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Another type of Mann iterative scheme for two mappings in a complete geodesic space

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Abstract

In this paper, we show a Δ-convergence theorem for a Mann iteration procedure in a complete geodesic space with two quasinonexpansive and Δ-demiclosed mappings. The proposed method is different from known procedures with respect to the order of taking the convex combination.

1 Introduction

The fixed point approximation has been studied in a variety of ways and its results are useful for the other studies. In 1953, Mann [1] introduced an iteration procedure for approximating fixed points of a nonexpansive mapping T in a Hilbert space. Later, Reich [2] discussed this iteration procedure in a uniformly convex Banach space whose norm is Fréchet differentiable. In 1998, Takahashi and Tamura [3] considered an iteration procedure with two nonexpansive mappings and obtained weak convergence theorems for this procedure in a uniformly convex Banach space which satisfies Opial's condition or whose norm is Fréchet differentiable. On the other hand, in 2008, Dhompongsa and Panyanak [4] proved the following theorem.

Theorem 1.1 *Let C be a bounded closed convex subset of a complete* CAT(0) *space and $T : C \to C$ a nonexpansive mapping. For any initial point x_0 in C, define the Mann iterative sequence $\{x_n\}$ by*

$$x_{n+1} = (1 - t_n)x_n \oplus t_n Tx_n, \quad n = 0, 1, 2, \dots,$$

where $\{t_n\}$ is a sequence in $[0,1]$, with the restrictions that $\sum_{n=0}^{\infty} t_n$ diverges and $\limsup_{n\to\infty} t_n < 1$. Then $\{x_n\}$ Δ-converges to a fixed point of T.

Further, in a CAT(1) space, Kimura *et al.* [5] proved the Δ-convergence theorem for a family of nonexpansive mappings including the following scheme:

$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n\bigl((1 - \beta_n)Sx_n \oplus \beta_n Tx_n\bigr).$$

In a Hilbert space H, the following equality holds for any $x, y, z \in H$:

$$\alpha x + (1 - \alpha)\bigl(\beta y + (1 - \beta)z\bigr) = \gamma\bigl(\delta x + (1 - \delta)y\bigr) + (1 - \gamma)z,$$

where $\alpha, \beta, \gamma, \delta \in]0,1[$ such that $\alpha = \gamma\delta$ and $\beta = \gamma(1-\delta)/(1-\gamma\delta)$. However, in CAT($\kappa$) spaces with $\kappa > 0$, it does not hold in general, that is, the value of the convex combination taken twice depends on their order. Thus, the following formulas are different in general:

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n\big((1-\beta_n)Sx_n \oplus \beta_n Tx_n\big),$$
$$x_{n+1} = (1-\alpha_n)\big(\beta_n x_n \oplus (1-\beta_n)Sx_n\big) \oplus \alpha_n\big((1-\beta_n)x_n \oplus \beta_n Tx_n\big). \qquad (1)$$

In this paper, we show an analogous result to Theorem 1.1 using the iterative scheme (1) in a complete CAT(1) space with two quasinonexpansive and Δ-demiclosed mappings. We also deal with the image recovery problem for two closed convex sets.

2 Preliminaries

Let X be a metric space. For $x, y \in X$, a mapping $c : [0,l] \to X$ is said to be a geodesic if c satisfies $c(0) = x$, $c(l) = y$ and $d(c(s), c(t)) = |s-t|$ for all $s, t \in [0,l]$. An image $[x,y]$ of c is called a geodesic segment joining x and y. For $r > 0$, X is said to be an r-geodesic metric space if, for any $x, y \in X$ with $d(x,y) < r$, there exists a geodesic segment $[x,y]$. In particular, if a segment $[x,y]$ is unique for any $x, y \in X$ with $d(x,y) < r$, then X is said to be a uniquely r-geodesic metric space. In what follows, we always assume $d(x,y) < \pi/2$ for any $x, y \in X$. Thus, we say X is a geodesic metric space instead of a $\pi/2$-geodesic metric space. For the more general case, see [6].

