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Integral inequalities of Hermite-Hadamard type for functions whose derivatives are α-preinvex

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Abstract
In the article, the authors introduce a new notion, 'α-preinvex function', establish an integral identity for the newly introduced function, and find some Hermite-Hadamard type integral inequalities for a function of which the power of the absolute value of the first derivative is α-preinvex.
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1 Introduction

Let us recall some definitions of various convex functions.

Definition 1 A function $f : I \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$ is said to be convex if

$$f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda)f(y) \tag{1}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 2 ([1]) For $f : [0,b] \to \mathbb{R}$ and $m \in (0,1]$, if

$$f\big(tx + m(1-t)y\big) \le tf(x) + m(1-t)f(y) \tag{2}$$

is valid for all $x, y \in [0,b]$ and $t \in [0,1]$, then we say that $f(x)$ is an m-convex function on $[0,b]$.

Definition 3 ([2]) For $f : [0,b] \to \mathbb{R}$ and $(\alpha, m) \in (0,1] \times (0,1]$, if

$$f\big(tx + m(1-t)y\big) \le t^\alpha f(x) + m\big(1-t^\alpha\big)f(y) \tag{3}$$

is valid for all $x, y \in [0,b]$ and $t \in [0,1]$, then we say that $f(x)$ is an (α, m)-convex function on $[0,b]$.

Definition 4 ([3–5]) A set $S \subseteq \mathbb{R}^n$ is said to be invex with respect to the map $\eta : S \times S \to \mathbb{R}^n$ if for every $x, y \in S$ and $t \in [0,1]$

$$y + t\eta(x,y) \in S. \tag{4}$$

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Definition 5 ([6]) Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. For every $x, y \in S$, the η-path P_{xv} joining the points x and $v = x + \eta(y, x)$ is defined by

$$P_{xv} = \{z \mid z = x + t\eta(y, x), t \in [0, 1]\}. \tag{5}$$

Definition 6 ([4]) Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. A function $f : S \to \mathbb{R}$ is said to be preinvex with respect to η, if for every $x, y \in S$ and $t \in [0, 1]$,

$$f(y + t\eta(x, y)) \le tf(x) + (1 - t)f(y). \tag{6}$$

Let us reformulate some inequalities of Hermite-Hadamard type for the above mentioned convex functions.

Theorem 1 ([7, Theorem 2.2]) *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping and $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b - a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b - a}{8}\left(\left|f'(a)\right| + \left|f'(b)\right|\right). \tag{7}$$

Theorem 2 ([8, 9]) *Let $f : \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ be m-convex and $m \in (0, 1]$. If $f \in L([a, b])$ for $0 \le a < b < \infty$, then*

$$\frac{1}{b - a}\int_a^b f(x)\,\mathrm{d}x \le \min\left\{\frac{f(a) + mf(b/m)}{2}, \frac{mf(a/m) + f(b)}{2}\right\}. \tag{8}$$

Theorem 3 ([10, Theorem 3.1]) *Let $I \supset \mathbb{R}_0$ be an open real interval and let $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L([a, b])$ for $0 \le a < b < \infty$. If $|f'|^q$ is (α, m)-convex on $[a, b]$ for some given numbers $m, \alpha \in (0, 1]$ and $q \ge 1$, then*

$$\begin{aligned}
&\left|\frac{f(a) + f(b)}{2} - \frac{1}{b - a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\quad \le \frac{b - a}{2}\left(\frac{1}{2}\right)^{1-1/q} \min\left\{\left[v_1\left|f'(a)\right|^q + v_2 m\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q},\right. \\
&\qquad \left.\left[v_2 m\left|f'\left(\frac{a}{m}\right)\right|^q + v_1\left|f'(b)\right|^q\right]^{1/q}\right\},
\end{aligned}$$

where

$$v_1 = \frac{1}{(\alpha + 1)(\alpha + 2)}\left(\alpha + \frac{1}{2^\alpha}\right) \quad \textit{and} \quad v_2 = \frac{1}{(\alpha + 1)(\alpha + 2)}\left(\frac{\alpha^2 + \alpha + 2}{2} - \frac{1}{2^\alpha}\right).$$

Theorem 4 ([4, Theorem 2.1]) *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta : A \times A \to \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a differentiable function. If $|f'|$ is preinvex on A, then for every $a, b \in A$ with $\theta(a, b) \ne 0$ we have*

$$\left|\frac{f(b) + f(b + \theta(a, b))}{2} - \frac{1}{\theta(a, b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}x\right| \le \frac{|\theta(a, b)|}{8}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]. \tag{9}$$

For more information on Hermite-Hadamard type inequalities for various convex functions, please refer to recently published articles [11–21] and closely related references therein.

