp\rangle + r_n^2\|Bx_n - Bp\|^2 \\ &\le \|x_n - p\|^2 - r_n(2\theta - r_n)\|Bx_n - Bp\|^2 \\ &\le \|x_n - p\|^2. \end{aligned} \tag{3.2}$$

From (3.2), we get

$$\begin{aligned} \|z_n - p\|^2 &= \left\|P_C(u_n - \lambda_n Au_n) - P_C(p - \lambda_n Ap)\right\|^2 \\ &\le \left\|u_n - p - \lambda_n(Au_n - Ap)\right\|^2 \\ &\le \|u_n - p\|^2 - \lambda_n(2\alpha - \lambda_n)\|Au_n - Ap\|^2 \\ &\le \|u_n - p\|^2 \\ &\le \|x_n - p\|^2. \end{aligned} \tag{3.3}$$

It follows from (3.3) that

$$\begin{aligned} \|y_n - p\| &= \left\|P_C\big[\beta_n Sx_n + (1 - \beta_n)x_n\big] - P_C p\right\| \\ &\le \left\|\beta_n Sx_n + (1 - \beta_n)z_n - p\right\| \\ &\le (1 - \beta_n)\|z_n - p\| + \beta_n\|Sx_n - p\| \\ &\le (1 - \beta_n)\|x_n - p\| + \beta_n\|Sx_n - Sp\| + \beta_n\|Sp - p\| \\ &\le \|x_n - p\| + \beta_n\|Sp - p\|. \end{aligned} \tag{3.4}$$

Since $\lim_{n\to\infty}\frac{\beta_n}{\alpha_n}=0$, without loss of generality, we can assume that $\beta_n\le\alpha_n$, for all $n\ge 1$. This gives us $\lim_{n\to\infty}\beta_n=0$.

Let $t_n=\alpha_n\rho Vx_n+(I-\alpha_n\mu F)T_ny_n$. Then we get

$$\begin{aligned}\|x_{n+1}-p\| &= \|P_Ct_n-P_Cp\|\\ &\le \|t_n-p\|\\ &= \bigl\|\alpha_n\rho Vx_n+(I-\alpha_n\mu F)T_ny_n-p\bigr\|\\ &\le \alpha_n\|\rho Vx_n-\mu Fp\|+\bigl\|(I-\alpha_n\mu F)T_ny_n-(I-\alpha_n\mu F)T_np\bigr\|\\ &\le \alpha_n\rho\gamma\|x_n-p\|+\alpha_n\|\rho Vp-\mu Fp\|\\ &\quad+(1-\alpha_n\nu)\bigl(\|y_n-p\|+a_n\bigr). \end{aligned}\tag{3.5}$$

From (3.4) and (3.5), we get

$$\begin{aligned}\|x_{n+1}-p\| &\le \alpha_n\rho\gamma\|x_n-p\|+\alpha_n\|\rho Vp-\mu Fp\|\\ &\quad+(1-\alpha_n\nu)\bigl(\|x_n-p\|+\beta_n\|Sp-p\|+a_n\bigr)\\ &\le \bigl(1-\alpha_n(\nu-\rho\gamma)\bigr)\|x_n-p\|\\ &\quad+\alpha_n\left(\|\rho Vp-\mu Fp\|+\|Sp-p\|+\frac{a_n}{\alpha_n}\right)\\ &\le \bigl(1-\alpha_n(\nu-\rho\gamma)\bigr)\|x_n-p\|\\ &\quad+\alpha_n(\nu-\rho\gamma)\left[\frac{1}{(\nu-\rho\gamma)}\left(\|\rho Vp-\mu Fp\|+\|Sp-p\|+\frac{a_n}{\alpha_n}\right)\right]. \end{aligned}\tag{3.6}$$

From condition (C2), there exists a constant $M_1>0$ such that

$$\|\rho Vp-\mu Fp\|+\|Sp-p\|+\frac{a_n}{\alpha_n}\le M_1,\quad \forall n\ge 1.$$

Thus, from (3.6) we have

$$\|x_{n+1}-p\|\le\bigl(1-\alpha_n(\nu-\rho\gamma)\bigr)\|x_n-p\|+\alpha_n(\nu-\rho\gamma)\frac{M_1}{(\nu-\rho\gamma)}.$$

By induction, we get

$$\|x_{n+1}-p\|\le\max\left\{\|x_1-p\|,\frac{M_1}{(\nu-\rho\gamma)}\right\}.$$

Hence, we find that $\{x_n\}$ is bounded. So, the sequences $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ are bounded. □

Lemma 9 *Assume that* (C1)-(C3) *hold. Let* $p\in\mathcal{F}$ *and* $\{x_n\}$ *be the sequence generated by* (3.1). *Then the follow hold*:

(i) $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

(ii) $w_w(x_n)\subset\operatorname{Fix}(T)$ *where* $w_w(x_n)$ *is the weak w-limit set of* $\{x_n\}$, *i.e.*, $w_w(x_n)=\{x:x_{n_i}\rightharpoonup x\}$.

Proof (i) Since the mappings P_C and $(I - \lambda_n A)$ are nonexpansive, we get

$$\begin{aligned}
\|z_n - z_{n-1}\| &= \big\|P_C(u_n - \lambda_n Au_n) - P_C(u_{n-1} - \lambda_{n-1}Au_{n-1})\big\| \\
&\le \big\|(u_n - \lambda_n Au_n) - (u_{n-1} - \lambda_{n-1}Au_{n-1})\big\| \\
&= \big\|u_n - u_{n-1} - \lambda_n(Au_n - Au_{n-1}) - (\lambda_n - \lambda_{n-1})Au_{n-1}\big\| \\
&\le \big\|u_n - u_{n-1} - \lambda_n(Au_n - Au_{n-1})\big\| + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\| \\
&\le \|u_n - u_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\|, \qquad (3.7)
\end{aligned}$$

and so

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \big\|P_C\big[\beta_n Sx_n + (1-\beta_n)z_n\big] \\
&\quad - P_C\big[\beta_{n-1}Sx_{n-1} - (1-\beta_{n-1})z_{n-1}\big]\big\| \\
&\le \big\|\beta_n Sx_n + (1-\beta_n)z_n - \beta_{n-1}Sx_{n-1} + (1-\beta_{n-1})z_{n-1}\big\| \\
&\le \big\|\beta_n(Sx_n - Sx_{n-1}) + (\beta_n - \beta_{n-1})Sx_{n-1} \\
&\quad + (1-\beta_n)(z_n - z_{n-1}) + (\beta_{n-1} - \beta_n)z_{n-1}\big\| \\
&\le \beta_n\|x_n - x_{n-1}\| + (1-\beta_n)\|z_n - z_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|\big(\|Sx_{n-1}\| + \|z_{n-1}\|\big) \\
&\le \beta_n\|x_n - x_{n-1}\| + (1-\beta_n)\big[\|u_n - u_{n-1}\| \\
&\quad + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\|\big] \\
&\quad + |\beta_n - \beta_{n-1}|\big(\|Sx_{n-1}\| + \|z_{n-1}\|\big). \qquad (3.8)
\end{aligned}$$

On the other hand, since $u_n = T_{r_n}(x_n - r_nBx_n)$ and $u_{n-1} = T_{r_{n-1}}(x_{n-1} - r_{n-1}Bx_{n-1})$, we have

$$G(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \qquad (3.9)$$

and

$$\begin{aligned}
&G(u_{n-1}, y) + \langle Bx_{n-1}, y - u_{n-1}\rangle \\
&\quad + \frac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \ge 0, \quad \forall y \in C. \qquad (3.10)
\end{aligned}$$

If we take $y = u_{n-1}$ and $y = u_n$ in (3.9) and (3.10), respectively, then we get

$$G(u_n, u_{n-1}) + \langle Bx_n, u_{n-1} - u_n\rangle + \frac{1}{r_n}\langle u_{n-1} - u_n, u_n - x_n\rangle \ge 0 \qquad (3.11)$$

and

$$\begin{aligned}
&G(u_{n-1}, u_n) + \langle Bx_{n-1}, u_n - u_{n-1}\rangle \\
&\quad + \frac{1}{r_{n-1}}\langle u_n - u_{n-1}, u_{n-1} - x_{n-1}\rangle \ge 0. \qquad (3.12)
\end{aligned}$$