Let X be a uniquely geodesic metric space. A geodesic triangle is defined by $\triangle(x,y,z) = [x,y] \cup [y,z] \cup [z,x]$. Let M be the two-dimensional unit sphere in $\mathbb{R}^3$. For $\bar{x}, \bar{y}, \bar{z} \in M$, a triangle $\triangle(\bar{x}, \bar{y}, \bar{z}) \subset M$ is called a comparison triangle of $\triangle(x,y,z)$ if $d(x,y) = d_M(\bar{x},\bar{y})$, $d(y,z) = d_M(\bar{y},\bar{z})$, $d(z,x) = d_M(\bar{z},\bar{x})$. Further, for any $x, y \in X$ and $t \in]0,1[$, if $z \in [x,y]$ satisfies $d(x,z) = (1-t)d(x,y)$ and $d(z,y) = td(x,y)$, then z is denoted by $z = tx \oplus (1-t)y$. A point $\bar{z} \in [\bar{x},\bar{y}]$ is called a comparison point of $z \in [x,y]$ if $d(x,z) = d_M(\bar{x},\bar{z})$. X is said to be a CAT(1) space if, for any $p, q \in \triangle(x,y,z) \subset X$ and its comparison points $\bar{p}, \bar{q} \in \triangle(\bar{x},\bar{y},\bar{z}) \subset M$, the inequality $d(p,q) \le d_M(\bar{p},\bar{q})$ holds.

Let X be a geodesic metric space and $\{x_n\}$ a bounded sequence of X. For $x \in X$, we put $r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n)$. The asymptotic radius of $\{x_n\}$ is defined by $r(\{x_n\}) = \inf_{x\in X} r(x,\{x_n\})$. Further, the asymptotic center of $\{x_n\}$ is defined by $AC(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}$. If, for any subsequences $\{x_{n_k}\}$ of $\{x_n\}$, $AC(\{x_{n_k}\}) = \{x_0\}$, *i.e.*, their asymptotic center consists of the unique element x_0, then we say $\{x_n\}$ Δ-converges to x_0 and we denote it by $x_n \overset{\Delta}{\rightharpoonup} x_0$.

Let X be a metric space. A mapping $T : X \to X$ is said to be a nonexpansive if T satisfies $d(Tx,Ty) \le d(x,y)$ for any $x, y \in X$. The set of fixed points of T is denoted by $F(T) = \{z \in X : Tz = z\}$. Further, a mapping $T : X \to X$ with $F(T) \neq \emptyset$ is said to be a quasinonexpansive if T satisfies $d(Tx,z) \le d(x,z)$ for any $x \in X$ and $z \in F(T)$. Moreover, T is said to be Δ-demiclosed if, for any bounded sequence $\{x_n\} \subset X$ and $x_0 \in X$ satisfying $d(x_n, Tx_n) \to 0$ and $x_n \overset{\Delta}{\rightharpoonup} x_0$, we have $x_0 \in F(T)$.

3 Tools for the main results

In this section, we introduce some tools for using the main theorem.

Theorem 3.1 (Kimura and Satô [7]) *Let $\triangle(x,y,z)$ be a geodesic triangle in a* CAT(1) *space such that $d(x,y)+d(y,z)+d(z,x)<2\pi$. Let $u=tx\oplus(1-t)y$ for some $t\in[0,1]$. Then*

$$\cos d(u,z)\sin d(x,y)\ge\cos d(x,z)\sin td(x,y)+\cos d(y,z)\sin(1-t)d(x,y).$$

Corollary 3.2 (Kimura and Satô [8]) *Let $\triangle(x,y,z)$ be a geodesic triangle in a* CAT(1) *space such that $d(x,y)+d(y,z)+d(z,x)<2\pi$. Let $u=tx\oplus(1-t)y$ for some $t\in[0,1]$. Then*

$$\cos d(u,z)\ge t\cos d(x,z)+(1-t)\cos d(y,z).$$

Theorem 3.3 (Espínola and Fernández-León [9]) *Let X be a complete* CAT(1) *space and $\{x_n\}$ a sequence in X. If $r(\{x_n\})<\pi/2$, then the following hold.*

(i) *$AC(\{x_n\})$ consists of exactly one point;*

(ii) *$\{x_n\}$ has a Δ-convergent subsequence.*

Theorem 3.4 (Kimura and Satô [8]) *Let X be a metric space and T a mapping from X into itself. If T is a nonexpansive with $F(T)\neq\emptyset$, then T is quasinonexpansive and Δ-demiclosed.*

The following lemmas are important properties of real numbers and they are easy to show. Thus, we omit the proofs.

