Upadhyay and Kumar *Journal of Inequalities and Applications* (2015) 2015:31
DOI 10.1186/s13660-014-0544-9

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Characterization of W^p-type of spaces involving fractional Fourier transform

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Abstract

The characterizations of W^p-type of spaces and mapping relations between W- and W^p-type of spaces are discussed by using the fractional Fourier transform. The uniqueness of the Cauchy problems is also investigated by using the same transform.

MSC: 46F12; 46E15

Keywords: fractional Fourier transform; convex functions; Gel'fand and Shilov spaces of type W; L_p-space

1 Introduction

The spaces of W-type were studied by Gurevich [1], Gel'fand and Shilov [2] and Friedman [3]. They investigated the behavior of the Fourier transformation on W-type spaces. The spaces of W-types are applied to the theory of partial differential equations. Pathak and Upadhyay [4] investigated the spaces W_M^p, $W_{M,a}^p$, $W^{\Omega,b,p}$, $W^{\Omega,p}$, $W_M^{\Omega,p}$, $W_{M,a}^{\Omega,b,p}$ in terms of L^p norms. Here M, Ω are certain continuous increasing convex functions and a, b are positive constants and $p \geq 1$. It was shown that the Fourier transformation F is to be a continuous linear mapping as follows: $F : W_{M,a}^p \to W^{\Omega,\frac{1}{a},r}$, $F : W^{\Omega,b,p} \to W_{M,\frac{1}{b}}^r$, $F : W_{M,a}^{\Omega,b,p} \to W_{M,\frac{1}{a}}^{\Omega,\frac{1}{a},r}$. Using the theory of the Hankel transform, Betancor and Rodriguez-Mesa [5] gave a new characterization of the space of We_μ^p-type and established the results $We_{M,a}^p = We_{M,a}$, $We^{p,\Omega,b} = We^{\Omega,b}$, $We_{M,a}^{p,\Omega,b} = We_{M,a}^{\Omega,b}$. Upadhyay [6] established the results of the following types: $W_{M,a}^p = W_{M,a}$, $W^{p,\Omega,b} = W^{\Omega,b}$, $W_{M,a}^{p,\Omega,b} = W_{M,a}^{\Omega,b}$ by exploiting the theory of Fourier transformations. Motivated by the work of Pathak and Upadhyay [4] and Upadhyay [6] we shall extend a similar type of results in n dimensions by using the theory of the fractional Fourier transformations. Let $\mathbb{R}^n$ be the usual Euclidean space given by

$$\mathbb{R}^n = \{(x_1, \ldots, x_n): x_j\text{'s are real numbers}\}.$$

Assume $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then the inner product of x and y is defined by

$$\langle x, y\rangle = x \cdot y = \sum_{j=1}^{n} x_j \cdot y_j \tag{1.1}$$

and the norm of x is defined by

$$|x| = \left(\sum_{j=1}^{n} x_j^2\right)^{\frac{1}{2}} = \left(x_1^2 + \cdots + x_n^2\right)^{\frac{1}{2}}. \tag{1.2}$$

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The L^p norm of a function f in $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, is denoted by $\|f\|_p$ and defined as

$$\|f\|_p = \left(\int_{\mathbb{R}^n} \left|f(x)\right|^p dx\right)^{\frac{1}{p}}. \tag{1.3}$$

The n-dimensional fractional Fourier transform (FrFT) with parameter α of $f(x)$ on $x \in \mathbb{R}^n$ is denoted by $(F_\alpha f)(\xi)$ [7, 8] and defined as

$$\hat{f}_\alpha(\xi) = (F_\alpha f)(\xi) = \int_{\mathbb{R}^n} K_\alpha(x,\xi) f(x)\,dx, \quad \xi \in \mathbb{R}^n, \tag{1.4}$$

where

$$K_\alpha(x,\xi) = \begin{cases} C_\alpha e^{\frac{i(|x|^2+|\xi|^2)\cot\alpha}{2} - i\langle x,\xi\rangle \csc\alpha} & \text{if } \alpha \neq n\pi, \\ \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-i\langle x,\xi\rangle} & \text{if } \alpha = \frac{\pi}{2}, \end{cases} \quad \forall n \in \mathbb{Z}$$

and

$$C_\alpha = (2\pi i \sin\alpha)^{\frac{-n}{2}} e^{\frac{in\alpha}{2}} = \frac{1}{[\pi(1-e^{-2i\alpha})]^{\frac{n}{2}}}. \tag{1.5}$$

The corresponding inversion formula is given by

$$f(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \overline{K_\alpha(x,\xi)} \hat{f}_\alpha(\xi)\,d\xi, \quad x \in \mathbb{R}^n, \tag{1.6}$$

where the kernel

$$\overline{K_\alpha(x,\xi)} = C_\alpha e^{-\frac{i(|x|^2+|\xi|^2)\cot\alpha}{2} + i\langle x,\xi\rangle \csc\alpha},$$

and C_α is defined by (1.5).

Now from the technique of [9, p.2], (1.1) can be written as

$$\begin{aligned} F_\alpha\big[f(x)\big](\xi) &= C_\alpha e^{\frac{i|\xi|^2\cot\alpha}{2}} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle\csc\alpha}\big[f(x)e^{\frac{i|x|^2 cot\alpha}{2}}\big]\,dx \\ &= (2\pi)^{\frac{n}{2}} C_\alpha e^{\frac{i|\xi|^2\cot\alpha}{2}} \big[f(x)e^{\frac{i|x|^2\cot\alpha}{2}}\hat{\big]}(\xi\csc\alpha). \end{aligned} \tag{1.7}$$

Replacing $f(x) = e^{-\frac{i|x|^2\cot\alpha}{2}}\phi(x)$ in (1.3), we obtain

$$F_\alpha\big[e^{-\frac{i|x|^2 cot\alpha}{2}}\phi(x)\big](\xi) = (2\pi)^{\frac{n}{2}} C_\alpha e^{\frac{i|\xi|^2\cot\alpha}{2}} \big[\phi(x)\hat{\big]}(\xi\csc\alpha). \tag{1.8}$$

Now substituting $\xi = w\sin\alpha$, where $w \in \mathbb{R}^n$ in (1.4), we obtain

$$F_\alpha\big[e^{-\frac{i|x|^2\cot\alpha}{2}}\phi(x)\big](w\sin\alpha) = (2\pi)^{\frac{n}{2}} C_\alpha e^{\frac{i|w\sin\alpha|^2\cot\alpha}{2}} \big[\phi(x)\hat{\big]}(w). \tag{1.9}$$

Let $\psi = F_\alpha[e^{-\frac{i|x|^2\cot\alpha}{2}}\phi(x)]$, then (1.6) can be written as

$$\psi(w\sin\alpha) = (2\pi)^{\frac{n}{2}} C_\alpha e^{\frac{i|w\sin\alpha|^2\cot\alpha}{2}} \big[\phi(x)\hat{\big]}(w). \tag{1.10}$$

Now we recall the definitions of W- and W^p-type of spaces from [2–4], which are given below. Let μ_j and w_j, $j = 1,\dots,n$, be continuous and increasing functions on $[0,\infty)$ with $\mu_j(0) = w_j(0) = 0$ and $\mu_j(\infty) = w_j(\infty) = \infty$.