It follows from (3.11), (3.12), and monotonicity of the function G that

$$\langle Bx_{n-1} - Bx_n, u_n - u_{n-1}\rangle + \left\langle u_n - u_{n-1}, \frac{u_{n-1} - x_{n-1}}{r_{n-1}} - \frac{u_n - x_n}{r_n}\right\rangle \geq 0.$$

The last inequality implies that

$$\begin{aligned}
0 &\leq \left\langle u_n - u_{n-1}, r_n(Bx_{n-1} - Bx_n) + \frac{r_n}{r_{n-1}}(u_{n-1} - x_{n-1}) - (u_n - x_n)\right\rangle \\
&= \left\langle u_{n-1} - u_n, u_n - u_{n-1} + \left(1 - \frac{r_n}{r_{n-1}}\right)u_{n-1} \right. \\
&\quad \left. + (x_{n-1} - r_nBx_{n-1}) - (x_n - r_nBx_n) - x_{n-1} + \frac{r_n}{r_{n-1}}x_{n-1}\right\rangle \\
&= \left\langle u_{n-1} - u_n, \left(1 - \frac{r_n}{r_{n-1}}\right)u_{n-1} + (x_{n-1} - r_nBx_{n-1}) \right. \\
&\quad \left. - (x_n - r_nBx_n) - x_{n-1} + \frac{r_n}{r_{n-1}}x_{n-1}\right\rangle - \|u_n - u_{n-1}\|^2 \\
&= \left\langle u_{n-1} - u_n, \left(1 - \frac{r_n}{r_{n-1}}\right)(u_{n-1} - x_{n-1}) \right. \\
&\quad \left. + (x_{n-1} - r_nBx_{n-1}) - (x_n - r_nBx_n)\right\rangle - \|u_n - u_{n-1}\|^2 \\
&\leq \|u_{n-1} - u_n\| \left\{ \left|1 - \frac{r_n}{r_{n-1}}\right| \|u_{n-1} - x_{n-1}\| \right. \\
&\quad \left. + \left\|(x_{n-1} - r_nBx_{n-1}) - (x_n - r_nBx_n)\right\| \right\} - \|u_n - u_{n-1}\|^2 \\
&\leq \|u_{n-1} - u_n\| \left\{ \left|1 - \frac{r_n}{r_{n-1}}\right| \|u_{n-1} - x_{n-1}\| \right. \\
&\quad \left. + \|x_{n-1} - x_n\| \right\} - \|u_n - u_{n-1}\|^2.
\end{aligned} \tag{3.13}$$

From (3.13), we have

$$\|u_{n-1} - u_n\| \leq \left|1 - \frac{r_n}{r_{n-1}}\right| \|u_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\|.$$

Without loss of generality, we can assume that there exists a real number μ such that $r_n > \mu > 0$ for all positive integers n. Then we obtain

$$\|u_{n-1} - u_n\| \leq \|x_{n-1} - x_n\| + \frac{1}{\mu}|r_{n-1} - r_n| \|u_{n-1} - x_{n-1}\|. \tag{3.14}$$

From (3.8) and (3.14), we get

$$\begin{aligned}
\|y_n - y_{n-1}\| &\leq \beta_n \|x_n - x_{n-1}\| \\
&\quad + (1 - \beta_n)\left[\|x_{n-1} - x_n\| + \frac{1}{\mu}|r_{n-1} - r_n| \|u_{n-1} - x_{n-1}\| \right.
\end{aligned}$$

$$
\begin{aligned}
&+ |\lambda_n - \lambda_{n-1}| \|Au_{n-1}\| \Bigg] + |\beta_n - \beta_{n-1}| \big(\|Sx_{n-1}\| + \|z_{n-1}\| \big) \\
&= \|x_n - x_{n-1}\| + (1-\beta_n) \Bigg[\frac{1}{\mu} |r_{n-1} - r_n| \|u_{n-1} - x_{n-1}\| \\
&+ |\lambda_n - \lambda_{n-1}| \|Au_{n-1}\| \Bigg] + |\beta_n - \beta_{n-1}| \big(\|Sx_{n-1}\| + \|z_{n-1}\| \big).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= \|P_C t_n - P_C t_{n-1}\| \\
&\le \|t_n - t_{n-1}\| \\
&= \big\| \alpha_n \rho V x_n + (I - \alpha_n \mu F) T_n y_n \\
&\quad - \alpha_{n-1} \rho V x_{n-1} + (I - \alpha_{n-1} \mu F) T_{n-1} y_{n-1} \big\| \\
&\le \big\| \alpha_n \rho V (x_n - x_{n-1}) + (\alpha_n - \alpha_{n-1}) \rho V x_{n-1} \\
&\quad + (I - \alpha_n \mu F) T_n y_n - (I - \alpha_n \mu F) T_n y_{n-1} \\
&\quad + T_n y_{n-1} - T_{n-1} y_{n-1} \\
&\quad + \alpha_{n-1} \mu F T_{n-1} y_{n-1} - \alpha_n \mu F T_n y_{n-1} \big\| \\
&\le \alpha_n \rho \gamma \|x_n - x_{n-1}\| + \gamma |\alpha_n - \alpha_{n-1}| \|V x_{n-1}\| \\
&\quad + (1 - \alpha_n \nu) \|T_n y_n - T_n y_{n-1}\| + \|T_n y_{n-1} - T_{n-1} y_{n-1}\| \\
&\quad + \mu \|\alpha_{n-1} F T_{n-1} y_{n-1} - \alpha_n F T_n y_{n-1}\| \\
&\le \alpha_n \rho \gamma \|x_n - x_{n-1}\| + \gamma |\alpha_n - \alpha_{n-1}| \|V x_{n-1}\| \\
&\quad + (1 - \alpha_n \nu) \big[\|y_n - y_{n-1}\| + a_n \big] + \|T_n y_{n-1} - T_{n-1} y_{n-1}\| \\
&\quad + \mu \big\| \alpha_{n-1} (F T_{n-1} y_{n-1} - F T_n y_{n-1}) - (\alpha_n - \alpha_{n-1}) F T_n y_{n-1} \big\| \\
&\le \alpha_n \rho \gamma \|x_n - x_{n-1}\| + \gamma |\alpha_n - \alpha_{n-1}| \|V x_{n-1}\| \\
&\quad + (1 - \alpha_n \nu) \Bigg\{ \|x_n - x_{n-1}\| \\
&\quad + (1 - \beta_n) \Bigg[\frac{1}{\mu} |r_{n-1} - r_n| \|u_{n-1} - x_{n-1}\| + |\lambda_n - \lambda_{n-1}| \|Au_{n-1}\| \Bigg] \\
&\quad + |\beta_n - \beta_{n-1}| \big(\|Sx_{n-1}\| + \|z_{n-1}\| \big) \Bigg\} \\
&\quad + (1 - \alpha_n \nu) a_n + \mathfrak{D}_B(T_n, T_{n-1}) \\
&\quad + \mu \alpha_{n-1} L \mathfrak{D}_B(T_n, T_{n-1}) + |\alpha_n - \alpha_{n-1}| \|F T_n y_{n-1}\| \\
&\le \big(1 - \alpha_n (\nu - \rho \gamma) \big) \|x_n - x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \big(\gamma \|V x_{n-1}\| + \|F T_n y_{n-1}\| \big) \\
&\quad + (1 + \mu \alpha_{n-1} L) \mathfrak{D}_B(T_n, T_{n-1}) + a_n \\
&\quad + \frac{1}{\mu} |r_{n-1} - r_n| \|u_{n-1} - x_{n-1}\| + |\lambda_n - \lambda_{n-1}| \|Au_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| \big(\|Sx_{n-1}\| + \|z_{n-1}\| \big)
\end{aligned}
$$

$$\begin{aligned}&\le \big(1-\alpha_n(\nu-\rho\gamma)\big)\|x_n-x_{n-1}\| + (1+\mu\alpha_{n-1}L)\mathfrak{D}_B(T_n,T_{n-1})\\&\quad + M_2\bigg(|\alpha_n-\alpha_{n-1}| + \frac{1}{\mu}|r_{n-1}-r_n|\\&\quad + |\lambda_n-\lambda_{n-1}| + |\beta_n-\beta_{n-1}|\bigg) + a_n,\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}M_2 = \max\Big\{&\sup_{n\ge1}\big(\gamma\|Vx_{n-1}\| + \|FT_ny_{n-1}\|\big), \sup_{n\ge1}\|u_{n-1}-x_{n-1}\|,\\&\sup_{n\ge1}\|Au_{n-1}\|, \sup_{n\ge1}\big(\|Sx_{n-1}\| + \|z_{n-1}\|\big)\Big\}.\end{aligned}$$