In this article, we will introduce a new notion 'α-preinvex function', establish an integral identity for such a kind of functions, and find some Hermite-Hadamard type integral inequalities for a function that the power of the absolute value of its first derivative is α-preinvex.

2 A new definition and a lemma

The so-called 'α-preinvex function' may be introduced as follows.

Definition 7 Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. A function $f : S \to \mathbb{R}$ is said to be α-preinvex with respect to η for $\alpha \in (0,1]$, if for every $x, y \in S$ and $t \in [0,1]$,

$$f\big(y + t\eta(x,y)\big) \le t^\alpha f(x) + \big(1 - t^\alpha\big)f(y). \tag{10}$$

Remark 1 If $\alpha = 1$ and $f(x)$ is an α-preinvex function, then $f(x)$ is a preinvex function.

For establishing our new integral inequalities of Hermite-Hadamard type for α-preinvex functions, we need the following integral identity.

Lemma 1 *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta : A \times A \to \mathbb{R}$ and let $a, b \in A$ with $\theta(a,b) \ne 0$. If $f : A \to \mathbb{R}$ is a differentiable function and f' is integrable on the θ-path P_{bc}: $c = b + \theta(a,b)$, then*

$$\begin{aligned}&\frac{1}{2}\left[\frac{f(b) + f(b+\theta(a,b))}{2} + f\left(\frac{2b+\theta(a,b)}{2}\right)\right] - \frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}x \\ &\quad = \frac{\theta(a,b)}{4}\int_0^1\left(\frac{1}{2} - t\right)\left[f'\left(b + \frac{1-t}{2}\theta(a,b)\right) + f'\left(b + \frac{2-t}{2}\theta(a,b)\right)\right]\mathrm{d}t.\end{aligned}$$

Proof Since $a, b \in A$ and A is an invex set with respect to θ, for every $t \in [0,1]$, we have $b + t\theta(a,b) \in A$. Integrating by parts gives

$$\begin{aligned}&\int_0^1\left(\frac{1}{2} - t\right)\left[f'\left(b + \frac{1-t}{2}\theta(a,b)\right) + f'\left(b + \frac{2-t}{2}\theta(a,b)\right)\right]\mathrm{d}t \\ &\quad = -\frac{2}{\theta(a,b)}\left[\left(\frac{1}{2} - t\right)f\left(b + \frac{1-t}{2}\theta(a,b)\right)\Big|_0^1 + \int_0^1 f\left(b + \frac{1-t}{2}\theta(a,b)\right)\mathrm{d}t\right. \\ &\qquad \left. + \left(\frac{1}{2} - t\right)f\left(b + \frac{2-t}{2}\theta(a,b)\right)\Big|_0^1 + \int_0^1 f\left(b + \frac{2-t}{2}\theta(a,b)\right)\mathrm{d}t\right] \\ &\quad = \frac{2}{\theta(a,b)}\left[\frac{1}{2}f(b) + \frac{1}{2}f\left(\frac{2b+\theta(a,b)}{2}\right) - \int_0^1 f\left(b + \frac{1-t}{2}\theta(a,b)\right)\mathrm{d}t\right] \\ &\qquad + \frac{2}{\theta(a,b)}\left[\frac{1}{2}f\left(\frac{2b+\theta(a,b)}{2}\right) + \frac{1}{2}f\big(b+\theta(a,b)\big) - \int_0^1 f\left(b + \frac{2-t}{2}\theta(a,b)\right)\mathrm{d}t\right] \\ &\quad = \frac{2}{\theta(a,b)}\left[\frac{f(b) + f(b+\theta(a,b))}{2} + f\left(\frac{2b+\theta(a,b)}{2}\right)\right] - \frac{4}{\theta^2(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}x.\end{aligned}$$

The proof of Lemma 1 is completed. □

3 Some new integral inequalities of Hermite-Hadamard type

We are now in a position to establish some Hermite-Hadamard type integral inequalities for a function that the power of the absolute value of its first derivative is α-preinvex.