We define

$$M_j(x_j) = \int_0^{x_j} \mu_j(\xi_j)\,d\xi_j \quad (x_j \geq 0), \tag{1.11}$$

$$\Omega_j(y_j) = \int_0^{y_j} w_j(\eta_j)\,d\eta_j \quad (y_j \geq 0), \tag{1.12}$$

where $j = 1,\dots,n$. The functions $M_j(x_j)$ and $\Omega_j(y_j)$ are continuous, increasing, and convex with $M_j(0) = \Omega_j(0) = 0$ and $M_j(\infty) = \Omega_j(\infty) = \infty$, we have

$$M_j(-x_j) = M_j(x_j), \qquad M_j(x_j) + M_j\bigl(x_j'\bigr) \leq M_j\bigl(x_j + x_j'\bigr), \tag{1.13}$$

$$\Omega_j(-y_j) = \Omega_j(y_j), \qquad \Omega_j(y_j) + \Omega_j\bigl(y_j'\bigr) \leq \Omega_j\bigl(y_j + y_j'\bigr). \tag{1.14}$$

We define

$$\mu(\xi) = \bigl(\mu_1(\xi_1)\bigr),\dots,\bigl(\mu_n(\xi_n)\bigr),$$
$$w(\eta) = \bigl(w_1(\eta_1)\bigr),\dots,\bigl(w_n(\eta_n)\bigr).$$

The space $W_{M,a}(\mathbb{R}^n)$ consists of all C^∞-complex valued functions $\phi(x)$ on $x \in \mathbb{R}^n$, which for any $\delta \in \mathbb{R}^n_+$ satisfy the inequality

$$\bigl|D_x^k \phi(x)\bigr| \leq C_{k,\delta} \exp\bigl[-M\bigl[(a-\delta)x\bigr]\bigr], \tag{1.15}$$

and the space $W^p_{M,a}(\mathbb{R}^n)$ consists of all infinitely differentiable functions $\phi(x)$ on $x \in \mathbb{R}^n$, which for any $\delta \in \mathbb{R}^n_+$ satisfy the inequality

$$\left(\int_{\mathbb{R}^n} \bigl|\exp\bigl[-M\bigl[(a-\delta)x\bigr]\bigr] D_x^k \phi(x)\bigr|^p\,dx\right)^{\frac{1}{p}} \leq C_{k,\delta,p}, \quad p \geq 1 \tag{1.16}$$

for each $k \in \mathbb{Z}^n_+$ where $D_x^k = D_{x_1}^{k_1} \cdots D_{x_n}^{k_n}$,

$$\exp\bigl[-M\bigl[(a-\delta)x\bigr]\bigr] = \exp\bigl[-M_1\bigl[(a_1-\delta_1)x_1\bigr] - \cdots - M_n\bigl[(a_n-\delta_n)x_n\bigr]\bigr]$$

and $a_1,\dots,a_n$, $C_{k,\delta,p}$, $C_{k,\delta}$ are positive constants depending on the function $\phi(x)$.

The space $W^{\Omega,b}(\mathbb{C}^n)$ consists of all entire analytic functions $\phi(z)$, where $z = x + iy$ and $x, y \in \mathbb{R}^n$, which for any $\rho \in \mathbb{R}^n_+$ satisfy the inequality

$$\bigl|z^k \phi(z)\bigr| \leq C_{k,\rho} \exp\bigl[\Omega\bigl[(b+\rho)y\bigr]\bigr], \quad k \in \mathbb{Z}^n_+, \tag{1.17}$$

where

$$z^k = z_1^{k_1} \cdots z_n^{k_n},$$

and $b_1,\dots,b_n$, $C_{k,\rho}$ are positive constants depending on the function $\phi(x)$ and the space $W^{\Omega,b,p}$ consists of all entire analytic functions $\phi(z)$ such that for $k\in\mathbb{Z}_+^n$, $\rho\in\mathbb{R}_+^n$, there exists a constant $C_{k,\rho,p}>0$ such that

$$\left(\int_{\mathbb{R}^n}\left|\exp\left[\Omega\left[(b+\rho)y\right]\right]z^k\phi(z)\right|^p dx\right)^{\frac{1}{p}}\le C_{k,\rho,p},\tag{1.18}$$

where

$$\exp\left[\Omega\left[(b+\rho)\right]y\right]=\exp\left[\Omega_1\left[(b_1+\rho_1)y_1\right]+\cdots+\Omega_n\left[(b_n+\rho_n)y_n\right]\right].$$

The space $W_{M,a}^{\Omega,b}(\mathbb{C}^n)$ consists of all entire analytic functions $\phi(z)$ such that there exist constants $\rho,\delta\in\mathbb{R}_+^n$ and $C_{\delta,\rho}>0$ such that

$$\left|\phi(z)\right|\le C_{\delta,\rho}\exp\left[-M\left[(a-\delta)x\right]+\Omega\left[(b+\rho)y\right]\right],\tag{1.19}$$

and the space $W_{M,a}^{\Omega,b,p}(\mathbb{C}^n)$ consists of all entire analytic functions $\phi(z)$ such that for $\rho,\delta\in\mathbb{R}_+^n$ and $C_{\rho,\delta,p}>0$,

$$\left(\int_{\mathbb{R}^n}\left|\exp\left[M\left[(a-\delta)x\right]-\Omega\left[(b+\rho)y\right]\right]\phi(z)\right|^p dx\right)^{\frac{1}{p}}\le C_{\rho,\delta,p},\tag{1.20}$$

where $\exp[M[(a-\delta)x]]$ and $\exp[-\Omega[(b+\rho)y]]$ have the usual meaning like (1.16) and (1.18), and the constants $C_{\rho,\delta,p}$, a, b, and ρ, δ depend only on the function $\phi(z)$.

Let $M_j(x_j)$ and $\Omega_j(y_j)$ be the functions defined by (1.11) and (1.12), respectively, the functions $\mu_j(\xi_j)$ and $w_j(\eta_j)$ which occur in these equations are mutually inverse, that is, $\mu_j(w_j(\eta_j))=\eta_j$ and $w_j(\mu_j(\xi_j))=\xi_j$, then the corresponding functions $M_j(x_j)$ and $\Omega_j(y_j)$ are said to be the dual in the sense of Young. In this case, the Young inequality,

$$x_jy_j\le M_j(x_j)+\Omega_j(y_j),\tag{1.21}$$

holds for any $x_j\ge$, $y_j\ge 0$.