Hence, we write

$$\|x_{n+1}-x_n\| \le \big(1-\alpha_n(\nu-\rho\gamma)\big)\|x_n-x_{n-1}\| + \alpha_n(\nu-\rho\gamma)\delta_n, \tag{3.16}$$

where

$$\begin{aligned}\delta_n = \frac{1}{(\nu-\rho\gamma)}\bigg[&(1+\mu\alpha_{n-1}L)\frac{\mathfrak{D}_B(T_n,T_{n-1})}{\alpha_n}\\&+ \frac{a_n}{\alpha_n} + M_2\bigg(\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n} + \frac{1}{\mu}\frac{|r_{n-1}-r_n|}{\alpha_n} + \frac{|\lambda_n-\lambda_{n-1}|}{\alpha_n} + \frac{|\beta_n-\beta_{n-1}|}{\alpha_n}\bigg)\bigg].\end{aligned}$$

From conditions (C2) and (C3), we get

$$\limsup_{n\to\infty}\delta_n \le 0. \tag{3.17}$$

So, it follows from (3.16), (3.17), and Lemma 7 that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.18}$$

(ii) First, we show that $\lim_{n\to\infty}\|u_n-x_n\| = 0$. Since $p\in\mathcal{F}$, from (3.2) and (3.3), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|t_n-p\|^2\\&= \big\|\alpha_n\rho Vx_n + (I-\alpha_n\mu F)T_ny_n - p\big\|^2\\&= \big\|\alpha_n\rho Vx_n - \alpha_n\mu Fp + (I-\alpha_n\mu F)T_ny_n - (I-\alpha_n\mu F)T_np\big\|^2\\&\le \alpha_n\|\rho Vx_n-\mu Fp\|^2 + (1-\alpha_n\nu)\big(\|y_n-p\| + a_n\big)^2\\&= \alpha_n\|\rho Vx_n-\mu Fp\|^2\\&\quad + (1-\alpha_n\nu)\big(\|y_n-p\|^2 + 2a_n\|y_n-p\| + a_n^2\big)\\&= \alpha_n\|\rho Vx_n-\mu Fp\|^2 + (1-\alpha_n\nu)\|y_n-p\|^2\\&\quad + 2(1-\alpha_n\nu)a_n\|y_n-p\| + (1-\alpha_n\nu)a_n^2\\&\le \alpha_n\|\rho Vx_n-\mu Fp\|^2 + (1-\alpha_n\nu)\big[\beta_n\|Sx_n-p\|^2\\&\quad + (1-\beta_n)\|z_n-p\|^2\big] + 2(1-\alpha_n\nu)a_n\|y_n-p\| + (1-\alpha_n\nu)a_n^2\end{aligned}$$

$$\begin{aligned}
&= \alpha_n\|\rho Vx_n - \mu Fp\|^2 + (1-\alpha_n\nu)\beta_n\|Sx_n - p\|^2 \\
&\quad + (1-\alpha_n\nu)(1-\beta_n)\|z_n - p\|^2 \\
&\quad + 2(1-\alpha_n\nu)a_n\|y_n - p\| + (1-\alpha_n\nu)a_n^2 \\
&\le \alpha_n\|\rho Vx_n - \mu Fp\|^2 + (1-\alpha_n\nu)\beta_n\|Sx_n - p\|^2 \\
&\quad + (1-\alpha_n\nu)(1-\beta_n)\big[\|x_n - p\|^2 - r_n(2\theta - r_n)\|Bx_n - Bp\|^2 \\
&\quad - \lambda_n(2\alpha - \lambda_n)\|Au_n - Ap\|^2\big] \\
&\quad + 2(1-\alpha_n\nu)a_n\|y_n - p\| + (1-\alpha_n\nu)a_n^2 \\
&\le \alpha_n\|\rho Vx_n - \mu Fp\|^2 + \beta_n\|Sx_n - p\|^2 + \|x_n - p\|^2 \\
&\quad - (1-\alpha_n\nu)(1-\beta_n)\big[r_n(2\theta - r_n)\|Bx_n - Bp\|^2 \\
&\quad + \lambda_n(2\alpha - \lambda_n)\|Au_n - Ap\|^2\big] \\
&\quad + 2(1-\alpha_n\nu)a_n\|y_n - p\| + (1-\alpha_n\nu)a_n^2.
\end{aligned} \tag{3.19}$$

Then, from (3.19), we get

$$\begin{aligned}
&(1-\alpha_n\nu)(1-\beta_n)\big\{r_n(2\theta - r_n)\|Bx_n - Bp\|^2 + \lambda_n(2\alpha - \lambda_n)\|Au_n - Ap\|^2\big\} \\
&\quad \le \alpha_n\|\rho Vx_n - \mu Fp\|^2 + \beta_n\|Sx_n - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\qquad + 2(1-\alpha_n\nu)a_n\|y_n - p\| + (1-\alpha_n\nu)a_n^2 \\
&\quad \le \alpha_n\|\rho Vx_n - \mu Fp\|^2 + \beta_n\|Sx_n - p\|^2 + \big(\|x_n - p\| + \|x_{n+1} - p\|\big)\|x_{n+1} - p\| \\
&\qquad + 2(1-\alpha_n\nu)a_n\|y_n - p\| + (1-\alpha_n\nu)a_n^2.
\end{aligned}$$

It follows from (3.18) and from conditions (C1) and (C2) that $\lim_{n\to\infty}\|Bx_n - Bp\| = 0$ and $\lim_{n\to\infty}\|Au_n - Ap\| = 0$.

Since T_{r_n} is firmly nonexpansive mapping, we have

$$\begin{aligned}
\|u_n - p\|^2 &= \big\|T_{r_n}(x_n - r_nBx_n) - T_{r_n}(p - r_nBp)\big\|^2 \\
&\le \big\langle u_n - p, (x_n - r_nBx_n) - (p - r_nBp)\big\rangle \\
&= \frac{1}{2}\big\{\|u_n - p\|^2 + \big\|(x_n - r_nBx_n) - (p - r_nBp)\big\|^2 \\
&\quad - \big\|u_n - p - \big[(x_n - r_nBx_n) - (p - r_nBp)\big]\big\|^2\big\}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\|u_n - p\|^2 &\le \big\|(x_n - r_nBx_n) - (p - r_nBp)\big\|^2 \\
&\quad - \big\|u_n - x_n - r_n(Bx_n - Bp)\big\|^2 \\
&\le \|x_n - p\|^2 - \big\|u_n - x_n - r_n(Bx_n - Bp)\big\|^2 \\
&\le \|x_n - p\|^2 - \|u_n - x_n\|^2 \\
&\quad + 2r_n\|u_n - x_n\|\|Bx_n - Bp\|.
\end{aligned} \tag{3.20}$$

Then, from (3.3), (3.19), and (3.20), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\|\rho Vx_n-\mu Fp\|^2+(1-\alpha_n\nu)\big[\beta_n\|Sx_n-p\|^2\\&\quad+(1-\beta_n)\|z_n-p\|^2\big]+2(1-\alpha_n\nu)a_n\|y_n-p\|+(1-\alpha_n\nu)a_n^2\\&\le \alpha_n\|\rho Vx_n-\mu Fp\|^2+(1-\alpha_n\nu)\big[\beta_n\|Sx_n-p\|^2\\&\quad+(1-\beta_n)\|u_n-p\|^2\big]+2(1-\alpha_n\nu)a_n\|y_n-p\|+(1-\alpha_n\nu)a_n^2\\&\le \alpha_n\|\rho Vx_n-\mu Fp\|^2+(1-\alpha_n\nu)\big[\beta_n\|Sx_n-p\|^2\\&\quad+(1-\beta_n)\big(\|x_n-p\|^2-\|u_n-x_n\|^2\\&\quad+2r_n\|u_n-x_n\|\|Bx_n-Bp\|\big)\big]\\&\quad+2(1-\alpha_n\nu)a_n\|y_n-p\|+(1-\alpha_n\nu)a_n^2\\&\le \alpha_n\|\rho Vx_n-\mu Fp\|^2+\beta_n\|Sx_n-p\|^2+\|x_n-p\|^2\\&\quad-(1-\alpha_n\nu)(1-\beta_n)\|u_n-x_n\|^2+2r_n\|u_n-x_n\|\|Bx_n-Bp\|\\&\quad+2(1-\alpha_n\nu)a_n\|y_n-p\|+(1-\alpha_n\nu)a_n^2.\end{aligned}$$