Lemma 3.5 *Let δ be a real number such that $-1<\delta<0$ and $\{b_n\}$, $\{c_n\}$ real sequences satisfying $\delta\le b_n\le 1$, $\delta\le c_n\le 1$ and $\liminf_{n\to\infty}b_nc_n\ge 1$. Then $\lim_{n\to\infty}b_n=\lim_{n\to\infty}c_n=1$.*

Lemma 3.6 *Let $s\in\,]0,\infty[$ and $\{b_n\}$, $\{c_n\}$ bounded real sequences satisfying $b_n\le 0$, $s<c_n$ and $\lim_{n\to\infty}b_n/c_n=0$. Then $\lim_{n\to\infty}b_n=0$.*

Lemma 3.7 *Let $\{b_n\}$ and $\{c_n\}$ be bounded real sequences satisfying $\lim_{n\to\infty}(b_n-c_n)=0$. Then $\liminf_{n\to\infty}b_n=\liminf_{n\to\infty}c_n$.*

4 The main result

In this section, we show the main result.

Theorem 4.1 *Let X be a complete* CAT(1) *space such that for any $u,v\in X$, $d(u,v)<\pi/2$. Let S and T be quasinonexpansive and Δ-demiclosed mappings from X into itself with $F(S)\cap F(T)\neq\emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences of $[a,b]\subset\,]0,1[$. Define a sequence $\{x_n\}\subset X$ by the following recurrence formula: $x_1\in X$ and*

$$\begin{cases} u_n=(1-\beta_n)x_n\oplus\beta_nSx_n,\\ v_n=(1-\gamma_n)x_n\oplus\gamma_nTx_n,\\ x_{n+1}=(1-\alpha_n)u_n\oplus\alpha_nv_n\end{cases}$$

for $n\in\mathbb{N}$. Then $\{x_n\}$ Δ-converges to a common fixed point of S and T.

Proof Let $z \in F(S) \cap F(T)$. By Corollary 3.2, we have

$$\begin{aligned}\cos d(u_n,z) &\geq (1-\beta_n)\cos d(x_n,z) + \beta_n \cos d(Sx_n,z)\\ &\geq (1-\beta_n)\cos d(x_n,z) + \beta_n \cos d(x_n,z)\\ &= \cos d(x_n,z),\\ \cos d(v_n,z) &\geq (1-\gamma_n)\cos d(x_n,z) + \gamma_n \cos d(Tx_n,z)\\ &\geq (1-\gamma_n)\cos d(x_n,z) + \gamma_n \cos d(x_n,z)\\ &= \cos d(x_n,z).\end{aligned}$$

Then, by Corollary 3.2 again, we have

$$\begin{aligned}\cos d(x_{n+1},z) &\geq (1-\alpha_n)\cos d(u_n,z) + \alpha_n \cos d(v_n,z)\\ &\geq (1-\alpha_n)\cos d(x_n,z) + \alpha_n \cos d(x_n,z)\\ &\geq \cos d(x_n,z).\end{aligned}$$

So, we get $d(x_{n+1},z) \leq d(x_n,z)$ for all $n \in \mathbb{N}$ and there exists $d_0 = \lim_{n\to\infty} d(x_n,z) \leq d(x_1,z) < \pi/2$.

Furthermore, by Theorem 3.1, we have

$$\begin{aligned}&\cos d(u_n,z)\sin d(x_n,Sx_n)\\ &\quad\geq \cos d(x_n,z)\sin(1-\beta_n)d(x_n,Sx_n) + \cos d(Sx_n,z)\sin \beta_n d(x_n,Sx_n)\\ &\quad\geq 2\cos d(x_n,z)\sin\frac{d(x_n,Sx_n)}{2}\cos\frac{(1-2\beta_n)d(x_n,Sx_n)}{2}\end{aligned}\tag{2}$$

and

$$\begin{aligned}&\cos d(v_n,z)\sin d(x_n,Tx_n)\\ &\quad\geq \cos d(x_n,z)\sin(1-\gamma_n)d(x_n,Tx_n) + \cos d(Tx_n,z)\sin \gamma_n d(x_n,Tx_n)\\ &\quad\geq 2\cos d(x_n,z)\sin\frac{d(x_n,Tx_n)}{2}\cos\frac{(1-2\gamma_n)d(x_n,Tx_n)}{2}.\end{aligned}\tag{3}$$