2 Characterization of W^p-type of spaces

In this section we study the characterization of W^p-type of spaces by using the fractional Fourier transformation.

Theorem 2.1 *Let $M(x)$ and $\Omega(y)$ be the pair of functions which are dual in the sense of Young. Then*

$$F_\alpha\left[W_{M,a}^p\right]\subset W^{\Omega,\frac{1}{a},r}\quad \textit{for any } 1\le p,r<\infty.\tag{2.1}$$

Proof Let $e^{-\frac{i|x|^2\cot\alpha}{2}}\phi(x)\in W_{M,a}^p(\mathbb{R}^n)$ and $\sigma=w+i\tau$. Then for any p and r, using the technique of [3, pp.20-21] and (1.10), we have

$$\left\|(\sigma\sin\alpha)^k\psi(\sigma\sin\alpha)\right\|_r=\left[\int_{\mathbb{R}^n}\left|(\sigma\sin\alpha)^k\psi(\sigma\sin\alpha)\right|^r dw\right]^{\frac{1}{r}}.$$

Now using the inequality $|\sigma|^{|k|} \le \frac{|\sigma|^{|k+2|}+|\sigma|^{|k|}}{|w|^2+1}$, we have

$$\begin{aligned}
&\left\|(\sigma\sin\alpha)^k\psi(\sigma\sin\alpha)\right\|_r \\
&\quad\le \left[|\sin\alpha|^{r|k|}\int_{\mathbb{R}^n}\left(\frac{|\sigma|^{|k+2|}+|\sigma|^{|k|}}{|w|^2+1}\right)^r\left|\psi(\sigma\sin\alpha)\right|^r dw\right]^{\frac{1}{r}} \\
&\quad\le \left[|\sin\alpha|^{r|k|}\int_{\mathbb{R}^n}\left(\frac{|\sigma|^{|k+2|}}{|w|^2+1}\right)^r\left|\psi(\sigma\sin\alpha)\right|^r dw\right]^{\frac{1}{r}} \\
&\qquad+\left[|\sin\alpha|^{r|k|}\int_{\mathbb{R}^n}\left(\frac{|\sigma|^{|k|}}{|w|^2+1}\right)^r\left|\psi(\sigma\sin\alpha)\right|^r dw\right]^{\frac{1}{r}}.
\end{aligned}$$

Using (1.7), we get

$$\begin{aligned}
&\left\|(\sigma\sin\alpha)^{|k|}\psi(\sigma\sin\alpha)\right\|_r \\
&\quad\le C_{\alpha,k+2}\left[\int_{\mathbb{R}^n}\frac{d\sigma}{(|w|^2+1)^r}\left(\int_{\mathbb{R}^n}\left|\exp\left[\langle x,\tau\rangle\right]\right|\left|D^{k+2}\phi(x)\right|dx\right)^r\right]^{\frac{1}{r}} \\
&\qquad+C_{\alpha,k}\left[\int_{\mathbb{R}^n}\frac{dw}{(|w|^2+1)^r}\left(\int_{\mathbb{R}^n}\left|\exp\left[\langle x,\tau\rangle\right]\right|\left|D^{k}\phi(x)\right|dx\right)^r\right]^{\frac{1}{r}} \\
&\quad\le \left[C_{\alpha,k+2}\left\{\int_{\mathbb{R}^n}\frac{dw}{(|w|^2+1)^r}\left(\int_{\mathbb{R}^n}\left|\exp\left[\langle x,\tau\rangle\right]\right|\left|D^{k+2}\phi(x)\right|dx\right)^r\right\}^{\frac{1}{r}}\right. \\
&\qquad\left.+C_{\alpha,k}\left\{\int_{\mathbb{R}^n}\frac{dw}{(|w|^2+1)^r}\left(\int_{\mathbb{R}^n}\left|\exp\left[\langle x,\tau\rangle\right]\right|\left|D^{k}\phi(x)\right|dx\right)^r\right\}^{\frac{1}{r}}\right] \\
&\quad\le C_{\alpha,k+2}\left[\int_{\mathbb{R}^n}\frac{dw}{(|w|^2+1)^r}\left(\left(\int_{\mathbb{R}^n}\left|\exp\left[M\left[(a-\delta)x\right]\right]D^{k+2}\phi(x)\right|^p dx\right)^{\frac{1}{p}}\right.\right. \\
&\qquad\left.\left.\times\left(\int_{\mathbb{R}^n}\left|\exp\left[|x||\tau|-M\left[(a-\delta)x\right]\right]\right|^{p'}dx\right)^{\frac{1}{p'}}\right)^r\right]^{\frac{1}{r}} \\
&\qquad+C_{\alpha,k}\left[\int_{\mathbb{R}^n}\frac{dw}{(|w|^2+1)^r}\left(\left(\int_{\mathbb{R}^n}\left|\exp\left[M\left[(a-\delta)x\right]\right]D^{k}\phi(x)\right|^p dx\right)^{\frac{1}{p}}\right.\right. \\
&\qquad\left.\left.\times\left(\int_{\mathbb{R}^n}\left|\exp\left[|x||\tau|-M\left[(a-\delta)x\right]\right]\right|^{p'}dx\right)^{\frac{1}{p'}}\right)^r\right]^{\frac{1}{r}}.
\end{aligned}$$

Now using the Young inequality (1.21) and the arguments of [3, p.23], we get

$$\left\|(\sigma\sin\alpha)^k\psi(\sigma\sin\alpha)\right\|_r \le D'_{k+2,\rho,\alpha,r}e^{\Omega[\frac{\tau}{\gamma}]}+D'_{k,\rho,\alpha,r}e^{\Omega[\frac{\tau}{\gamma}]}.$$

In the above expression, we set $\frac{1}{\gamma}=(\frac{1}{a}+\rho)$, since $\gamma=a-\delta$ and ρ is arbitrarily small together with δ. Therefore, we have

$$\left\|(\sigma\sin\alpha)^k\psi(\sigma\sin\alpha)\right\|_r \le C^r_{k,\rho,|\sin\alpha|}e^{\Omega[\frac{\tau}{\gamma}]}. \qquad \square$$

Theorem 2.2 *Let $M(x)$ and $\Omega(y)$ be the pair of functions which are dual in the sense of Young. Then*

$$F_\alpha\left[W^{\Omega,b,p}\right]\subset W^r_{M,\frac{1}{b}} \quad \textit{for any } 1\le p,r<\infty.$$

Proof Let $e^{-\frac{i|z|^2 cot\alpha}{2}}\phi(z) \in W^{\Omega,b,p}(\mathbb{C}^n)$ and $\sigma = w + i\tau$. Then from the arguments of [8, Theorem 2.2], we have

$$\begin{aligned}\left|D_w^k\psi(w\sin\alpha)\right| &\le C_{k,r,\eta,s,\rho}\exp\left[-\langle y,w\rangle + \Omega(b+\rho)y\right]\\ &\le C_{k,r,\eta,s,\rho}\exp\left[-y_1w_1 + \Omega_1\left[(b_1+\rho_1)y_1\right]\right.\\ &\qquad \left.-\cdots - y_nw_n + \Omega_n\left[(b_n+\rho_n)y_n\right]\right].\end{aligned}$$