Theorem 5 *Let $A \subseteq \mathbb{R}$ be an invex subset with respect to $\theta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\theta(a,b) \neq 0$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function, f' is integrable on the θ-path P_{bc}: $c = b + \theta(a,b)$, and $\alpha \in (0,1]$. If $|f'|^q$ is α-preinvex on A for $q \geq 1$, then*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(b)+f(b+\theta(a,b))}{2}+f\left(\frac{2b+\theta(a,b)}{2}\right)\right]-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)\,\mathrm{d}x\right| \\
&\quad\leq \frac{|\theta(a,b)|}{16}\left[\frac{1}{(\alpha+1)(\alpha+2)2^{2\alpha}}\right]^{1/q}\Big\{\big[2\big(1+\alpha 2^{\alpha}\big)\big|f'(a)\big|^q \\
&\qquad+\big(2^{2\alpha+1}-2+\alpha\big(3\times 2^{\alpha}-2\big)2^{\alpha}+\alpha^2 2^{2\alpha}\big)\big|f'(b)\big|^q\big]^{1/q} \\
&\qquad+\big[\big(\alpha\big(2^{\alpha+1}-1\big)2^{\alpha+1}+2\times 3^{\alpha+2}-\big(2^{\alpha}+1\big)2^{\alpha+3}\big)\big|f'(a)\big|^q \\
&\qquad+\big(\alpha^2 2^{2\alpha}+\big(\alpha\big(1-2^{\alpha-1}\big)+4+5\times 2^{\alpha}\big)2^{\alpha+1}-2\times 3^{\alpha+2}\big)\big|f'(b)\big|^q\big]^{1/q}\Big\}.
\end{aligned}$$

Proof Since $b + t\theta(a,b) \in A$ for every $t \in [0,1]$, by Lemma 1 and Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(b)+f(b+\theta(a,b))}{2}+f\left(\frac{2b+\theta(a,b)}{2}\right)\right]-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)\,\mathrm{d}x\right| \\
&\quad\leq \frac{|\theta(a,b)|}{4}\int_0^1\left|\frac{1}{2}-t\right|\left[\left|f'\left(b+\frac{1-t}{2}\theta(a,b)\right)\right|+\left|f'\left(b+\frac{2-t}{2}\theta(a,b)\right)\right|\right]\mathrm{d}t \\
&\quad\leq \frac{|\theta(a,b)|}{4}\left(\int_0^1\left|\frac{1}{2}-t\right|\mathrm{d}t\right)^{1-1/q}\left\{\left[\int_0^1\left|\frac{1}{2}-t\right|\left|f'\left(b+\frac{1-t}{2}\theta(a,b)\right)\right|^q\mathrm{d}t\right]^{1/q}\right. \\
&\qquad\left.+\left[\int_0^1\left|\frac{1}{2}-t\right|\left|f'\left(b+\frac{2-t}{2}\theta(a,b)\right)\right|^q\mathrm{d}t\right]^{1/q}\right\}. \qquad (11)
\end{aligned}$$

Using the α-preinvexity of $|f'|^q$, we have

$$\begin{aligned}
&\int_0^1\left|\frac{1}{2}-t\right|\left|f'\left(b+\frac{1-t}{2}\theta(a,b)\right)\right|^q\mathrm{d}t \\
&\quad\leq \int_0^1\left|\frac{1}{2}-t\right|\left[\left(\frac{1-t}{2}\right)^{\alpha}\big|f'(a)\big|^q+\left(1-\left(\frac{1-t}{2}\right)^{\alpha}\right)\big|f'(b)\big|^q\right]\mathrm{d}t \\
&\quad=\frac{1}{(\alpha+1)(\alpha+2)2^{2(\alpha+1)}}\big[2\big(1+\alpha 2^{\alpha}\big)\big|f'(a)\big|^q \\
&\qquad+\big(2^{2\alpha+1}-2+\alpha\big(3\times 2^{\alpha}-2\big)2^{\alpha}+\alpha^2 2^{2\alpha}\big)\big|f'(b)\big|^q\big]
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\left|\frac{1}{2}-t\right|\left|f'\left(b+\frac{2-t}{2}\theta(a,b)\right)\right|^q\mathrm{d}t \\
&\quad\leq \int_0^1\left|\frac{1}{2}-t\right|\left[\left(\frac{2-t}{2}\right)^{\alpha}\big|f'(a)\big|^q+\left(1-\left(\frac{2-t}{2}\right)^{\alpha}\right)\big|f'(b)\big|^q\right]\mathrm{d}t
\end{aligned}$$