The last inequality implies that

$$\begin{aligned}&(1-\alpha_n\nu)(1-\beta_n)\|u_n-x_n\|^2\\&\quad\le \alpha_n\|\rho Vx_n-\mu Fp\|^2+\beta_n\|Sx_n-p\|^2\\&\qquad+\|x_n-p\|^2-\|x_{n+1}-p\|^2+2r_n\|u_n-x_n\|\|Bx_n-Bp\|\\&\qquad+2(1-\alpha_n\nu)a_n\|y_n-p\|+(1-\alpha_n\nu)a_n^2\\&\quad\le \alpha_n\|\rho Vx_n-\mu Fp\|^2+\beta_n\|Sx_n-p\|^2\\&\qquad+\big(\|x_n-p\|+\|x_{n+1}-p\|\big)\|x_{n+1}-x_n\|\\&\qquad+2r_n\|u_n-x_n\|\|Bx_n-Bp\|\\&\qquad+2(1-\alpha_n\nu)a_n\|y_n-p\|+(1-\alpha_n\nu)a_n^2.\end{aligned}$$

Since $\lim_{n\to\infty}\|Bx_n-Bp\|=0$ and $\{\|y_n-p\|\}$ is a bounded sequence, by using (3.18) and conditions (C1), (C2), we obtain

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{3.21}$$

On the other hand, since a metric projection P_C satisfies

$$\langle u-v,P_Cu-P_Cv\rangle\ge\|P_Cu-P_Cv\|^2,$$

we write

$$\begin{aligned}\|z_n-p\|^2&=\big\|P_C(u_n-\lambda_nAu_n)-P_C(p-\lambda_nAp)\big\|^2\\&\le\big\langle z_n-p,(u_n-\lambda_nAu_n)-(p-\lambda_nAp)\big\rangle\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\big\{\|z_n - p\|^2 + \big\|u_n - p(Au_n - Ap)\big\|^2 \\
&\quad - \big\|u_n - p - \lambda_n(Au_n - Ap) - (z_n - p)\big\|^2\big\} \\
&\leq \frac{1}{2}\big\{\|z_n - p\|^2 + \|u_n - p\|^2 \\
&\quad - \big\|u_n - z_n - \lambda_n(Au_n - Ap)\big\|^2\big\} \\
&\leq \frac{1}{2}\big\{\|z_n - p\|^2 + \|u_n - p\|^2 \\
&\quad - \|u_n - z_n\|^2 + 2\lambda_n\langle u_n - z_n, Au_n - Ap\rangle\big\} \\
&\leq \frac{1}{2}\big\{\|z_n - p\|^2 + \|u_n - p\|^2 - \|u_n - z_n\|^2 \\
&\quad + 2\lambda_n\|u_n - z_n\|\|Au_n - Ap\|\big\}.
\end{aligned}
$$

So, we get

$$
\begin{aligned}
\|z_n - p\|^2 &\leq \|u_n - p\|^2 - \|u_n - z_n\|^2 \\
&\quad + 2\lambda_n\|u_n - z_n\|\|Au_n - Ap\| \\
&\leq \|x_n - p\|^2 - \|u_n - z_n\|^2 \\
&\quad + 2\lambda_n\|u_n - z_n\|\|Au_n - Ap\|. \qquad (3.22)
\end{aligned}
$$

By using (3.19) and (3.22), we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \alpha_n\|\rho Vx_n - \mu Fp\|^2 + (1 - \alpha_n\nu)\big[\beta_n\|Sx_n - p\|^2 \\
&\quad + (1 - \beta_n)\|z_n - p\|^2\big] + 2(1 - \alpha_n\nu)a_n\|y_n - p\| + (1 - \alpha_n\nu)a_n^2 \\
&\leq \alpha_n\|\rho Vx_n - \mu Fp\|^2 + (1 - \alpha_n\nu)\big[\beta_n\|Sx_n - p\|^2 \\
&\quad + (1 - \beta_n)\big(\|x_n - p\|^2 - \|u_n - z_n\|^2 \\
&\quad + 2\lambda_n\|u_n - z_n\|\|Au_n - Ap\|\big)\big] \\
&\quad + 2(1 - \alpha_n\nu)a_n\|y_n - p\| + (1 - \alpha_n\nu)a_n^2 \\
&\leq \alpha_n\|\rho Vx_n - \mu Fp\|^2 + \beta_n\|Sx_n - p\|^2 + \|x_n - p\|^2 \\
&\quad - (1 - \alpha_n\nu)\beta_n\|u_n - z_n\|^2 + 2\lambda_n\|u_n - z_n\|\|Au_n - Ap\| \\
&\quad + 2(1 - \alpha_n\nu)a_n\|y_n - p\| + (1 - \alpha_n\nu)a_n^2.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
(1 - \alpha_n\nu)\beta_n\|u_n - z_n\|^2 &\leq \alpha_n\|\rho Vx_n - \mu Fp\|^2 + \beta_n\|Sx_n - p\|^2 \\
&\quad + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad + 2\lambda_n\|u_n - z_n\|\|Au_n - Ap\| \\
&\quad + 2(1 - \alpha_n\nu)a_n\|y_n - p\| + (1 - \alpha_n\nu)a_n^2 \\
&\leq \alpha_n\|\rho Vx_n - \mu Fp\|^2 + \beta_n\|Sx_n - p\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(\|x_n - p\| + \|x_{n+1} - p\|\right)\|x_{n+1} - x_n\| \\
&+ 2\lambda_n\|u_n - z_n\|\|Au_n - Ap\| \\
&+ 2(1-\alpha_n\nu)a_n\|y_n - p\| + (1-\alpha_n\nu)a_n^2.
\end{aligned}
$$

Since $\lim_{n\to\infty}\|Au_n - Ap\| = 0$ and $\{\|y_n - p\|\}$ is a bounded sequence, by using (3.18) and conditions (C1), (C2), we obtain

$$\lim_{n\to\infty}\|u_n - z_n\| = 0. \tag{3.23}$$

Also, from (3.21) and (3.23), we have

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{3.24}$$

On the other hand, we get

$$
\begin{aligned}
\|x_n - y_n\| &\le \|x_n - u_n\| + \|u_n - z_n\| + \|z_n - y_n\| \\
&= \|x_n - u_n\| + \|u_n - z_n\| + \beta_n(Sx_n - z_n).
\end{aligned}
$$

Since $\lim_{n\to\infty}\beta_n = 0$, again from (3.21) and (3.23), we obtain

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.25}$$

Now, we show that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Before that we need to show that $\lim_{n\to\infty}\|x_n - T_nx_n\| = 0$:

$$
\begin{aligned}
\|x_n - T_nx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_nx_n\| \\
&\le \|x_n - x_{n+1}\| + \|P_Ct_n - P_CT_nx_n\| \\
&\le \|x_n - x_{n+1}\| + \left\|\alpha_n\rho Vx_n + (I-\alpha_n\mu F)T_ny_n - T_nx_n\right\| \\
&\le \|x_n - x_{n+1}\| + \left\|\alpha_n(\rho Vx_n - \mu FT_ny_n) + T_ny_n - T_nx_n\right\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\|\rho Vx_n - \mu FT_ny_n\| + \|y_n - x_n\| + a_n.
\end{aligned}
$$

Since $a_n \to 0$, by using (3.18), (3.25), and condition (C1), we obtain

$$\lim_{n\to\infty}\|x_n - T_nx_n\| = 0. \tag{3.26}$$

Hence, from (3.26) and condition (C3), we have

$$
\begin{aligned}
\|x_n - Tx_n\| &\le \|x_n - T_nx_n\| + \|T_nx_n - Tx_n\| \\
&\le \|x_n - T_nx_n\| + \mathfrak{D}_B(T_n, T) \to 0 \quad \text{as } n\to\infty.
\end{aligned}
$$

Since $\{x_n\}$ is bounded, there exists a weak convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let $x_{n_k} \rightharpoonup w$ as $k\to\infty$. From the Opial condition, we get $x_n \rightharpoonup w$. So, it follows from Lemma 6 that $w \in \mathrm{Fix}(T)$. Therefore, $w_w(x_n) \subset \mathrm{Fix}(T)$. □

Theorem 1 *Assume that* (C1)-(C3) *hold. Then the sequence* $\{x_n\}$ *generated by* (3.1) *converges strongly to* $x^* \in \mathcal{F}$*, which is the unique solution of the variational inequality*

$$\langle (\rho V - \mu F)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \mathcal{F}. \tag{3.27}$$

Proof Since the mapping T is defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$, by Lemma 3, T is a nonexpansive mapping, and $\mathrm{Fix}(T) \neq \emptyset$. Moreover, since the operator $\mu F - \rho V$ is $(\mu\eta - \rho\gamma)$-strongly monotone by Lemma 4, we get the uniqueness of the solution of the variational inequality (3.27). Let us denote this solution by $x^* \in \mathrm{Fix}(T) = \mathcal{F}$.