From the arguments of [4, p.737] we have

$$\left|D_w^k\psi(w\sin\alpha)\right| \le C_{k,\rho,r,\eta,s}\exp\left[-M\left[\left(\frac{1}{b}-\delta\right)w\right] - M\left[\frac{\rho^2 w}{b^3}\right]\right].$$

Hence,

$$\left\|\exp\left[M\left[\left(\frac{1}{b}-\delta\right)w\right]\right]D_w^k\phi(w\sin\alpha)\right\|_r \le C^r_{k,\rho,r,\eta,s}.$$

This implies that

$$\psi(w\sin\alpha) \in W^r_{M,\frac{1}{b}}. \qquad \square$$

Theorem 2.3 *Let $M_0(x)$ and $\Omega_0(y)$ be the functions which are dual in the sense of Young to the functions $M(x)$ and $\Omega(y)$, respectively. Then*

$$F_\alpha\left[W^{\Omega,b,p}_{M,a,}\right] \subset W^{\Omega_0,\frac{1}{a},r}_{M_0,\frac{1}{b}} \quad \textit{for any } 1 \le p,r < \infty.$$

Proof Let $e^{-\frac{i|x|^2 cot\alpha}{2}}\phi(x) \in W^{\Omega,b,p}_{M,a}(\mathbb{R}^n)$ and $\sigma = w + i\tau$. Then by the technique of [2, pp.23-24] and (1.10), we have

$$\begin{aligned}\left|\psi(\sigma\sin\alpha)\right| &\le \left|(2\pi i\sin\alpha)^{\frac{-n}{2}}e^{\frac{in\alpha}{2}}\right|\int_{\mathbb{R}^n}\left|e^{\overline{i\langle\sigma,z\rangle}}\right|\left|\phi(z)\right|dx\\ &\le \left|(2\pi i\sin\alpha)^{\frac{-n}{2}}e^{\frac{in\alpha}{2}}\right|\int_{\mathbb{R}^n}\left|e^{-\langle w,y\rangle-\langle\tau,x\rangle}\right|\left|\phi(z)\right|dx\\ &\le DC_{\rho,\delta}\left|e^{-\langle w,y\rangle}\right|\int_{\mathbb{R}^n}\left|e^{-\langle\tau,x\rangle}\right|\exp\left[-M\left[(a-\delta)x\right]+\Omega\left[(b+\rho)y\right]\right]dx\\ &\le C_{\rho,\delta,\alpha}e^{-\langle w,y\rangle}\int_{\mathbb{R}^n}\exp\left[-M\left[(a-\delta)x\right]+\Omega\left[(b+\rho)y\right]\right]\exp\left[|\tau||x|\right]dx\\ &= C_{\rho,\delta,\alpha}\exp\left[-\langle w,y\rangle + \Omega\left[(b+\rho)y\right]\right]\int_{\mathbb{R}^n}\exp\left[-M\left[(a-\delta)x\right]+|\tau||x|\right]dx.\end{aligned}$$

Now using the arguments of [4, p.738], we have

$$\left|\psi(\sigma\sin\alpha)\right| \le C_{\rho,\delta,\alpha}\exp\left[-M_0\left[\left(\frac{1}{b}-\delta_0\right)w\right] + \Omega_0\left[\left(\frac{1}{a}+\rho_0\right)\tau\right] - M_0\left[\frac{\rho_0^2 w}{b^3}\right]\right].$$

Hence,

$$\left\|\exp\left[M_0\left[\left(\frac{1}{b}-\delta_0\right)w\right] - \Omega_0\left[\left(\frac{1}{a}+\rho_0\right)\tau\right]\right]\psi(\sigma\sin\alpha)\right\|_r \le C'^r_{\alpha,\delta,\rho}. \qquad \square$$

3 Relation between W- and W^p-types of spaces

In this section the mapping relations between W- and W^p-types of spaces are discussed.

Theorem 3.1 *Let $M(x)$, $\Omega(y)$ be the pair of functions which are dual in the sense of Young. Then*

$$W^p_{M,a} = W_{M,a}, \quad 1 \le p < \infty.$$

Proof Now, for showing the above theorem we shall prove the following lemma. □

Lemma 3.2 *Let $1 \le p < \infty$. Then $W^p_{M,a} \subset W_{M,a}$.*

Proof Let $e^{-\frac{|x|^2 \cot\alpha}{2}}\phi(x) \in W^p_{M,a}(\mathbb{R}^n)$ and $\sigma = w + i\tau$. Then from the arguments of Theorem 2.1, we get

$$F_\alpha\big(W^p_{M,a}\big) \subset W^{\Omega,\frac{1}{a}}. \tag{3.1}$$

From the inverse property of the fractional Fourier transform, we have

$$W^p_{M,a} \subset F_\alpha^{-1}\big(W^{\Omega,\frac{1}{a}}\big). \tag{3.2}$$

Now, let $\hat{\phi}_\alpha(\sigma) \in W^{\Omega,\frac{1}{a}}$. Then by the technique of [2, pp.21-22] and (1.6), we have

$$\phi(x) = C_\alpha \int_{\mathbb{R}^n} e^{-\frac{i(|x|^2+|\sigma|^2)\cot\alpha}{2} + i\langle x,\sigma\rangle \csc\alpha} \hat{\phi}_\alpha(\sigma)\, dw.$$