Let $d_n = d(x_n,z)$, $s_n = d(x_n,Sx_n)/2$ and $t_n = d(x_n,Tx_n)/2$ for $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $s_{n_0} = t_{n_0} = 0$, then we have $x_{n_0} \in F(S) \cap F(T)$ and since

$$\begin{aligned}x_{n_0+1} &= (1-\alpha_{n_0})\big((1-\beta_{n_0})x_{n_0} \oplus \beta_{n_0} Sx_{n_0}\big) \oplus \alpha_{n_0}\big((1-\gamma_{n_0})x_{n_0} \oplus \gamma_{n_0} Tx_{n_0}\big)\\ &= (1-\alpha_{n_0})x_{n_0} \oplus \alpha_{n_0}x_{n_0}\\ &= x_{n_0},\end{aligned}$$

and the proof is finished. So, we may assume $s_n \neq 0$ or $t_n \neq 0$ for all $n \in \mathbb{N}$.

If $s_n = 0$ and $t_n \neq 0$, then we have $u_n = x_n$. From (2), (3), and Corollary 3.2, we get

$$\begin{aligned}&2\cos d_{n+1}\sin t_n \cos t_n\\ &\quad= \cos d_{n+1}\sin 2t_n\end{aligned}$$

$$\begin{aligned}&\geq (1-\alpha_n)\cos d(u_n,z)\sin 2t_n + \alpha_n \cos d(v_n,z)\sin 2t_n \\ &\geq 2(1-\alpha_n)\cos d_n \sin t_n \cos t_n + 2\alpha_n \cos d_n \sin t_n \cos(1-2\gamma_n)t_n.\end{aligned}$$

Dividing by $2\sin t_n > 0$, we get

$$\cos d_{n+1}\cos t_n \geq (1-\alpha_n)\cos d_n \cos t_n + \alpha_n \cos d_n \cos(1-2\gamma_n)t_n. \tag{4}$$

If $t_n = 0$ and $s_n \neq 0$, then we have $v_n = x_n$. In a similar way as above, we get

$$\cos d_{n+1}\cos s_n \geq (1-\alpha_n)\cos d_n \cos(1-2\beta_n)s_n + \alpha_n \cos d_n \cos s_n. \tag{5}$$

If $s_n \neq 0$ and $t_n \neq 0$, then from (2), (3), and Corollary 3.2, we get

$$\begin{aligned}&\cos d_{n+1}\sin 2s_n \sin 2t_n \\ &\quad\geq (1-\alpha_n)\cos d(u_n,z)\sin 2s_n \sin 2t_n + \alpha_n \cos d(v_n,z)\sin 2s_n \sin 2t_n \\ &\quad\geq 4\cos d_n \sin s_n \sin t_n \bigl((1-\alpha_n)\cos t_n \cos(1-2\beta_n)s_n + \alpha_n \cos s_n \cos(1-2\gamma_n)t_n\bigr).\end{aligned}$$

Dividing by $4\sin s_n \sin t_n > 0$, we get

$$\begin{aligned}&\cos d_{n+1}\cos s_n \cos t_n \\ &\quad\geq (1-\alpha_n)\cos d_n \cos t_n \cos(1-2\beta_n)s_n + \alpha_n \cos d_n \cos s_n \cos(1-2\gamma_n)t_n.\end{aligned} \tag{6}$$

Therefore, (4) and (5) can be reduced to the inequality (6) and it is equivalent to

$$\left(\frac{\varepsilon_n \cos s_n}{\alpha_n \cos(1-2\beta_n)s_n} - \frac{1-\alpha_n}{\alpha_n}\right)\left(\frac{\varepsilon_n \cos t_n}{(1-\alpha_n)\cos(1-2\gamma_n)t_n} - \frac{\alpha_n}{1-\alpha_n}\right) \geq 1,$$

where $\varepsilon_n = \cos d_{n+1}/\cos d_n$ for $n \in \mathbb{N}$. It follows that $\lim_{n\to\infty}\varepsilon_n = \cos d_0/\cos d_0 = 1$. Since $\{\alpha_n\} \subset [a,b] \subset]0,1[$ for $n \in \mathbb{N}$, we get

$$\liminf_{n\to\infty}\left(\frac{\cos s_n}{\alpha_n \cos(1-2\beta_n)s_n} - \frac{1-\alpha_n}{\alpha_n}\right)\left(\frac{\cos t_n}{(1-\alpha_n)\cos(1-2\gamma_n)t_n} - \frac{\alpha_n}{1-\alpha_n}\right) \geq 1. \tag{7}$$