Now, we divide our proof into three steps.

Step 1. From Lemma 8, since $\{x_n\}$ is bounded, there exists an element w such that $x_n \rightharpoonup w$. First, we show that $w \in \mathcal{F} = \mathrm{Fix}(T) \cap \Omega \cap \mathrm{GEP}(G)$. It follows from Lemma 9 that $w \in \mathrm{Fix}(T) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. Next we show that $w \in \Omega$. Let $N_C v$ be the normal cone to C at $v \in C$, *i.e.*,

$$N_C v = \{ w \in H : \langle v - u, w \rangle \geq 0, \ \forall u \in C \}.$$

Let

$$Hv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then H is maximal monotone mapping. Let $(v,u) \in G(H)$. Since $u - Av \in N_C v$ and $z_n \in C$, we get

$$\langle v - z_n, u - Av \rangle \geq 0. \tag{3.28}$$

On the other hand, from the definition of z_n, we have

$$\langle v - z_n, z_n - u_n - \lambda_n A u_n \rangle \geq 0$$

and hence,

$$\left\langle v - z_n, \frac{z_n - u_n}{\lambda_n} + A u_n \right\rangle \geq 0.$$

Therefore, using (3.28), we get

$$\begin{aligned} \langle v - z_{n_i}, u \rangle &\geq \langle v - z_{n_i}, Av \rangle \\ &\geq \langle v - z_{n_i}, Av \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}} + A u_{n_i} \right\rangle \\ &= \left\langle v - z_{n_i}, Av - A u_{n_i} - \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\ &= \langle v - z_{n_i}, Av - A z_{n_i} \rangle + \langle v - z_{n_i}, A z_{n_i} - A u_{n_i} \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\ &\geq \langle v - z_{n_i}, A z_{n_i} - A u_{n_i} \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle. \end{aligned} \tag{3.29}$$

By using (3.21), (3.23), and (3.24), we get $u_{n_i} \rightharpoonup w$ and $z_{n_i} \rightharpoonup w$ for $i \to \infty$. Hence, from (3.29) we have

$$\langle v - w, u \rangle \geq 0.$$

Since H is maximal monotone, we have $w \in H^{-1}0$ and hence $w \in \Omega$.

Finally, we show that $w \in \mathrm{GEP}(G)$. By using $u_n = T_{r_n}(x_n - r_n B x_n)$, we get

$$G(u_n, y) + \langle Bx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C.$$

Also, from the monotonicity of G, we have

$$\langle Bx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq G(y, u_n), \quad \forall y \in C,$$

and

$$\langle Bx_{n_k}, y - u_{n_k} \rangle + \left\langle y - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}} \right\rangle \geq G(y, u_{n_k}), \quad \forall y \in C. \tag{3.30}$$

Let $y \in C$ and $y_t = ty + (1-t)w$, for $t \in (0,1]$. Then $y_t \in C$. From (3.30), we get

$$\begin{aligned} \langle By_t, y_t - u_{n_k} \rangle &\geq \langle By_t, y_t - u_{n_k} \rangle - \langle Bx_{n_k}, y_t - u_{n_k} \rangle \\ &\quad - \left\langle y_t - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}} \right\rangle + G(y_t, u_{n_k}) \\ &= \langle By_t - Bx_{n_k}, y_t - u_{n_k} \rangle + \langle Bu_{n_k} - Bx_{n_k}, y_t - u_{n_k} \rangle \\ &\quad - \left\langle y_t - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}} \right\rangle + G(y_t, u_{n_k}). \end{aligned} \tag{3.31}$$

Since B is Lipschitz continuous, using (3.21) we obtain $\lim_{k\to\infty} \|Bu_{n_k} - Bx_{n_k}\| = 0$. It follows from (3.31), $u_{n_k} \rightharpoonup w$ and the monotonicity of B that

$$\langle By_t, y_t - w \rangle \geq G(y_t, w). \tag{3.32}$$

Therefore, from assumptions (A1)-(A4) and (3.32), we have

$$\begin{aligned} 0 &= G(y_t, y_t) \leq tG(y_t, y) + (1-t)G(y_t, w) \\ &\leq tG(y_t, y) + (1-t)\langle By_t, y_t - w \rangle \\ &\leq tG(y_t, y) + (1-t)t\langle By_t, y - w \rangle. \end{aligned}$$

The last inequality implies that

$$G(y_t, y) + (1-t)\langle By_t, y - w \rangle \geq 0.$$

If we take the limit $t \to 0^+$, we get

$$G(w, y) + \langle Bw, y - w \rangle \geq 0, \quad \forall y \in C.$$

Hence, we have $w \in \mathrm{GEP}(G)$. Thus, we obtain $w \in \mathcal{F} = \mathrm{Fix}(T) \cap \Omega \cap \mathrm{GEP}(G)$.

Step 2. We show that $\limsup_{n\to\infty}\langle(\rho V-\mu F)x^*, x_n-x^*\rangle\leq 0$, where x^* is the unique solution of variational inequality (3.27). Since the sequence $\{x_n\}$ is bounded, it has a weak convergent subsequence $\{x_{n_k}\}$ such that

$$\limsup_{n\to\infty}\langle(\rho V-\mu F)x^*, x_n-x^*\rangle = \limsup_{k\to\infty}\langle(\rho V-\mu F)x^*, x_{n_k}-x^*\rangle.$$

Let $x_{n_k}\rightharpoonup w$, as $k\to\infty$. It follows from Step 1 that $w\in\mathcal{F}$. Hence

$$\limsup_{n\to\infty}\langle(\rho V-\mu F)x^*, x_n-x^*\rangle = \langle(\rho V-\mu F)x^*, w-x^*\rangle\leq 0.$$

Step 3. Finally, we show that the sequence $\{x_n\}$ generated by (3.1) converges strongly to the point x^*. By using the iteration (3.1), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \langle P_C t_n - x^*, x_{n+1}-x^*\rangle\\ &= \langle P_C t_n - t_n, x_{n+1}-x^*\rangle + \langle t_n - x^*, x_{n+1}-x^*\rangle.\end{aligned}\tag{3.33}$$

Since the metric projection P_C satisfies the inequality

$$\langle x-P_Cx, y-P_Cx\rangle\leq 0,\quad \forall x\in H, y\in C,$$

from (3.33), we get

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq \langle t_n-x^*, x_{n+1}-x^*\rangle\\ &= \langle\alpha_n\rho Vx_n + (I-\alpha_n\mu F)T_ny_n - x^*, x_{n+1}-x^*\rangle\\ &= \langle\alpha_n(\rho Vx_n-\mu Fx^*) + (I-\alpha_n\mu F)T_ny_n\\ &\quad - (I-\alpha_n\mu F)T_nx^*, x_{n+1}-x^*\rangle\\ &= \alpha_n\rho\langle Vx_n - Vx^*, x_{n+1}-x^*\rangle + \alpha_n\langle\rho Vx^*-\mu Fx^*, x_{n+1}-x^*\rangle\\ &\quad + \langle(I-\alpha_n\mu F)T_ny_n - (I-\alpha_n\mu F)T_nx^*, x_{n+1}-x^*\rangle.\end{aligned}$$