Therefore,

$$\begin{aligned}
\phi^{(k)}(x) &= C_\alpha \int_{\mathbb{R}^n} D_x^k\big(e^{-\frac{i|x|^2\cot\alpha}{2} + i\langle x,\sigma\rangle\csc\alpha}\big) e^{-\frac{i|\sigma|^2\cot\alpha}{2}} \hat{\phi}_\alpha(\sigma)\, dw \\
&= C_\alpha \int_{\mathbb{R}^n} \sum_{r \le k} \binom{k}{r} \big(D_x^r e^{-\frac{i|x|^2\cot\alpha}{2}}\big)\big(D_x^{k-r} e^{i\langle x,\sigma\rangle\csc\alpha}\big) e^{-\frac{i|\sigma|^2\cot\alpha}{2}} \hat{\phi}_\alpha(\sigma)\, dw \\
&= C_\alpha \sum_{r \le k} \binom{k}{r} \int_{\mathbb{R}^n} \sum_{\eta \le r} C_\eta(\cot\alpha) x^\eta e^{-\frac{i|x|^2\cot\alpha}{2}} (i\sigma \csc\alpha)^{k-r} \\
&\quad \times e^{i\langle x,\sigma\rangle\csc\alpha} e^{-\frac{i|\sigma|^2\cot\alpha}{2}} \hat{\phi}_\alpha(\sigma)\, dw \\
&= C_\alpha \sum_{r \le k} \binom{k}{r} \sum_{\eta \le r} C_\eta(\cot\alpha) \int_{\mathbb{R}^n} e^{-\frac{i|x|^2\cot\alpha}{2}} (i\sigma\csc\alpha)^{k-r} \\
&\quad \times \big(D_\sigma^\eta e^{i\langle x,\sigma\rangle\csc\alpha}\big) e^{-\frac{i|\sigma|^2\cot\alpha}{2}} \hat{\phi}_\alpha(\sigma)\, dw \\
&= C_\alpha \sum_{r \le k} \binom{k}{r} \sum_{\eta \le r} C_\eta(\cot\alpha)(i\csc\alpha)^{-|\eta|+|k-r|}(-1)^{|\eta|} \int_{\mathbb{R}^n} e^{-\frac{i|x|^2\cot\alpha}{2}} e^{i\langle x,\sigma\rangle\csc\alpha} \\
&\quad \times D_\sigma^\eta\big(\sigma^{k-r} e^{-\frac{i|\sigma|^2\cot\alpha}{2}} \hat{\phi}_\alpha(\sigma)\big)\, dw \\
&= C_\alpha \sum_{r \le k} \binom{k}{r} \sum_{\eta \le r} C_\eta(\cot\alpha)(i\csc\alpha)^{-|\eta|+|k-r|}(-1)^{|\eta|} \int_{\mathbb{R}^n} e^{-\frac{i|x|^2\cot\alpha}{2}} e^{i\langle x,\sigma\rangle\csc\alpha}
\end{aligned}$$

$$\begin{aligned}
&\times \sum_{\beta\leq\eta}\binom{\eta}{\beta}\big(D_\sigma^\eta e^{-\frac{i|\sigma|^2\cot\alpha}{2}}\big)\big(D_\sigma^{\eta-\beta}\sigma^{k-r}\hat{\phi}_\alpha(\sigma)\big)\,dw \\
&= C_\alpha \sum_{r\leq k}\binom{k}{r}\sum_{\eta\leq r} C_\eta(\cot\alpha)(i\csc\alpha)^{-|\eta|+|k-r|}(-1)^{|\eta|}\int_{\mathbb{R}^n} e^{-\frac{i|x|^2\cot\alpha}{2}} e^{i\langle x,\sigma\rangle\csc\alpha} \\
&\times \sum_{\beta\leq\eta}\binom{\eta}{\beta}\Big(\sum_{\lambda\leq\eta} C_\lambda(\cot\alpha)\sigma^\lambda\Big)e^{-\frac{i|\sigma|^2\cot\alpha}{2}} \\
&\times \sum_{m\leq\eta-\beta}\binom{\eta-\beta}{m}\big(D_\sigma^m\sigma^{k-r}\big)\big(D_\sigma^{\eta-\beta-m}\hat{\phi}_\alpha(\sigma)\big)\,dw.
\end{aligned}$$

Hence,

$$\begin{aligned}
\big|\phi^{(k)}(x)\big| &\leq |C_\alpha|\sum_{r\leq k}\binom{k}{r}\sum_{\eta\leq r}\big|C_\eta(\cot\alpha)\big|\,|\csc\alpha|^{-|\eta|+|k-r|} \\
&\times \sum_{\beta\leq\eta}\binom{\eta}{\beta}\frac{(k-r)!}{(k-r-m)!}\sum_{\lambda\leq\beta}\big|C_\lambda(\cot\alpha)\big| \\
&\times \sum_{m\leq\eta-\beta}\binom{\eta-\beta}{m}\int_{\mathbb{R}^n}\big|e^{i\langle x,\sigma\rangle\csc\alpha}\big|\big|\sigma^{k-r-m+\lambda}D_\sigma^{\eta-\beta-m}\hat{\phi}_\alpha(\sigma)\big|\,dw.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\big|\phi^{(k)}(x)\big| &\leq |C_\alpha|\sum_{r\leq k}\binom{k}{r}\sum_{\eta\leq r}\big|C_\eta(\cot\alpha)\big|\,|\csc\alpha|^{-|\eta|+|k-r|} \\
&\times \sum_{\beta\leq\eta}\binom{\eta}{\beta}\frac{(k-r)!}{(k-r-m)!}\sum_{\lambda\leq\beta}\big|C_\lambda(\cot\alpha)\big| \\
&\times \sum_{m\leq\eta-\beta}\binom{\eta-\beta}{m}\int_{\mathbb{R}^n} e^{-\langle x,\tau\rangle\csc\alpha}\left(\frac{|\sigma|^{|k-r-m+\lambda|+2}+|\sigma|^{|k-r-m+\lambda|}}{(|w|^2+1)}\right) \\
&\times \big|D_\sigma^{\eta-\beta-m}\hat{\phi}_\alpha(\sigma)\big|\,dw \\
&\leq C'_{\alpha,k}\exp\left[-\langle x,\tau\rangle\csc\alpha+\Omega\left[\left(\frac{1}{a}+\rho\right)\tau\right]\right]. \qquad (3.3)
\end{aligned}$$

Now using the arguments of [4, p.23], we get

$$\big|\phi^{(k)}(x)\big| \leq C'_{\alpha,k}\exp\big[-M\big[(a-\delta)x\csc\alpha\big]\big] \tag{3.4}$$

for arbitrarily small δ together with ρ. Hence the above expression gives

$$F_\alpha^{-1}\big(W^{\Omega,\frac{1}{a}}\big) \subset W_{M,a}. \tag{3.5}$$

Thus (3.2) and (3.5) imply that

$$W_{M,a}^p \subset W_{M,a}. \tag{3.6}$$

□

Lemma 3.3 *Let* $1 \leq p < \infty$. *Then* $W_{M,a} \subset W_{M,a}^p$.