$$= \frac{1}{(\alpha+1)(\alpha+2)2^{2(\alpha+1)}}\big[\big(\alpha\big(2^{\alpha+1}-1\big)2^{\alpha+1}+2\times 3^{\alpha+2}-\big(2^{\alpha}+1\big)2^{\alpha+3}\big) \times \big|f'(a)\big|^q+\big(\alpha^2 2^{2\alpha}+\big(\alpha\big(1-2^{\alpha-1}\big)+4+5\times 2^{\alpha}\big)2^{\alpha+1}-2\times 3^{\alpha+2}\big)\big|f'(b)\big|^q\big].$$

Substituting the above two inequalities into (11) yields

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(b)+f(b+\theta(a,b))}{2}+f\left(\frac{2b+\theta(a,b)}{2}\right)\right]-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)\,\mathrm{d}x\right| \\ &\quad\le \frac{|\theta(a,b)|}{4}\left(\int_0^1\left|\frac{1}{2}-t\right|\mathrm{d}t\right)^{1-1/q}\left\{\left[\int_0^1\left|\frac{1}{2}-t\right|\left(\left(\frac{1-t}{2}\right)^{\alpha}\big|f'(a)\big|^q\right.\right.\right. \\ &\qquad\left.\left.+\left(1-\left(\frac{1-t}{2}\right)^{\alpha}\right)\big|f'(b)\big|^q\right)\mathrm{d}t\right]^{1/q} \\ &\qquad\left.+\left[\int_0^1\left|\frac{1}{2}-t\right|\left(\left(\frac{2-t}{2}\right)^{\alpha}\big|f'(a)\big|^q+\left(1-\left(\frac{2-t}{2}\right)^{\alpha}\right)\big|f'(b)\big|^q\right)\mathrm{d}t\right]^{1/q}\right\} \\ &\quad= \frac{|\theta(a,b)|}{4}\left(\frac{1}{4}\right)^{1-1/q}\left[\frac{1}{(\alpha+1)(\alpha+2)2^{2(\alpha+1)}}\right]^{1/q}\big\{\big[2\big(1+\alpha 2^{\alpha}\big)\big|f'(a)\big|^q \\ &\qquad+\big(2^{2\alpha+1}-2+\alpha\big(3\times 2^{\alpha}-2\big)2^{\alpha}+\alpha^2 2^{2\alpha}\big)\big|f'(b)\big|^q\big]^{1/q} \\ &\qquad+\big[\big(\alpha\big(2^{\alpha+1}-1\big)2^{\alpha+1}+2\times 3^{\alpha+2}-\big(2^{\alpha}+1\big)2^{\alpha+3}\big)\big|f'(a)\big|^q \\ &\qquad+\big(\alpha^2 2^{2\alpha}+\big(\alpha\big(1-2^{\alpha-1}\big)+4+5\times 2^{\alpha}\big)2^{\alpha+1}-2\times 3^{\alpha+2}\big)\big|f'(b)\big|^q\big]^{1/q}\big\}.\end{aligned}$$

The proof of Theorem 5 is completed. □

Corollary 1 *Under the conditions of Theorem* 5, *if* $\alpha = 1$, *we have*

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(b)+f(b+\theta(a,b))}{2}+f\left(\frac{2b+\theta(a,b)}{2}\right)\right]-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)\,\mathrm{d}x\right| \\ &\quad\le \frac{|\theta(a,b)|}{16}\left\{\left[\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right]^{1/q}+\left[\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right]^{1/q}\right\}.\end{aligned}$$

Theorem 6 *Let* $A \subseteq \mathbb{R}$ *be an invex subset with respect to* $\theta : A \times A \to \mathbb{R}$ *and* $a, b \in A$ *with* $\theta(a,b) \neq 0$. *Suppose that* $f : A \to \mathbb{R}$ *is a differentiable function*, f' *is integrable on the* θ*-path* P_{bc}: $c = b + \theta(a,b)$, *and* $\alpha \in (0,1]$. *If* $|f'|^q$ *is* α*-preinvex on* A *for* $q > 1$, *then*