Hence, from Lemma 5, we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq \alpha_n\rho\gamma\|x_n-x^*\|\|x_{n+1}-x^*\| + \alpha_n\langle\rho Vx^*-\mu Fx^*, x_{n+1}-x^*\rangle\\ &\quad + (1-\alpha_n\nu)(\|y_n-x^*\| + a_n)\|x_{n+1}-x^*\|\\ &\leq \alpha_n\rho\gamma\|x_n-x^*\|\|x_{n+1}-x^*\| + \alpha_n\langle\rho Vx^*-\mu Fx^*, x_{n+1}-x^*\rangle\\ &\quad + (1-\alpha_n\nu)(\beta_n\|x_n-x^*\| + \beta_n\|Sx^*-x^*\|\\ &\quad + (1-\beta_n)\|z_n-x^*\| + a_n)\|x_{n+1}-x^*\|\\ &\leq \alpha_n\rho\gamma\|x_n-x^*\|\|x_{n+1}-x^*\| + \alpha_n\langle\rho Vx^*-\mu Fx^*, x_{n+1}-x^*\rangle\\ &\quad + (1-\alpha_n\nu)(\beta_n\|x_n-x^*\| + \beta_n\|Sx^*-x^*\|\\ &\quad + (1-\beta_n)\|x_n-x^*\| + a_n)\|x_{n+1}-x^*\|\\ &\leq (1-\alpha_n(\nu-\rho\gamma))\|x_n-x^*\|\|x_{n+1}-x^*\|\end{aligned}$$

$$
\begin{aligned}
&+ \alpha_n\langle \rho Vx^* - \mu Fx^*, x_{n+1} - x^*\rangle \\
&+ (1-\alpha_n\nu)\beta_n\|Sx^* - x^*\|\|x_{n+1} - x^*\| \\
&+ (1-\alpha_n\nu)a_n\|x_{n+1} - x^*\| \\
\le{}& \frac{(1-\alpha_n(\nu-\rho\gamma))}{2}\left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\right) \\
&+ \alpha_n\langle \rho Vx^* - \mu Fx^*, x_{n+1} - x^*\rangle \\
&+ (1-\alpha_n\nu)\beta_n\|Sx^* - x^*\|\|x_{n+1} - x^*\| \\
&+ (1-\alpha_n\nu)a_n\|x_{n+1} - x^*\|.
\end{aligned}
$$

The last inequality implies that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 \le{}& \frac{(1-\alpha_n(\nu-\rho\gamma))}{(1+\alpha_n(\nu-\rho\gamma))}\|x_n - x^*\|^2 \\
&+ \frac{2\alpha_n}{(1+\alpha_n(\nu-\rho\gamma))}\langle \rho Vx^* - \mu Fx^*, x_{n+1} - x^*\rangle \\
&+ \frac{2\beta_n}{(1+\alpha_n(-\rho\gamma))}\|Sx^* - x^*\|\|x_{n+1} - x^*\| \\
&+ \frac{2a_n}{(1+\alpha_n(\nu-\rho\gamma))}\|x_{n+1} - x^*\| \\
\le{}& \bigl(1-\alpha_n(\nu-\rho\gamma)\bigr)\|x_n - x^*\|^2 + \alpha_n(\nu-\rho\gamma)\theta_n,
\end{aligned}
$$

$$
\theta_n = \frac{2}{(1+\alpha_n(\nu-\rho\gamma))(\nu-\rho\gamma)}\left[\langle \rho Vx^* - \mu Fx^*, x_{n+1} - x^*\rangle + \frac{\beta_n}{\alpha_n}M_3 + \frac{a_n}{\alpha_n}\|x_{n+1} - x^*\|\right],
$$

and

$$
\sup_{n\ge 1}\bigl\{\|Sx^* - x^*\|\|x_{n+1} - x^*\|\bigr\} \le M_3.
$$

Since $\frac{\beta_n}{\alpha_n} \to 0$ and $\frac{a_n}{\alpha_n} \to 0$, we get

$$
\limsup_{n\to\infty}\theta_n \le 0.
$$

So, it follows from Lemma 7 that the sequence $\{x_n\}$ generated by (3.1) converges strongly to $x^* \in \mathcal{F}$ which is the unique solution of variational inequality (3.27). □

Putting $A = 0$ in Theorem 1, we have the following corollary.

Corollary 1 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $B : C \to H$ be θ-inverse strongly monotone mapping, $G : C \times C \to \mathbb{R}$ be a bifunction satisfying assumptions* (A1)-(A4), *$S : C \to H$ be a nonexpansive mapping and $\{T_n\}$ be a sequence of nearly nonexpansive mappings with the sequence $\{a_n\}$ such that $\mathcal{F} := \mathrm{Fix}(T) \cap \Omega \cap \mathrm{GEP}(G) \neq \emptyset$ where $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$ and $\mathrm{Fix}(T) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. Let $V : C \to H$ be a γ-Lipschitzian mapping, $F : C \to H$ be a L-Lipschitzian and η-strongly monotone operator such that these coefficients satisfy $0 < \mu < \frac{2\eta}{L^2}$, $0 \le \rho\gamma < \nu$, where*

$\nu = 1 - \sqrt{1 - \mu(2\eta - \mu L^2)}$. *For an arbitrarily initial value* $x_1 \in C$, *consider the sequence* $\{x_n\}$ *in* C *generated by*

$$\begin{cases} G(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C[\beta_n S x_n + (1 - \beta_n)u_n], \\ x_{n+1} = P_C[\alpha_n \rho V x_n + (I - \alpha_n \mu F)T_n y_n], \quad n \geq 1, \end{cases} \tag{3.34}$$

where $\{r_n\} \subset (0, 2\theta)$, $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0,1]$ *satisfying the conditions* (C1)-(C3) *except the condition* $\lim_{n\to\infty} \frac{|\lambda_n - \lambda_{n-1}|}{\alpha_n} = 0$. *Then the sequence* $\{x_n\}$ *generated by* (3.34) *converges strongly to* $x^* \in \mathcal{F}$, *where* x^* *is the unique solution of variational inequality* (3.27).

In Theorem 1, if we take $A = 0$ and $\beta_n = 0$ for all $n \geq 1$, then we have the following corollary.

Corollary 2 *Let* C *be a nonempty, closed, and convex subset of a real Hilbert space* H. *Let* $B : C \to H$ *be* θ*-inverse strongly monotone mapping,* $G : C \times C \to \mathbb{R}$ *be a bifunction satisfying assumptions* (A1)-(A4), $\{T_n\}$ *be a sequence of nearly nonexpansive mappings with the sequence* $\{a_n\}$ *such that* $\mathcal{F} := \mathrm{Fix}(T) \cap \Omega \cap \mathrm{GEP}(G) \neq \emptyset$ *where* $Tx = \lim_{n\to\infty} T_n x$ *for all* $x \in C$ *and* $\mathrm{Fix}(T) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. *Let* $V : C \to H$ *be a* γ*-Lipschitzian mapping,* $F : C \to H$ *be a* L*-Lipschitzian and* η*-strongly monotone operator such that these coefficients satisfy* $0 < \mu < \frac{2\eta}{L^2}$, $0 \leq \rho\gamma < \nu$, *where* $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu L^2)}$. *For an arbitrarily initial value* $x_1 \in C$, *consider the sequence* $\{x_n\}$ *in* C *generated by*

$$\begin{cases} G(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = P_C[\alpha_n \rho V x_n + (I - \alpha_n \mu F)T_n u_n], \quad n \geq 1, \end{cases} \tag{3.35}$$

where $\{r_n\} \subset (0, 2\theta)$, $\{\alpha_n\}$ *is a sequence in* $[0,1]$ *satisfying the conditions* (C1)-(C3) *except the conditions* $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = 0$, $\lim_{n\to\infty} \frac{|\lambda_n - \lambda_{n-1}|}{\alpha_n} = 0$ *and* $\lim_{n\to\infty} \frac{|\beta_n - \beta_{n-1}|}{\alpha_n} = 0$. *Then the sequence* $\{x_n\}$ *generated by* (3.35) *converges strongly to* $x^* \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \Omega \cap \mathrm{GEP}(G)$, *where* x^* *is the unique solution of variational inequality* (3.27).

Putting $A = 0$ and $B = 0$, we have the following corollary, which gives us an iterative scheme to find a common solution of an equilibrium problem and a hierarchical fixed point problem.