Proof Let $e^{-\frac{i|x|^2\cot\alpha}{2}}\phi(x) \in W_{M,a}(\mathbb{R}^n)$ and $\sigma = w + i\tau \in \mathbb{C}^n$. Then from [8, Theorem 2.1], it follows that

$$F_\alpha(W_{M,a}) \subset W^{\Omega,\frac{1}{a}}. \tag{3.7}$$

Now by the inverse property of the fractional Fourier transform we have

$$W_{M,a} \subset F_\alpha^{-1}\bigl(W^{\Omega,\frac{1}{a}}\bigr). \tag{3.8}$$

Again let $\hat{\phi}_\alpha(\sigma) \in W^{\Omega,\frac{1}{a}}$. Then from (3.3) we have

$$\bigl|\phi^k(x)\bigr| \le C'_{\alpha,k}\exp\biggl[-\langle x,\tau\rangle\csc\alpha + \Omega\biggl[\biggl(\frac{1}{a}+\rho\biggr)\tau\biggr]\biggr]. \tag{3.9}$$

Using (1.21) and [6, p.385] we get

$$\bigl|\phi^k(x)\bigr| \le C'_{\alpha,k}\exp\bigl[-M\bigl[(a-\delta)x\csc\alpha\bigr] - M\bigl[a^3\rho^2x\csc\alpha\bigr]\bigr]. \tag{3.10}$$

Therefore,

$$\bigl\|\exp\bigl[M\bigl[(a-\delta)x\csc\alpha\bigr]\bigr]\phi^k(x)\bigr\|_p \le C'_{\alpha,k}\bigl\|e^{-M[a^3\rho^2x\csc\alpha]}\bigr\|_p. \tag{3.11}$$

This implies that

$$F_\alpha^{-1}\bigl(W^{\Omega,\frac{1}{a}}\bigr) \subset W^p_{M,a}. \tag{3.12}$$

From (3.8) and (3.12) we find

$$W_{M,a} \subset W^p_{M,a}. \tag{3.13}$$

Now from (3.6) and (3.13) we get the result

$$W^p_{M,a} = W_{M,a}. \tag{3.14}$$

□

Theorem 3.4 *Let* $M(x)$ *and* $\Omega(y)$ *be the same functions as in Theorem* 3.1. *Then*

$$W^{\Omega,b,p} = W^{\Omega,b}, \quad 1 \le p < \infty. \tag{3.15}$$

Proof Let $e^{-\frac{i|z|^2\cot\alpha}{2}}\phi(z) \in W^{\Omega,b,p}$. Then from Theorem 2.2 it follows that

$$F_\alpha\bigl(W^{\Omega,b,p}\bigr) \subset W_{M,\frac{1}{a}}. \tag{3.16}$$

By the inverse property of the fractional Fourier transform, we have

$$W^{\Omega,b,p} \subset F_\alpha^{-1}(W_{M,\frac{1}{a}}). \tag{3.17}$$

Now let $\hat{\phi}_\alpha(x) \in W_{M,\frac{1}{b}}$. Then from the technique of [2, pp.20-21], we have

$$\begin{aligned}
&(i\sigma \csc\alpha)^k \phi(\sigma) \\
&\quad = C_\alpha \int_{\mathbb{R}^n} e^{-\frac{i(|x|^2+|\sigma|^2)\cot\alpha}{2}} \left(D_x^k e^{i\langle x,\sigma\rangle \csc\alpha}\right) \hat{\phi}_\alpha(x)\,dx \\
&\quad = C_\alpha (-1)^{|k|} \int_{\mathbb{R}^n} e^{i\langle x,\sigma\rangle \csc\alpha} \left(D_x^k e^{-\frac{i|x|^2\cot\alpha}{2}} \hat{\phi}_\alpha(x)\right) e^{-\frac{i|\sigma|^2\cot\alpha}{2}}\,dx \\
&\quad = C_\alpha (-1)^{|k|} \int_{\mathbb{R}^n} e^{i\langle x,\sigma\rangle \csc\alpha} \sum_{r\leq k} \binom{k}{r} \left(D_x^r e^{-\frac{i|x|^2\cot\alpha}{2}}\right) \left(D_x^{k-r} \hat{\phi}_\alpha(x)\right) e^{-\frac{i|\sigma|^2\cot\alpha}{2}}\,dx \\
&\quad = C_\alpha (-1)^{|k|} \int_{\mathbb{R}^n} e^{i\langle x,\sigma\rangle \csc\alpha} \sum_{r\leq k} \binom{k}{r} \sum_{\beta\leq r} C_\beta(\cot\alpha) e^{-\frac{i|x|^2\cot\alpha}{2}} e^{-\frac{i|\sigma|^2\cot\alpha}{2}} \left(x^\beta D_x^{k-r} \hat{\phi}_\alpha(x)\right)dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|(\sigma \csc\alpha)^k \phi(s)\right| &\leq |C_\alpha| \sum_{r\leq k} \binom{k}{r} \sum_{\beta\leq r} \left|C_\beta(\cot\alpha)\right| \int_{\mathbb{R}^n} \left|e^{-\langle x,\tau\rangle \csc\alpha}\right| \left|x^\beta D_x^{k-r} \hat{\phi}_\alpha(x)\right|dx \\
&\leq |C_\alpha| \sum_{r\leq k} \binom{k}{r} \sum_{\beta\leq r} \left|C_\beta(\cot\alpha)\right| \int_{\mathbb{R}^n} e^{[|x||\tau \csc\alpha| - M[(\frac{1}{b}-\delta)x]]}\,dx.
\end{aligned}$$

Now using the arguments of [2, p.21], we get

$$\left|(\sigma \csc\alpha)^k \phi(\sigma)\right| \leq C_{\alpha,k} \exp\left[\Omega\left[(b+\rho)\tau \csc\alpha\right]\right],$$

where ρ is arbitrarily small together with δ. Thus we have

$$F_\alpha^{-1}(W_{M,\frac{1}{b}}) \subset W^{\Omega,b}. \tag{3.18}$$

Therefore (3.17) and (3.18) yield

$$W^{\Omega,b,p} \subset W^{\Omega,b}. \tag{3.19}$$

Again we take $e^{-\frac{i|z|^2\cot\alpha}{2}} \phi(z) \in W^{\Omega,b}(\mathbb{C}^n)$. Then from [8, Theorem 2.2], we have

$$F_\alpha\left(W^{\Omega,b}\right) \subset W_{M,\frac{1}{b}}. \tag{3.20}$$

By the inverse property of the fractional Fourier transform, we have

$$W^{\Omega,b} \subset F_\alpha^{-1}(W_{M,\frac{1}{b}}). \tag{3.21}$$

Furthermore, we take $\hat{\phi}(x) \in W_{M,\frac{1}{b}}(\mathbb{R}^n)$. Then from the arguments of [2, pp.20-21], we have