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(b)+f(b+\theta(a,b))}{2}+f\left(\frac{2b+\theta(a,b)}{2}\right)\right]-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)\,\mathrm{d}x\right| \\ &\quad\le \frac{|\theta(a,b)|}{8}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left[\frac{1}{(\alpha+1)2^{\alpha}}\right]^{1/q}\big\{\big[\big|f'(a)\big|^q+\big((\alpha+1)2^{\alpha}-1\big)\big|f'(b)\big|^q\big]^{1/q} \\ &\qquad+\big[\big(2^{\alpha+1}-1\big)\big|f'(a)\big|^q+\big(1-(1-\alpha)2^{\alpha}\big)\big|f'(b)\big|^q\big]^{1/q}\big\}.\end{aligned}$$

Proof Since $b + t\theta(a,b) \in A$ for every $t \in [0,1]$, by Lemma 1, Hölder's inequality, and the α-preinvexity of $|f'|^q$, we have

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(b)+f(b+\theta(a,b))}{2}+f\left(\frac{2b+\theta(a,b)}{2}\right)\right]-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)\,\mathrm{d}x\right| \\ &\quad\le \frac{|\theta(a,b)|}{4}\int_0^1\left|\frac{1}{2}-t\right|\left[\left|f'\left(b+\frac{1-t}{2}\theta(a,b)\right)\right|+\left|f'\left(b+\frac{2-t}{2}\theta(a,b)\right)\right|\right]\mathrm{d}t\end{aligned}$$

$$\leq \frac{|\theta(a,b)|}{4}\left(\int_0^1\left|\frac{1}{2}-t\right|^{q/(q-1)}\,\mathrm{d}t\right)^{1-1/q}\left\{\left[\int_0^1\left|f'\left(b+\frac{1-t}{2}\theta(a,b)\right)\right|^q\,\mathrm{d}t\right]^{1/q}\right.$$
$$\left.+\left[\int_0^1\left|f'\left(b+\frac{2-t}{2}\theta(a,b)\right)\right|^q\,\mathrm{d}t\right]^{1/q}\right\}$$
$$\leq \frac{|\theta(a,b)|}{4}\left(\int_0^1\left|\frac{1}{2}-t\right|^{q/(q-1)}\,\mathrm{d}t\right)^{1-1/q}$$
$$\times\left\{\left[\int_0^1\left(\left(\frac{1-t}{2}\right)^\alpha\left|f'(a)\right|^q+\left(1-\left(\frac{1-t}{2}\right)^\alpha\right)\left|f'(b)\right|^q\right)\mathrm{d}t\right]^{1/q}\right.$$
$$\left.+\left[\int_0^1\left(\left(\frac{2-t}{2}\right)^\alpha\left|f'(a)\right|^q+\left(1-\left(\frac{2-t}{2}\right)^\alpha\right)\left|f'(b)\right|^q\right)\mathrm{d}t\right]^{1/q}\right\}$$
$$=\frac{|\theta(a,b)|}{8}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left[\frac{1}{(\alpha+1)2^\alpha}\right]^{1/q}\left\{\left[\left|f'(a)\right|^q+\left((\alpha+1)2^\alpha-1\right)\left|f'(b)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[\left(2^{\alpha+1}-1\right)\left|f'(a)\right|^q+\left(1-(1-\alpha)2^\alpha\right)\left|f'(b)\right|^q\right]^{1/q}\right\}.$$

The proof of Theorem 6 is complete. □

Corollary 2 *Under the conditions of Theorem* 6, *if* $\alpha = 1$, *we have*

$$\left|\frac{1}{2}\left[\frac{f(b)+f(b+\theta(a,b))}{2}+f\left(\frac{2b+\theta(a,b)}{2}\right)\right]-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)\,\mathrm{d}x\right|$$
$$\leq \frac{|\theta(a,b)|}{8}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left\{\left[\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right]^{1/q}+\left[\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right]^{1/q}\right\}.$$

