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Fixed point results for generalized (α,ψ)-Meir-Keeler contractive mappings and applications

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Abstract
In this paper, we introduce a new type of a generalized-(α,ψ)-Meir-Keeler contractive mapping and establish some interesting theorems on the existence of fixed points for such mappings via admissible mappings. Applying our results, we derive fixed point theorems in ordinary metric spaces and metric spaces endowed with an arbitrary binary relation.
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1 Introduction

Fixed points and fixed point theorems have countless applications and have become a major theoretical tool in many fields such as differential equations, mathematical economics, game theory, dynamics, optimal control, functional analysis, and operator theory. In particular, the well-known Banach contraction principle is one of the forceful tools in nonlinear analysis, which states that every contraction self-mapping T on complete metric spaces (X,d) (*i.e.*, $d(Tx,Ty)\leq kd(x,y)$ for all $x,y\in X$, where $k\in[0,1)$) has a unique fixed point. Due to its simplicity and importance, this classical principle has been generalized by several authors in different directions.

In 1969 Meir and Keeler [1] established a fixed point theorem in a metric space (X,d) for mappings satisfying the condition that for each $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that

$$\varepsilon\leq d(x,y)<\varepsilon+\delta(\varepsilon)\quad\text{implies}\quad d(Tx,Ty)<\varepsilon \tag{1.1}$$

for all $x,y\in X$. This condition is called the Meir-Keeler contractive type condition. Afterward, many authors extended and improved this condition and established fixed point results for new generalized conditions, see Maiti and Pal [2], Park and Rhoades [3], Mongkolkeha and Kumam [4] and others.

On the other hand, Samet *et al.* [5] introduced the notions of α, ψ contractive and α-admissible mapping in metric spaces. They also proved a fixed point theorem for α, ψ contractive mappings in complete metric spaces using the concept of α-admissible mapping.

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Theorem 1 ([5]) *Let (X, d) be a complete metric space and $T : X \to X$ be (α, ψ)-contractive mapping. Suppose that*

(i) *T is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*;
(iii) *T is continuous.*

Then there exists $u \in X$ such that $Tu = u$.

Theorem 2 ([5]) *Let (X, d) be a complete metric space and $T : X \to X$ be (α, ψ)-contractive mapping. Suppose that*

(i) *T is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*;
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$ then $\alpha(x_n, x) \geq 1$ for all n.*

Then, there exists $u \in X$ such that $Tu = u$.

These results can be used as an efficient tool to study various fixed point results such as fixed point results in partially ordered spaces, fixed point results for cyclic mappings, multidimensional fixed point results (coupled fixed point results, tripled fixed point results, quadrupled fixed point *etc.*). Moreover, such type of fixed point results are helpful to solve several problems and equations like the boundary value problem, differential equations, nonlinear integral equations *etc.* In the recent literature, a wide-ranging discussion of fixed point theorems for admissible mappings had the interest of several mathematicians, for example, see [6–12].

In this paper, we introduce new type of contractive mapping based on Meir-Keeler type contractive condition. For such mappings, we study and establish fixed point theorems via admissible mappings. Moreover, we present some applications of our new results.

2 Preliminaries

In the sequel, $\mathbb{N}$ denote the set of positive integers. Let Ψ stands for the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the nth iterate of ψ.

Remark 3 For every function $\psi : [0, \infty) \to [0, \infty)$ the following holds:

if ψ is nondecreasing, then for each $t > 0$,

$$\lim_{n\to\infty} \psi^n(t) = 0 \quad \Longrightarrow \quad \psi(t) < t \quad \Longrightarrow \quad \psi(0) = 0.$$

Therefore, if $\psi \in \Psi$, then for each $t > 0$, $\psi(t) < t$ and $\psi(0) = 0$.

Example 4 Let $\psi_1, \psi_2 : [0, \infty) \to [0, \infty)$ be defined in the following way:

$$\psi_1 = \frac{1}{2}t \quad \text{and} \quad \psi_2(t) = \begin{cases} \frac{t}{3} & \text{if } 0 \leq t < 1, \\ \frac{t}{5} & \text{if } t \geq 1. \end{cases}$$

It is clear that $\psi_1, \psi_2 \in \Psi$. Notice that ψ_1, ψ_2 are examples of continuous and discontinuous functions in Ψ.

Definition 5 ([5]) Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an (α, ψ)-contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Tx, Ty) \le \psi\big(d(x, y)\big), \tag{2.1}$$

for all $x, y \in X$.

Remark 6 If $T : X \to X$ satisfies the Banach contraction principle in a metric space (X, d), then T is an (α, ψ)-contractive mapping, where $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$ for all $t > 0$, where $k \in [0, 1)$.

Definition 7 ([5]) Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that T is α-admissible when if $x, y \in X$ such that $\alpha(x, y) \ge 1$ then we have $\alpha(Tx, Ty) \ge 1$.

Example 8 Let $X = (0, \infty)$. Define $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Tx = \ln(x + 1)$ for all $x \in X$ and

$$\alpha(x, y) = \begin{cases} e & \text{if } x \ge y, \\ 0 & \text{if } x < y. \end{cases}$$

Then, T is α-admissible.