$$\begin{aligned}
&\left\|(i\sigma \csc\alpha)^k \phi(\sigma)\right\|_p \\
&\quad = \left(\int_{\mathbb{R}^n} \left|(i\sigma \csc\alpha)^k \phi(\sigma)\right|^p dw\right)^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_{\mathbb{R}^n}\left(\frac{|i\sigma\csc\alpha|^{|k|+2}+|i\sigma\csc\alpha|^{|k|}}{(|w|^2+1)}\right)^p\left|\phi(\sigma)\right|^p dw\right)^{\frac{1}{p}}\\
&\leq \left(\int_{\mathbb{R}^n}\left(\frac{|i\sigma\csc\alpha|^{|k|+2}}{(|w|^2+1)}\right)^p\left|\phi(\sigma)\right|^p dw\right)^{\frac{1}{p}}+\left(\int_{\mathbb{R}^n}\left(\frac{|i\sigma\csc\alpha|^{|k|}}{(|w|^2+1)}\right)^p\left|\phi(\sigma)\right|^p dw\right)^{\frac{1}{p}}\\
&\leq \left(\int_{\mathbb{R}^n}\frac{dw}{(|w|^2+1)^p}\left(C_{\alpha,k}\int_{\mathbb{R}^n}\left|e^{-\langle x,\tau\rangle\csc\alpha}\hat{\phi}_\alpha(x)\right|dx\right)^p\right)^{\frac{1}{p}}\\
&\quad+\left(\int_{\mathbb{R}^n}\frac{dw}{(|w|^2+1)^p}\left(C'_{\alpha,k}\int_{\mathbb{R}^n}\left|e^{-\langle x,\tau\rangle\csc\alpha}\hat{\phi}_\alpha(x)\right|dx\right)^p\right)^{\frac{1}{p}}\\
&\leq \left(C_{\alpha,k}C_{k+2,\delta}+C'_{\alpha,k}C_{k,\delta}\right)\left(\int_{\mathbb{R}^n}\frac{dw}{(|w|^2+1)^p}\right.\\
&\quad\times\left.\left(\int_{\mathbb{R}^n}\left|\exp\left[-\langle x,\tau\rangle\csc\alpha-M\left[\left(\frac{1}{b}-\delta\right)x\right]\right]\right|dx\right)^p\right)^{\frac{1}{p}}\\
&\leq C_{\alpha,k,\delta}\left(\int_{\mathbb{R}^n}\left|\exp\left[-\langle x,\tau\rangle\csc\alpha-M\left[\left(\frac{1}{b}-\delta\right)x\right]\right]\right|dx\right)\left(\int_{\mathbb{R}^n}\frac{dw}{(|w|^2+1)^p}\right)^{\frac{1}{p}}\\
&\leq C_{\alpha,k,\delta}C_p\int_{\mathbb{R}^n}\left|\exp\left[-\langle x,\tau\rangle\csc\alpha-M\left[\left(\frac{1}{b}-\delta\right)x\right]\right]\right|dx.
\end{aligned}
\tag{3.22}
$$

Now using the Young inequality (1.21) and from the arguments of [2, p.21], we get

$$\left\|(i\sigma\csc\alpha)^k\phi(\sigma)\right\|_p\leq C_{\alpha,k,\delta,p}\exp\left[\Omega\left[(b+\rho)\tau\right]\right].$$

This implies that

$$F_\alpha^{-1}(W_{M,\frac{1}{b}})\subset W^{\Omega,b,p}. \tag{3.23}$$

Now (3.21) and (3.23) give

$$W^{\Omega,b}\subset W^{\Omega,b,p}. \tag{3.24}$$

Finally, (3.19) and (3.24) give

$$W^{\Omega,b}=W^{\Omega,b,p}. \tag{3.25}$$

□

Theorem 3.5 *Let $\Omega_0(y)$ and $M_0(x)$ be the functions which are dual in the sense of Young to the functions $M(x)$ and $\Omega(y)$, respectively. Then*

$$W_{M,a}^{\Omega,b,p}=W_{M,a}^{\Omega,b},\quad 1\leq p<\infty. \tag{3.26}$$

Proof Let $e^{-\frac{i|x|^2\cot\alpha}{2}}\phi(x)\in W_{M,a}^{\Omega,b,p}$. Then from Theorem 2.3, it follows that

$$F_\alpha\left(W_{M,a}^{\Omega,b,p}\right)\subset W_{M_0,\frac{1}{b}}^{\Omega_0,\frac{1}{a}}. \tag{3.27}$$

By the inverse property of the fractional Fourier transform we get

$$W_{M,a}^{\Omega,b,p} \subset F_\alpha^{-1}\left(W_{M_0,\frac{1}{b}}^{\Omega_0,\frac{1}{a}}\right). \tag{3.28}$$

Now let $\hat{\phi}_\alpha(z) \in W_{M_0,\frac{1}{b}}^{\Omega_0,\frac{1}{a}}$. Then from the arguments of [2, p.24], we get

$$\phi(\sigma + i\tau) = C_\alpha \int_{\mathbb{R}^n} e^{-\frac{i(|z|^2+|\sigma|^2)\cot\alpha}{2} + i\langle\sigma,z\rangle\csc\alpha} \hat{\phi}_\alpha(z)\,dx.$$

Therefore,

$$\begin{aligned}
&\left|\phi(\sigma + i\tau)\right| \\
&\quad\le |C_\alpha| \int_{\mathbb{R}^n} \left|\exp\left[-\langle w,y\rangle\csc\alpha - \langle\tau,x\rangle\csc\alpha\right]\right| \left|\hat{\phi}_\alpha(z)\right|dx \\
&\quad\le C_\alpha C_{\delta,\rho} \int_{\mathbb{R}^n} \left|\exp\left[-\langle w,y\rangle\csc\alpha - \langle\tau,x\rangle\csc\alpha\right]\right| \\
&\qquad\times \left|\exp\left[-M_0\left[\left(\frac{1}{b} - \delta\right)x\right] + \Omega_0\left[\left(\frac{1}{a} + \rho\right)y\right]\right]\right|dx \\
&\quad\le C_{\delta,\rho,\alpha} \exp\left[\Omega_0\left[\left(\frac{1}{a} + \rho\right)y\right] - \langle w,y\rangle\csc\alpha\right] \\
&\qquad\times \int_{\mathbb{R}^n} \exp\left[-M_0\left[\left(\frac{1}{b} - \delta\right)x\right] + \langle\tau,x\rangle\csc\alpha\right]dx.
\end{aligned}$$

Now using (1.21), we have

$$\begin{aligned}
\left|\phi(\sigma + i\tau)\right| &\le C'_{\delta,\rho,\alpha} \exp\left[-M\left[\frac{w\csc\alpha}{\frac{1}{a} + \rho}\right] + \Omega\left[\frac{\tau\csc\alpha}{(\frac{1}{b} + 2\delta)}\right]\right] \\
&\le C'_{\delta,\rho,\alpha} \exp\left[-M\left[\left(\frac{1}{a} - \delta_0\right)w\csc\alpha\right] + \Omega\left[\left(\frac{1}{b} + \rho_0\right)\tau\csc\alpha\right]\right],
\end{aligned}$$

where ρ_0 and δ_0 are arbitrarily small together with ρ and δ, respectively. This shows that

$$F_\alpha^{-1}\left(W_{M_0,a}^{\Omega_0,b,}\right) \subset W_{M,a}^{\Omega,b}. \tag{3.29}$$