Corollary 3 *Let* C *be a nonempty, closed, and convex subset of a real Hilbert space* H. *Let* $G : C \times C \to \mathbb{R}$ *be a bifunction satisfying assumptions* (A1)-(A4), $S : C \to H$ *be a nonexpansive mapping and* $\{T_n\}$ *be a sequence of nearly nonexpansive mappings with the sequence* $\{a_n\}$ *such that* $\mathcal{F} := \mathrm{Fix}(T) \cap \Omega \cap \mathrm{GEP}(G) \neq \emptyset$ *where* $Tx = \lim_{n\to\infty} T_n x$ *for all* $x \in C$ *and* $\mathrm{Fix}(T) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. *Let* $V : C \to H$ *be a* γ*-Lipschitzian mapping,* $F : C \to H$ *be a* L*-Lipschitzian and* η*-strongly monotone operator such that these coefficients satisfy* $0 < \mu < \frac{2\eta}{L^2}$, $0 \leq \rho\gamma < \nu$, *where* $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu L^2)}$. *For an arbitrarily initial value* x_1, *define the sequence* $\{x_n\}$ *in* C *generated by*

$$\begin{cases} G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C[\beta_n S x_n + (1 - \beta_n)u_n], \\ x_{n+1} = P_C[\alpha_n \rho V x_n + (I - \alpha_n \mu F)T_n y_n], \quad n \geq 1, \end{cases} \tag{3.36}$$

where $\{r_n\} \subset (0,\infty)$, $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0,1]$ *satisfying the conditions* (C1)-(C3) *except the condition* $\lim_{n\to\infty} \frac{|\lambda_n-\lambda_{n-1}|}{\alpha_n} = 0$. *Then the sequence* $\{x_n\}$ *generated by* (3.36) *converges strongly to* $x^* \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{EP}(G)$, *where* x^* *is the unique solution of variational inequality* (3.27).

Corollary 4 *Let* C *be a nonempty, closed, and convex subset of a real Hilbert space* H. *Let* $A, B : C \to H$ *be* α, θ*-inverse strongly monotone mappings, respectively.* $G : C \times C \to \mathbb{R}$ *be a bifunction satisfying assumptions* (A1)-(A4), $S : C \to H$ *be a nonexpansive mapping and* $\{T_n\}$ *be a sequence of nonexpansive mappings such that* $\mathcal{F} := \mathrm{Fix}(T) \cap \Omega \cap \mathrm{GEP}(G) \neq \emptyset$ *where* $Tx = \lim_{n\to\infty} T_n x$ *for all* $x \in C$ *and* $\mathrm{Fix}(T) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. *Let* $V : C \to H$ *be a* γ*-Lipschitzian mapping,* $F : C \to H$ *be a L-Lipschitzian and* η*-strongly monotone operator such that these coefficients satisfy* $0 < \mu < \frac{2\eta}{L^2}$, $0 \le \rho\gamma < \nu$, *where* $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu L^2)}$. *For an arbitrarily initial value* $x_1 \in C$, *consider the sequence* $\{x_n\}$ *in* C *generated by* (3.1) *where* $\{\lambda_n\} \subset (0, 2\alpha)$, $\{r_n\} \subset (0, 2\theta)$, $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0,1]$ *satisfying the conditions* (C1)-(C3) *of Theorem* 1 *except the condition* $\lim_{n\to\infty} \frac{a_n}{\alpha_n} = 0$. *Then the sequence* $\{x_n\}$ *converges strongly to* $x^* \in \mathcal{F}$, *where* x^* *is the unique solution of variational inequality* (3.27).

Remark 1 Our results can be reduced to some corresponding results in the following ways:

(1) In our iterative process (3.35), if we take $G(x,y) = 0$ for all $x, y \in C$, $B = 0$, and $r_n = 1$ for all $n \ge 1$, then we derive the iterative process

$$x_{n+1} = P_C\bigl[\alpha_n \rho V x_n + (I - \alpha_n \mu F) T_n x_n\bigr], \quad n \ge 1,$$

which is studied by Sahu *et al.* [4]. Therefore, Theorem 1 generalizes the main result of Sahu *et al.* [4, Theorem 3.1]. So, our results extend the corresponding results of Ceng *et al.* [25] and of many other authors.

(2) If we take S as a nonexpansive self-mapping on C and $T_n = T$ for all $n \ge 1$ such that T is a nonexpansive mapping in (3.1), then it is clear that our iterative process generalizes the iterative process of Wang and Xu [28]. Hence, Theorem 1 generalizes the main result of Wang and Xu [28, Theorem 3.1]. So, our results extend and improve the corresponding results of [11, 27].

(3) The problem of finding the solution of variational inequality (3.27) is equivalent to finding the solutions of hierarchical fixed point problem

$$\bigl\langle (I - S)x^*, x^* - x\bigr\rangle \le 0, \quad \forall x \in \mathcal{F},$$

where $S = I - (\rho V - \mu F)$.

Example 1 Let $H = \mathbb{R}$ and $C = [0,1]$. Let $G : C \times C \to \mathbb{R}$, $G(x,y) = y^2 + xy - 2x^2$, $S = I$, $A : C \to H$, $Ax = 2x$, $B : C \to H$, $Bx = 3x - 1$, $Vx = 4x + 2$, $Fx = 5x$, and

$$T_n x = \begin{cases} 1 - x, & \text{if } x \in [0,1), \\ a_n, & \text{if } x = 1, \end{cases}$$

for all $x \in C$. It is clear that $G(x,y)$ is a bifunction satisfying the assumptions (A1)-(A4), S is nonexpansive mapping, A is $\frac{1}{4}$-inverse strongly monotone mapping, B is $\frac{1}{6}$-inverse strongly monotone mapping, V is γ-Lipschitzian mapping with $\gamma = 4$, F is L-Lipschitzian and η-strongly monotone operator with $L = \eta = 5$ and $\{T_n\}$ is a sequence of nearly nonexpansive mappings with respect to the sequence $a_n = \frac{1}{2n^2-1}$. Define sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0,1]$ by $\alpha_n = \frac{1}{n}$ and $\beta_n = \frac{1}{n^2+2}$ for all $n \geq 1$ and take $\mu = \rho = \frac{1}{5}$, $\nu = 1$, $r_n = \frac{1}{n+3}$, and $\lambda_n = \frac{1}{n+2}$. It is easy to see that all conditions of Theorem 1 are satisfied. First, we find the sequence $\{u_n\}$ which satisfies the following generalized equilibrium problem for all $y \in C$:

$$G(u_n,y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0.$$

For all $n \geq 1$, we get

$$\begin{aligned}
&G(u_n,y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \\
&\quad\Rightarrow\quad y^2 + u_n y - 2u_n^2 + (3x_n - 1)(y - u_n) + \frac{1}{r_n}(y - u_n)(u_n - x_n) \geq 0 \\
&\quad\Rightarrow\quad y^2 r_n + y(u_n r + 3x_n r_n + u_n - r_n - x_n) - 2u_n^2 r_n - 3x_n u_n r_n + u_n r_n - u_n^2 + u_n x_n \geq 0.
\end{aligned}$$

Put $K(y) = y^2 r_n + y(u_n r + 3x_n r_n + u_n - r_n - x_n) - 2u_n^2 r_n - 3x_n u_n r_n + u_n r_n - u_n^2 + u_n x_n$. Then K is a quadratic function of y with coefficients $a = r_n$, $b = u_n r_n + 3x_n r_n + u_n - r_n - x_n$, and $c = -2u_n^2 r_n - 3x_n u_n r_n + u_n r_n - u_n^2 + u_n x_n$. Next, we compute the discriminant Δ of K as follows:

$$\begin{aligned}
\Delta &= b^2 - 4ac \\
&= (u_n r + 3x_n r_n + u_n - r_n - x_n)^2 \\
&\quad - 4r_n\bigl(-2u_n^2 r_n - 3x_n u_n r_n + u_n r_n - u_n^2 + u_n x_n\bigr) \\
&= (u_n - r_n - x_n + 3r_n u_n + 3r_n x_n)^2.
\end{aligned}$$

We know that $K(y) \geq 0$ for all $y \in C = [0,1]$. If it has most one solution in $[0,1]$, so $\Delta \leq 0$ and hence $u_n = \frac{r_n + x_n(1-3r_n)}{1+3r_n} = \frac{1+nx_n}{n+6}$. By using this equation, the sequence $\{x_n\}$ generated by the iterative scheme (3.1) becomes

$$\begin{cases}
y^2 + u_n y - 2u_n^2 + (3x_n - 1)(y - u_n) + (n+3)(y - u_n)(u_n - x_n) \geq 0, \quad \forall y \in C, \\
z_n = u_n - \frac{2}{n+2}u_n, \\
y_n = \frac{1}{n^2+2}x_n + (1 - \frac{1}{n^2+2})z_n, \\
x_{n+1} = \frac{1}{5n}(4x_n + 2) + (1 - \frac{1}{n})(1 - y_n), \quad \forall n \geq 1,
\end{cases} \tag{3.37}$$

for all $n \geq 1$, and it converges strongly to $x^* = 0.5$ which is the unique common fixed point of the sequence $\{T_n\}$ and the unique solution of the variational inequality (1.6) over $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. Some of the values of the iterative scheme (3.37) for the different initial values $x_1 = 0.1$, $x_1 = 0.4$, and $x_1 = 0.7$ are as in Table 1.