Then we show that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the following hold:

$$-\frac{1}{2} \leq \frac{\cos s_n}{\alpha_n \cos(1-2\beta_n)s_n} - \frac{1-\alpha_n}{\alpha_n} \leq 1 \tag{8}$$

and

$$-\frac{1}{2} \leq \frac{\cos t_n}{(1-\alpha_n)\cos(1-2\gamma_n)t_n} - \frac{\alpha_n}{1-\alpha_n} \leq 1. \tag{9}$$

First, we show the right inequality of (8). Since $\{\beta_n\} \subset [a,b] \subset]0,1[$ for $n \in \mathbb{N}$, we get $\cos s_n \leq \cos|1-2\beta_n|s_n = \cos(1-2\beta_n)s_n$. Hence we get

$$\frac{\cos s_n}{\alpha_n \cos(1-2\beta_n)s_n} - \frac{1-\alpha_n}{\alpha_n} \leq \frac{1}{\alpha_n} - \frac{1-\alpha_n}{\alpha_n} = 1.$$

By the same method as above, the right inequality of (9) also holds. Next, let us show the left inequality of (8). If it does not hold, then letting

$$\sigma_n = \frac{\cos s_n}{\alpha_n \cos(1-2\beta_n)s_n} - \frac{1-\alpha_n}{\alpha_n} \quad \text{and} \quad \tau_n = \frac{\cos t_n}{(1-\alpha_n)\cos(1-2\gamma_n)t_n} - \frac{\alpha_n}{1-\alpha_n},$$

we can find a subsequence $\{\sigma_{n_i}\} \subset \{\sigma_n\}$ such that $\sigma_{n_i} < -1/2$ for $i \in \mathbb{N}$ and $\lim_{i\to\infty} \sigma_{n_i} = \sigma \leq -1/2$. Since $\{\alpha_n\}, \{\gamma_n\} \subset [a,b] \subset]0,1[$ and $\{t_n\} \subset [0,\pi/4[\subset [0,\pi/2[$, we have $\{\tau_n\}$ is bounded. Therefore, by taking a subsequence again if necessary, we may assume that $\{\tau_{n_i}\}$ converges to $\tau \in \mathbb{R}$. Then, by (7), we get $\sigma\tau = \lim_{i\to\infty} \sigma_{n_i}\tau_{n_i} \geq \liminf_{n\to\infty} \sigma_n\tau_n \geq 1$. Hence we may assume that $\tau_{n_i} < 0$ for all $i \in \mathbb{N}$. Since $\sqrt{2}/2 < \cos s_n$, $\sqrt{2}/2 < \cos t_n$, $0 < \cos(1-2\beta_n)s_n \leq 1$, $0 < \cos(1-2\gamma_n)t_n \leq 1$ and $\{\alpha_n\} \subset [a,b] \subset]0,1[$, we also have

$$0 < \frac{\sqrt{2}}{2b} \leq \frac{\cos s_n}{\alpha_n \cos(1-2\beta_n)s_n} \quad \text{and} \quad 0 < \frac{\sqrt{2}}{2(1-a)} \leq \frac{\cos t_n}{(1-\alpha_n)\cos(1-2\gamma_n)t_n}. \tag{10}$$

Let ρ be a real number such that

$$0 < \rho < \min\left\{\frac{\sqrt{2}}{2b}, \frac{\sqrt{2}}{2(1-a)}, \frac{1-b}{b} + \frac{a}{1-a}\right\}. \tag{11}$$

Then, by (10), we get

$$\rho - \frac{1-\alpha_{n_i}}{\alpha_{n_i}} \leq \sigma_{n_i} < 0 \quad \text{and} \quad \rho - \frac{\alpha_{n_i}}{1-\alpha_{n_i}} \leq \tau_{n_i} < 0. \tag{12}$$