Theorem 7 *Let* $A \subseteq \mathbb{R}$ *be an invex subset with respect to* $\theta : A\times A \to \mathbb{R}$ *and* $a,b \in A$ *with* $\theta(a,b) \neq 0$. *Suppose that* $f : A \to \mathbb{R}$ *is a differentiable function*, f' *is integrable on the* θ-*path* P_{bc}: $c = b+\theta(a,b)$, *and* $\alpha \in (0,1]$. *If* $|f'|^q$ *is* α-*preinvex on* A *for* $q > 1$ *and* $q \geq r > 0$, *then*

$$\left|\frac{1}{2}\left[\frac{f(b)+f(b+\theta(a,b))}{2}+f\left(\frac{2b+\theta(a,b)}{2}\right)\right]-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)\,\mathrm{d}x\right|$$
$$\leq \frac{|\theta(a,b)|}{4}\left(\frac{q-1}{2q-r-1}\right)^{1-1/q}\left(\frac{1}{2}\right)^{(q-r)/q}$$
$$\times\left\{\left[\left(\frac{1}{(2r+1)2^{2r+1}}+\frac{1}{(2\alpha+1)2^{2\alpha+1}}\right)\left|f'(a)\right|^q+\frac{3}{(r+1)2^{r+2}}\left|f'(b)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[\left(\frac{1}{(2r+1)2^{2r+1}}+\frac{2^{2\alpha+1}-1}{(2\alpha+1)2^{2\alpha+1}}\right)\left|f'(a)\right|^q+\frac{1}{(r+1)2^{r+2}}\left|f'(b)\right|^q\right]^{1/q}\right\}. \tag{12}$$

Proof Since $b+t\theta(a,b) \in A$ for every $t \in [0,1]$, by Lemma 1, Hölder's inequality, and the α-preinvexity of $|f'|^q$, we have

$$\left|\frac{1}{2}\left[\frac{f(b)+f(b+\theta(a,b))}{2}+f\left(\frac{2b+\theta(a,b)}{2}\right)\right]-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)\,\mathrm{d}x\right|$$
$$\leq \frac{|\theta(a,b)|}{4}\int_0^1\left|\frac{1}{2}-t\right|\left[\left|f'\left(b+\frac{1-t}{2}\theta(a,b)\right)\right|+\left|f'\left(b+\frac{2-t}{2}\theta(a,b)\right)\right|\right]\mathrm{d}t$$

$$\begin{aligned}
&\leq \frac{|\theta(a,b)|}{4}\left(\int_0^1\left|\frac{1}{2}-t\right|^{(q-r)/(q-1)}\mathrm{d}t\right)^{1-1/q}\left\{\left[\int_0^1\left|\frac{1}{2}-t\right|^r\left|f'\left(b+\frac{1-t}{2}\theta(a,b)\right)\right|^q\mathrm{d}t\right]^{1/q}\right.\\
&\quad\left.+\left[\int_0^1\left|\frac{1}{2}-t\right|^r\left|f'\left(b+\frac{2-t}{2}\theta(a,b)\right)\right|^q\mathrm{d}t\right]^{1/q}\right\}\\
&\leq \frac{|\theta(a,b)|}{4}\left(\frac{q-1}{2q-r-1}\right)^{1-1/q}\left(\frac{1}{2}\right)^{(q-r)/q}\\
&\quad\times\left\{\left[\int_0^1\left|\frac{1}{2}-t\right|^r\left(\left(\frac{1-t}{2}\right)^\alpha|f'(a)|^q+\left(1-\left(\frac{1-t}{2}\right)^\alpha\right)|f'(b)|^q\right)\mathrm{d}t\right]^{1/q}\right.\\
&\quad\left.+\left[\int_0^1\left|\frac{1}{2}-t\right|^r\left(\left(\frac{2-t}{2}\right)^\alpha|f'(a)|^q+\left(1-\left(\frac{2-t}{2}\right)^\alpha\right)|f'(b)|^q\right)\mathrm{d}t\right]^{1/q}\right\}. \qquad (13)
\end{aligned}$$