Table 1 Some of the values of the iterative scheme (3.37)

	x_1 = 1.000000E–01	x_1 = 4.000000E–01	x_1 = 7.000000E–01
x_2	4.800000E–01	7.200000E–01	9.600000E–01
x_3	6.520000E–01	6.280000E–01	6.040000E–01
x_4	5.392000E–01	5.488000E–01	5.584000E–01
x_5	5.534400E–01	5.481600E–01	5.428800E–01
x_6	5.257984E–01	5.291776E–01	5.325568E–01
x_7	5.319411E–01	5.295757E–01	5.272102E–01
x_8	5.191295E–01	5.208866E–01	5.226438E–01
x_9	5.226747E–01	5.213129E–01	5.199510E–01
x_{10}	5.151936E–01	5.162830E–01	5.173725E–01
⋮	⋮	⋮	⋮
x_{100}	5.015339E–01	5.015208E–01	5.015075E–01
⋮	⋮	⋮	⋮
x_{1000}	5.001506E–01	5.001503E–01	5.001503E–01

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

1. Agarwal, RP, O'Regan, D, Sahu, DR: Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. J. Nonlinear Convex Anal. **8**(1), 61-79 (2007)
2. Agarwal, RP, O'Regan, D, Sahu, DR: Fixed Point Theory for Lipschitzian-Type Mappings with Applications, Topological Fixed Point Theory and Its Applications. Springer, New York (2009)
3. Wong, NC, Sahu, DR, Yao, JC: A generalized hybrid steepest-descent method for variational inequalities in Banach spaces. Fixed Point Theory Appl. **2011**, Article ID 754702 (2011)
4. Sahu, DR, Kang, SM, Sagar, V: Approximation of common fixed points of a sequence of nearly nonexpansive mappings and solutions of variational inequality problems. J. Appl. Math. **2012**, Article ID 902437 (2012)
5. Sanhan, S, Inchan, I, Sanhan, W: Weak and strong convergence theorem of iterative scheme for generalized equilibrium problem and fixed point problems of asymptotically strict pseudo-contraction mappings. Appl. Math. Sci. **5**, 1977-1992 (2011)
6. Kangtunyakarn, A: Strong convergence theorem for a generalized equilibrium problem and system of variational inequalities problem and infinite family of strict pseudo-contractions. Fixed Point Theory Appl. **2011**, 23 (2011) doi:10.1186/1687.1812.2011.23
7. Min, L, Shisheng, Z: A new iterative method for common states of generalized equilibrium problem, fixed point problem of infinite κ-strict pseudo-contractive mappings, and quasi-variational inclusion problem. Acta Math. Sci. **32B**(2), 499-519 (2012)
8. Wang, Y, Xu, HK, Yin, X: Strong convergence theorems for generalized equilibrium, variational inequalities and nonlinear operators. Arab. J. Math. **1**, 549-568 (2012)
9. Razani, A, Yazdı, M: A new iterative method for generalized equilibrium and fixed point problem of nonexpansive mappings. Bull. Malays. Math. Soc. **35**(4), 1049-1061 (2012)
10. Cianciaruso, F, Marino, G, Muglia, L, Yao, Y: On a two-steps algorithm for hierarchical fixed point problems and variational inequalities. J. Inequal. Appl. **2009**, 13 (2009)
11. Tian, M: A general iterative algorithm for nonexpansive mappings in Hilbert spaces. Nonlinear Anal., Theory Methods Appl. **73**(3), 689-694 (2010)
12. Yao, Y, Cho, YJ, Liou, YC: Iterative algorithms for hierarchical fixed points problems and variational inequalities. Math. Comput. Model. **52**(9-10), 1697-1705 (2010)
13. Gu, G, Wang, S, Cho, YJ: Strong convergence algorithms for hierarchical fixed points problems and variational inequalities. J. Appl. Math. **2011**, 1-17 (2011)
14. Yao, Y, Chen, R: Regularized algorithms for hierarchical fixed-point problems. Nonlinear Anal. **74**, 6826-6834 (2011)
15. Tian, M, Huang, LH: Iterative methods for constrained convex minimization problem in Hilbert spaces. Fixed Point Theory Appl. **2013**, 105 (2013)
16. Yao, Y, Liou, YC: Weak and strong convergence of Krasnoselski-Mann iteration for hierarchical fixed point problems. Inverse Problems **24**, 015015 (2008)
17. Xu, HK: Viscosity method for hierarchical fixed point approach to variational inequalities. Taiwan. J. Math. **14**(2), 463-478 (2010)

18. Marino, G, Xu, HK: Explicit hierarchical fixed point approach to variational inequalities. J. Optim. Theory Appl. **149**(1), 61-78 (2011)
19. Bnouhachem, A, Noor, MA: An iterative method for approximating the common solutions of a variational inequality, a mixed equilibrium problem and a hierarchical fixed point problem. J. Inequal. Appl. **2013**, 490 (2013)
20. Bnouhachem, A, Chen, Y: An iterative method for a common solution of a generalized mixed equilibrium problems, variational inequalities, and a hierarchical fixed point problems. Fixed Point Theory Appl. **2014**, 155 (2014)
21. Ceng, LC, Ansari, QH, Yao, JC: Hybrid pseudoviscosity approximation schemes for equilibrium problems, and fixed point problems of infinitely many nonexpansive mappings. Nonlinear Anal. Hybrid Syst. **4**, 743-754 (2010)
22. Ceng, LC, Ansari, QH, Schaible, S, Yao, JC: Iterative methods for generalized equilibrium problems, systems of general generalized equilibrium problems and fixed point problems for nonexpansive mappings in Hilbert space. Fixed Point Theory **12**(2), 293-308 (2011)
23. Ceng, LC, Ansari, QH: Hybrid extragradient-like methods for generalized mixed equilibrium problems, systems of generalized equilibrium problems and optimization problems. J. Glob. Optim. **53**, 69-96 (2012)
24. Latif, A, Ceng, LC, Ansari, QH: Multi-step hybrid viscosity method for systems of variational inequalities defined over sets of solutions of an equilibrium problem and fixed point problems. Fixed Point Theory Appl. **2012**, 186 (2012)
25. Ceng, LC, Ansari, QH, Yao, JC: Some iterative methods for finding fixed points and for solving constrained convex minimization problems. Nonlinear Anal. **74**, 5286-5302 (2011)
26. Sahu, DR, Kang, SM, Sagar, V: Iterative methods for hierarchical common fixed point problems and variational inequalities. Fixed Point Theory Appl. **2013**, 299 (2013)
27. Marino, G, Xu, HK: A general iterative method for nonexpansive mappings in Hilbert spaces. J. Math. Anal. Appl. **318**, 43-52 (2006)
28. Wang, Y, Xu, W: Strong convergence of a modified iterative algorithm for hierarchical fixed point problems and variational inequalities. Fixed Point Theory Appl. **2013**, 121 (2013)
29. Blum, E, Oettli, W: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123-145 (1994)
30. Combettes, PL, Hirstoaga, A: Equilibrium programming in Hilbert spaces. J. Nonlinear Convex Anal. **6**, 117-136 (2005)
31. Yamada, I: The hybrid steepest-descent method for variational inequality problems over the intersection of the fixed point sets of nonexpansive mappings. In: Butnariu, D, Censor, Y, Reich, S (eds.) Inherently Parallel Algorithms and Optimization and Their Applications, pp. 473-504. North-Holland, Amsterdam (2001)
32. Goebel, K, Kirk, WA: Topics on Metric Fixed-Point Theory. Cambridge University Press, Cambridge (1990)
33. Xu, HK, Kim, TH: Convergence of hybrid steepest-descent methods for variational inequalities. J. Optim. Theory Appl. **119**(1), 185-201 (2003)