Then, by (11) and (12), we have

$$\begin{aligned}
\sigma_{n_i}\tau_{n_i} &\leq \left(\rho - \frac{1-\alpha_{n_i}}{\alpha_{n_i}}\right)\left(\rho - \frac{\alpha_{n_i}}{1-\alpha_{n_i}}\right) \\
&= \rho^2 - \left(\frac{1-\alpha_{n_i}}{\alpha_{n_i}} + \frac{\alpha_{n_i}}{1-\alpha_{n_i}}\right)\rho + 1 \\
&\leq \rho^2 - \left(\frac{1-b}{b} + \frac{a}{1-a}\right)\rho + 1 \\
&= \rho\left(\rho - \left(\frac{1-b}{b} + \frac{a}{1-a}\right)\right) + 1.
\end{aligned}$$

Thus, as $i \to \infty$, we have

$$1 \leq \sigma\tau \leq \rho\left(\rho - \left(\frac{1-b}{b} + \frac{a}{1-a}\right)\right) + 1 < 1. \tag{13}$$

This is a contradiction. We also obtain the left inequality of (9) in a similar way. Hence we get

$$\lim_{n\to\infty}\left(\frac{\cos s_n}{\alpha_n \cos(1-2\beta_n)s_n} - \frac{1-\alpha_n}{\alpha_n}\right) = \lim_{n\to\infty}\left(\frac{\cos t_n}{(1-\alpha_n)\cos(1-2\gamma_n)t_n} - \frac{\alpha_n}{1-\alpha_n}\right) = 1 \tag{14}$$

by Lemma 3.5, (8), and (9). Furthermore, from (14), we get

$$\lim_{n\to\infty} \frac{\cos s_n - \cos(1-2\beta_n)s_n}{\alpha_n \cos(1-2\beta_n)s_n} = 0. \tag{15}$$

By Lemma 3.6 and (15), we get

$$\lim_{n\to\infty}\bigl(\cos s_n - \cos(1-2\beta_n)s_n\bigr) = 0. \tag{16}$$

Moreover, by Lemma 3.7 and (16), we get

$$\liminf_{n\to\infty}\cos s_n = \liminf_{n\to\infty}\cos(1-2\beta_n)s_n = \liminf_{n\to\infty}\cos|1-2\beta_n|s_n.$$

Hence we get

$$\limsup_{n\to\infty} s_n = \limsup_{n\to\infty}\bigl(|1-2\beta_n|s_n\bigr) \le \limsup_{n\to\infty}|1-2\beta_n| \limsup_{n\to\infty} s_n,$$

and we have

$$0 \ge \Bigl(1 - \limsup_{n\to\infty}|1-2\beta_n|\Bigr)\limsup_{n\to\infty} s_n = \liminf_{n\to\infty}\bigl(1-|1-2\beta_n|\bigr)\limsup_{n\to\infty} s_n.$$

Since $\{\beta_n\} \subset [a,b] \subset]0,1[$ for $n \in \mathbb{N}$, we have $\liminf_{n\to\infty}(1-|1-2\beta_n|) > 0$ and thus, $\limsup_{n\to\infty} s_n \le 0$. Therefore, we get $\limsup_{n\to\infty} s_n = 0$ and we also get $\limsup_{n\to\infty} t_n = 0$. It implies $d(x_n, Sx_n) \to 0$ and $d(x_n, Tx_n) \to 0$.

Next, let $\{y_k\}$ be a subsequence of $\{x_n\}$. Since $r(\{x_n\}) \le d_0 < \pi/2$, by Theorem 3.3(i), there exists a unique asymptotic center x_0 of $\{y_k\}$. Moreover, since $r(\{y_k\}) < \pi/2$, by Theorem 3.3(ii), there exists a subsequence $\{z_l\}$ of $\{y_k\}$ such that $z_l \overset{\Delta}{\rightharpoonup} z_0 \in X$. Further, since $d(z_l, Sz_l) \to 0$, $d(z_l, Tz_l) \to 0$ and S, T are Δ-demiclosed, we have $z_0 \in F(S) \cap F(T)$. Then we can show that $x_0 = z_0$, *i.e.*, $x_0 \in F(S) \cap F(T)$. If not, from the uniqueness of the asymptotic centers x_0, z_0 of $\{y_k\}$, $\{z_l\}$, respectively, due to Theorem 3.3(i), we have

$$\begin{aligned}
\limsup_{k\to\infty} d(y_k, x_0) &< \limsup_{k\to\infty} d(y_k, z_0) \\
&= \lim_{n\to\infty} d(x_n, z_0) \\
&= \limsup_{l\to\infty} d(z_l, z_0) \\
&< \limsup_{l\to\infty} d(z_l, x_0) \\
&\le \limsup_{k\to\infty} d(y_k, x_0).
\end{aligned}$$