Since $x^r y^\alpha \leq \frac{x^{2r}+y^{2\alpha}}{2}$ and $z \leq z^\alpha$ for $x, y \geq 0$ and $0 \leq z \leq 1$, we obtain

$$\begin{aligned}
&\int_0^1\left|\frac{1}{2}-t\right|^r\left[\left(\frac{1-t}{2}\right)^\alpha|f'(a)|^q+\left(1-\left(\frac{1-t}{2}\right)^\alpha\right)|f'(b)|^q\right]\mathrm{d}t\\
&\quad\leq \int_0^1\left[\frac{1}{2}\left(\left|\frac{1}{2}-t\right|^{2r}+\left(\frac{1-t}{2}\right)^{2\alpha}\right)|f'(a)|^q+\left(\frac{1+t}{2}\right)\left|\frac{1}{2}-t\right|^r|f'(b)|^q\right]\mathrm{d}t\\
&\quad=\left[\frac{1}{(2r+1)2^{2r+1}}+\frac{1}{(2\alpha+1)2^{2\alpha+1}}\right]|f'(a)|^q+\frac{3}{(r+1)2^{r+2}}|f'(b)|^q \qquad (14)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\left|\frac{1}{2}-t\right|^r\left[\left(\frac{2-t}{2}\right)^\alpha|f'(a)|^q+\left(1-\left(\frac{2-t}{2}\right)^\alpha\right)|f'(b)|^q\right]\mathrm{d}t\\
&\quad\leq \int_0^1\left[\frac{1}{2}\left(\left|\frac{1}{2}-t\right|^{2r}+\left(\frac{2-t}{2}\right)^{2\alpha}\right)|f'(a)|^q+\frac{t}{2}\left|\frac{1}{2}-t\right|^r|f'(b)|^q\right]\mathrm{d}t\\
&\quad=\left[\frac{1}{(2r+1)2^{2r+1}}+\frac{2^{2\alpha+1}-1}{(2\alpha+1)2^{2\alpha+1}}\right]|f'(a)|^q+\frac{1}{(r+1)2^{r+2}}|f'(b)|^q. \qquad (15)
\end{aligned}$$

Substituting (14) and (15) into (13) results in (12). The proof of Theorem 7 is complete. □

Corollary 3 *Under the conditions of Theorem* 7, *if* $\alpha = 1$, *we have*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(b)+f(b+\theta(a,b))}{2}+f\left(\frac{2b+\theta(a,b)}{2}\right)\right]-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)\,\mathrm{d}x\right|\\
&\quad\leq \frac{|\theta(a,b)|}{4}\left(\frac{q-1}{2q-r-1}\right)^{1-1/q}\left(\frac{1}{2}\right)^{(q-r)/q}\\
&\quad\times\left\{\left[\left(\frac{1}{(2r+1)2^{2r+1}}+\frac{1}{24}\right)|f'(a)|^q+\frac{3}{(r+1)2^{r+2}}|f'(b)|^q\right]^{1/q}\right.\\
&\quad\left.+\left[\left(\frac{1}{(2r+1)2^{2r+1}}+\frac{7}{24}\right)|f'(a)|^q+\frac{|f'(b)|^q}{(r+1)2^{r+2}}\right]^{1/q}\right\}.
\end{aligned}$$

Theorem 8 *Let* $A \subseteq \mathbb{R}$ *be an invex subset with respect to* $\theta : A \times A \to \mathbb{R}$ *and* $a, b \in A$ *with* $\theta(a,b) \neq 0$. *Suppose that* $f : A \to \mathbb{R}$ *is a differentiable function and* f' *is integrable on the*

θ-path P_{bc}: $c = b + \theta(a,b)$. If $|f'|^q$ is preinvex on A for $q > 1$ and $q \geq r > 0$, then

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(b)+f(b+\theta(a,b))}{2}+f\left(\frac{2b+\theta(a,b)}{2}\right)\right]-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)\,\mathrm{d}x\right|\\&\quad\leq\frac{|\theta(a,b)|}{8}\left(\frac{q-1}{2q-r-1}\right)^{1-1/q}\left(\frac{1}{r+1}\right)^{1/q}\\&\qquad\times\left\{\left[\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right]^{1/q}+\left[\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right]^{1/q}\right\}.\end{aligned}$$

Proof Since $b + t\theta(a,b) \in A$ for every $t \in [0,1]$, by Lemma 1, Hölder's inequality, and the preinvexity of $|f'|^q$, we have