Thus from (3.28) and (3.29), we get

$$W_{M,a}^{\Omega,b,p} \subset W_{M,a}^{\Omega,b}. \tag{3.30}$$

Similarly it is easy to show that

$$W_{M,a}^{\Omega,b} \subset W_{M,a}^{\Omega,b,p}. \tag{3.31}$$

Finally, (3.30) and (3.31) imply that

$$W_{M,a}^{\Omega,b,p} = W_{M,a}^{\Omega,b}. \tag{3.32}$$

□

4 Uniqueness class of a Cauchy problem

In this section we apply the theory of the fractional Fourier transform which is discussed in (1.4) and (1.6) to establish a uniqueness theorem for the Cauchy problem:

$$\frac{\partial u(x,t)}{\partial t} = P(i\triangle_x)u(x,t), \quad \forall (x,t) \in \mathbb{R}^n \times [0,T], \tag{4.1}$$

$$u(x,0) = u_0(x), \tag{4.2}$$

where

$$\triangle_x^k = \triangle_{x_1}^{k_1} \cdots \triangle_{x_n}^{k_n} \tag{4.3}$$

$$= \left(\frac{\partial}{\partial x_1} - ix_1 \cot\alpha\right)^{k_1} \cdots \left(\frac{\partial}{\partial x_n} - ix_n \cot\alpha\right)^{k_n} \tag{4.4}$$

is a differential operator and $u(x,t)$ is an $N \times 1$ column vector. Here P is an $N \times N$ polynomial matrix with constant coefficients of order k. A similar problem has been investigated by Gel'fand and Shilov [2], and Friedman [3] by exploiting the theory of Fourier transforms. Also, Pathak [10] studied the uniqueness of the Cauchy problem by using the theory of the Hankel transform.

Theorem 4.1 *The Cauchy problem* (4.1) *and* (4.2) *possesses a unique solution $u(x,t)$ in the space $(W_{M_0,\frac{1}{b-\theta}}^{\Omega_0,\frac{1}{a-\theta}})'$ for the interval $0 \leq t \leq T$, $T < (2cp_0)^{-1}(d/2)^{p_0}$, $\theta < a$, and for any initial function $u_0(x)$ belonging to the same space, where p_0 is the reduced order of the system* (4.1) *and* (4.2) *with $i\triangle_x$ replaced by $i\frac{\partial}{\partial x}$ and c being a constant depending on P.*

Proof From the fundamental result [3, p.177], the Cauchy problem (4.1) and (4.2) will have a solution in the space Φ_1' for $0 \leq t \leq T$ if there exists a solution of the adjoint problem,

$$\frac{\partial}{\partial t}\phi(x,t) = \tilde{P}\big(i\triangle_x^*\big)\phi(x,t), \tag{4.5}$$

$$\phi(x,t_0) = \phi_0(x) \in \Phi, \tag{4.6}$$

in the space Φ_1 for $0 \leq t \leq t_0$, where t_0 is any point in the interval $0 \leq t \leq T$, $\tilde{P}$ is the adjoint of P and $\triangle_x^*$ is the conjugate of $\triangle_x$.

Applying the fractional Fourier transform to (4.5) and (4.6), we get

$$\frac{d}{dt}\Psi_\alpha(\sigma,t) = \tilde{P}(\sigma \csc\alpha)\Psi(\sigma,t), \tag{4.7}$$

$$\Psi_\alpha(\sigma,t_0) = \Psi_{\alpha,0}(\sigma), \tag{4.8}$$

where $\Psi_\alpha(\sigma,t) = (F_\alpha\phi)(x,t)$. A formal solution of (4.7) and (4.8) is given by

$$\Psi_\alpha(\sigma,t) = \exp\big[(t-t_0)\tilde{P}(\sigma \csc\alpha)\big]\Psi_{\alpha,0}(\sigma). \tag{4.9}$$

Let us write

$$Q(\sigma \csc\alpha, t_0, t) = \exp\big[(t-t_0)\tilde{P}(\sigma \csc\alpha)\big], \tag{4.10}$$

consisting of entire analytic functions of σ where $\sigma = w + i\tau$. Since p_0 is the reduced order of the system (4.1) and (4.2), using the inequality

$$|\sigma \csc\alpha|^{p_0} \le 2^{p_0}\left(|w \csc\alpha|^{p_0} + |\tau \csc\alpha|^{p_0}\right)$$

and the arguments of [2, p.53] in (4.10) we obtain

$$\left|Q(\sigma \csc\alpha, t_0, t)\right| \le C \exp\left[(p_0)^{-1}\theta^{p_0}\left(|w \csc\alpha|^{p_0} + |\tau \csc\alpha|^{p_0}\right)\right]$$

under the assumptions $t_0 \le t \le t_0 + T$ and $2^{p_0+1}cT < (p_0)^{-1}\theta^{p_0}$.

If we set

$$M(w \csc\alpha) = |w \csc\alpha|^{p_0}/p_0, \qquad \Omega(\tau \csc\alpha) = |\tau \csc\alpha|^{p_0}/p_0,$$

then

$$\left|Q(\sigma \csc\alpha, t_0, t)\right| \le C \exp\left[M(\theta \cdot w \csc\alpha) + \Omega(\theta \cdot \tau \csc\alpha)\right].$$

Now, let us assume that

$$\phi_0(x) \in \Phi = W^{\Omega_0, \frac{1}{a}}_{M_0, \frac{1}{b}}.$$

Then

$$\psi_{\alpha,0}(\sigma) = (F_\alpha \phi_0)(x) \in W^{\Omega,b}_{M,a}.$$

We now apply the theorem [2, p.54] for given a. One can always choose the time interval $0 \le t \le T$ so small that the inequality $\theta < a$ holds; for such values of T the matrix $Q(\sigma \csc\alpha, t_0, t)$ will be a multiplier in the space $W^{\Omega,b}_{M,a}$ which maps this space into the space $W^{\Omega,b+\theta}_{M,a-\theta}$ taking T sufficiently small. Thus the Cauchy problem (4.7) and (4.8) has a unique solution in $W^{\Omega,b+\theta}_{M,a-\theta}$. Also we can show that

$$F^{-1}_\alpha\left[W^{\Omega,b+\theta}_{M,a-\theta}\right] = \Phi_1 = W^{\Omega_0, \frac{1}{a-\theta}}_{M_0, \frac{1}{b+\theta}},$$

and the Cauchy problem (4.5) and (4.6) has a unique solution in $W^{\Omega_0, \frac{1}{a-\theta}}_{M_0, \frac{1}{b+\theta}}$.

Now using the arguments of [3, Theorem 6, p.177], we get the complete proof. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The main idea of this paper was proposed by SKU. AK prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

Acknowledgements

The first author is thankful to DST-CIMS, Banaras Hindu University, Varanasi, India for providing the research facilities and the second author is also thankful to DST-CIMS, Banaras Hindu University, Varanasi, India for awarding the Junior Research Fellowship since December 2012.

Received: 25 November 2014 Accepted: 23 December 2014 Published online: 28 January 2015

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