This is a contradiction. Hence we get $x_0 \in F(S) \cap F(T)$. Next, we show that for any subsequences of $\{x_n\}$, their asymptotic center consists of the unique element. Let $\{u_k\}$, $\{v_k\}$ be subsequences of $\{x_n\}$, $x_0 \in AC(\{u_k\})$ and $x_0' \in AC(\{v_k\})$. We show $x_0 = x_0'$ by using contradiction. Assume $x_0 \ne x_0'$. Then $x_0' \notin AC(\{u_k\})$ and $x_0 \notin AC(\{v_k\})$ by Theorem 3.3(i). It follows that

$$\begin{aligned}
\limsup_{k\to\infty} d(u_k, x_0) &< \limsup_{k\to\infty} d\bigl(u_k, x_0'\bigr) \\
&= \lim_{n\to\infty} d\bigl(x_n, x_0'\bigr) \\
&= \limsup_{k\to\infty} d\bigl(v_k, x_0'\bigr)
\end{aligned}$$

$$
\begin{aligned}
&< \limsup_{k\to\infty} d(v_k, x_0) \\
&= \lim_{n\to\infty} d(x_n, x_0) \\
&= \limsup_{k\to\infty} d(u_k, x_0).
\end{aligned}
$$

This is a contradiction. Hence we get $x_0 = x_0'$. Therefore, we have $\{x_n\}$ Δ-converges to a common fixed point of S and T. $\square$

By Theorem 3.4, we know that a nonexpansive mapping having a fixed point satisfies the assumptions in Theorem 4.1. Thus, we get the following result.

Corollary 4.2 *Let X be a complete* CAT(1) *space such that for any $u, v \in X$, $d(u,v) < \pi/2$. Let S and T be nonexpansive mappings of X into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[a,b] \subset]0,1[$. Define a sequence $\{x_n\} \subset X$ as the following recurrence formula: $x_1 \in X$ and*

$$
\begin{cases}
u_n = (1-\beta_n)x_n \oplus \beta_n S x_n, \\
v_n = (1-\gamma_n)x_n \oplus \gamma_n T x_n, \\
x_{n+1} = (1-\alpha_n)u_n \oplus \alpha_n v_n
\end{cases}
$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to a common fixed point of S and T.

5 An application to the image recovery

The image recovery problem is formulated as to find the nearest point in the intersection of family of closed convex subsets from a given point by using corresponding metric projection of each subset. In this section, we consider this problem for two subsets of a complete CAT(1) space.

Theorem 5.1 *Let X be a complete* CAT(1) *space such that for any $u, v \in X$, $d(u,v) < \pi/2$. Let C_1 and C_2 be nonempty closed convex subsets of X such that $C_1 \cap C_2 \neq \emptyset$. Let P_1 and P_2 be metric projections onto C_1 and C_2, respectively. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[a,b] \subset]0,1[$. Define a sequence $\{x_n\} \subset X$ by the following recurrence formula: $x_1 \in X$ and*

$$
\begin{cases}
u_n = (1-\beta_n)x_n \oplus \beta_n P_1 x_n, \\
v_n = (1-\gamma_n)x_n \oplus \gamma_n P_2 x_n, \\
x_{n+1} = (1-\alpha_n)u_n \oplus \alpha_n v_n
\end{cases}
$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to a fixed point of the intersection of C_1 and C_2.

Proof We see that P_1 and P_2 are quasinonexpansive [9] and Δ-demiclosed [8]. Further, we also get $F(P_1) = C_1$ and $F(P_2) = C_2$. Thus, letting $S = P_1$ and $T = P_2$ in Theorem 4.1, we obtain the desired result. $\square$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

The authors have contributed in this work on an equal basis. All authors read and approved the final manuscript.

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