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(b)+f(b+\theta(a,b))}{2}+f\left(\frac{2b+\theta(a,b)}{2}\right)\right]-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)\,\mathrm{d}x\right|\\&\quad\leq\frac{|\theta(a,b)|}{4}\int_0^1\left|\frac{1}{2}-t\right|\left[\left|f'\left(b+\frac{1-t}{2}\theta(a,b)\right)\right|+\left|f'\left(b+\frac{2-t}{2}\theta(a,b)\right)\right|\right]\mathrm{d}t\\&\quad\leq\frac{|\theta(a,b)|}{4}\left(\int_0^1\left|\frac{1}{2}-t\right|^{(q-r)/(q-1)}\mathrm{d}t\right)^{1-1/q}\\&\qquad\times\left\{\left[\int_0^1\left|\frac{1}{2}-t\right|^r\left|f'\left(b+\frac{1-t}{2}\theta(a,b)\right)\right|^q\mathrm{d}t\right]^{1/q}\right.\\&\qquad\left.+\left[\int_0^1\left|\frac{1}{2}-t\right|^r\left|f'\left(b+\frac{2-t}{2}\theta(a,b)\right)\right|^q\mathrm{d}t\right]^{1/q}\right\}\\&\quad\leq\frac{|\theta(a,b)|}{4}\left(\frac{q-1}{2q-r-1}\right)^{1-1/q}\left(\frac{1}{2}\right)^{(q-r)/q}\\&\qquad\times\left\{\left[\int_0^1\left|\frac{1}{2}-t\right|^r\left(\frac{1-t}{2}\left|f'(a)\right|^q+\frac{1+t}{2}\left|f'(b)\right|^q\right)\mathrm{d}t\right]^{1/q}\right.\\&\qquad\left.+\left[\int_0^1\left|\frac{1}{2}-t\right|^r\left(\frac{2-t}{2}\left|f'(a)\right|^q+\frac{t}{2}\left|f'(b)\right|^q\right)\mathrm{d}t\right]^{1/q}\right\}\\&\quad=\frac{|\theta(a,b)|}{4}\left(\frac{q-1}{2q-r-1}\right)^{1-1/q}\left(\frac{1}{2}\right)^{(q-r)/q}\\&\qquad\times\left\{\left[\frac{1}{(r+1)2^{r+2}}\left|f'(a)\right|^q+\frac{3}{(r+1)2^{r+2}}\left|f'(b)\right|^q\right]^{1/q}\right.\\&\qquad\left.+\left[\frac{3}{(r+1)2^{r+2}}\left|f'(a)\right|^q+\frac{1}{(r+1)2^{r+2}}\left|f'(b)\right|^q\right]^{1/q}\right\}.\end{aligned}$$

The proof of Theorem 8 is complete. □

Corollary 4 *Under the conditions of Theorem* 8, *if $r = q$, we have*

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(b)+f(b+\theta(a,b))}{2}+f\left(\frac{2b+\theta(a,b)}{2}\right)\right]-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)\,\mathrm{d}x\right|\\&\quad\leq\frac{|\theta(a,b)|}{8}\left(\frac{1}{q+1}\right)^{1/q}\left\{\left[\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right]^{1/q}+\left[\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right]^{1/q}\right\}.\end{aligned}$$

Theorem 9 *Let $A \subseteq \mathbb{R}$ be an invex subset with respect to $\theta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\theta(a,b) \neq 0$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function, f' is integrable on the θ-path P_{bc}: $c = b + \theta(a,b)$, and $\alpha \in (0,1]$. If f is α-preinvex on A, then*

$$\frac{1}{\theta(a,b)} \int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}x \le \min\left\{\frac{f(a) + \alpha f(b)}{\alpha + 1}, \frac{\alpha f(a) + f(b)}{\alpha + 1}\right\}. \tag{16}$$

Proof Since $b + t\theta(a,b) \in A$ for $0 \le t \le 1$, letting $x = (1-t)b + t(b + \theta(a,b)) = b + t\theta(a,b)$ for $0 \le t \le 1$ and using the α-preinvexity of f, we have

$$\begin{aligned}\frac{1}{\theta(a,b)} \int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}x &= \int_0^1 f\bigl(b + t\theta(a,b)\bigr)\,\mathrm{d}t \\ &\le \int_0^1 \bigl[t^\alpha f(a) + \bigl(1 - t^\alpha\bigr) f(b)\bigr]\,\mathrm{d}t = \frac{f(a) + \alpha f(b)}{\alpha + 1}.\end{aligned}$$

The proof of Theorem 9 is complete. □

Corollary 5 *Under the conditions of Theorem* 9, *if* $\alpha = 1$, *we have*

$$\frac{1}{\theta(a,b)} \int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}x \le \frac{f(a) + f(b)}{2}.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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