Example 9 Let $X = [1, \infty)$. Define $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Tx = x^2$ for all $x \in X$ and

$$\alpha(x, y) = \begin{cases} x + y & \text{if } x \ge y, \\ 0 & \text{if } x < y. \end{cases}$$

Then T is α-admissible.

Remark 10 ([5]) Every nondecreasing self-mapping T is α-admissible.

3 Main results

In this section, introducing the class of (α, ψ)-Meir-Keeler contractive mappings and the class of generalized-(α, ψ)-Meir-Keeler contractive mappings, we study the existence of fixed points for mappings via admissible mappings.

Definition 11 Let (X, d) be a metric space and $T : X \to X$. The mapping T is called an (α, ψ)-Meir-Keeler contractive mapping if there exist two functions $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$ satisfying the following condition:

for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \le \psi\big(d(x, y)\big) < \varepsilon + \delta(\varepsilon) \quad \text{implies} \quad \alpha(x, y)d(Tx, Ty) < \varepsilon. \tag{3.1}$$

Remark 12 It is easily shown that if $T : X \to X$ is an (α, ψ)-Meir-Keeler type contraction, then

$$\alpha(x, y)d(Tx, Ty) < \psi\big(d(x, y)\big) \tag{3.2}$$

for all $x, y \in X$ when $x \neq y$. Also, if $x = y$ then $d(Tx, Ty) = 0$ and thus $\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y))$. Therefore, if $T : X \to X$ is a (α, ψ)-Meir-Keeler type contraction, then

$$\alpha(x,y)d(Tx,Ty) \leq \psi\bigl(d(x,y)\bigr) \tag{3.3}$$

for all $x, y \in X$.

Definition 13 Let (X, d) be a metric space and $T : X \to X$. The mapping T is called a generalized-(α, ψ)-Meir-Keeler contractive mapping if there exist two functions $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$ satisfying the following condition:

for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \psi\bigl(M(x,y)\bigr) < \varepsilon + \delta(\varepsilon) \quad \text{implies} \quad \alpha(x,y)d(Tx,Ty) < \varepsilon, \tag{3.4}$$

where

$$M(x,y) = \max\biggl\{d(x,y), d(Tx,x), d(Ty,y), \frac{1}{2}\bigl[d(Tx,y) + d(x,Ty)\bigr]\biggr\}.$$

Remark 14 If $T : X \to X$ is a generalized-(α, ψ)-Meir-Keeler type contraction, then

$$\alpha(x,y)d(Tx,Ty) < \psi\bigl(M(x,y)\bigr) \tag{3.5}$$

for all $x, y \in X$ when $x \neq y$. Also, if $x = y$ then $d(Tx, Ty) = 0$ and thus $\alpha(x,y)d(Tx,Ty) \leq \psi(M(x,y))$. Therefore, if $T : X \to X$ is a generalized-(α, ψ)-Meir-Keeler type contraction, then

$$\alpha(x,y)d(Tx,Ty) \leq \psi\bigl(M(x,y)\bigr) \tag{3.6}$$

for all $x, y \in X$.

Theorem 15 *Let (X, d) be a complete metric space and $T : X \to X$ be a generalized-(α, ψ)-Meir-Keeler contractive mapping. Suppose that*

(i) *T is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*;
(iii) *T is continuous.*

Then there exists $u \in X$ such that $Tu = u$.

Proof Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$. Note that such a point x_0 exists due to condition (ii). We define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \geq 0$. If $x_{n_0} = x_{n_0+1}$ for some n_0, then clearly x_{n_0} is a fixed point of T. Hence, throughout the proof, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since T is α-admissible, we have

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \quad \Rightarrow \quad \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

By induction, we obtain

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \text{for all } n = 0, 1, \ldots. \tag{3.7}$$

From (3.7) and Remark (14), it follows that for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \alpha(x_n, x_{n-1}) d(Tx_n, Tx_{n-1}) \\ &< \psi\big(M(x_n, x_{n-1})\big) \\ &= \psi\bigg(\max\bigg\{d(x_n, x_{n-1}), d(Tx_n, x_n), d(Tx_{n-1}, x_{n-1}), \\ &\quad \frac{1}{2}\big(d(Tx_n, x_{n-1}) + d(Tx_{n-1}, x_n)\big)\bigg\}\bigg) \\ &= \psi\bigg(\max\bigg\{d(x_n, x_{n-1}), d(x_{n+1}, x_n), d(x_n, x_{n-1}), \frac{1}{2} d(x_{n+1}, x_{n-1})\bigg\}\bigg) \\ &\leq \psi\bigg(\max\bigg\{d(x_n, x_{n-1}), d(x_{n+1}, x_n), d(x_n, x_{n-1}), \\ &\quad \frac{1}{2}\big(d(x_{n+1}, x_n) + d(x_n, x_{n-1})\big)\bigg\}\bigg) \\ &\leq \psi\big(\max\big\{d(x_n, x_{n-1}), d(x_{n+1}, x_n)\big\}\big). \end{aligned} \tag{3.8}$$

If $\max\{d(x_n, x_{n-1}), d(x_{n+1}, x_n)\} = d(x_{n+1}, x_n)$, from (3.8) and Remark 3, we have

$$d(x_{n+1}, x_n) < \psi\big(d(x_{n+1}, x_n)\big) < d(x_{n+1}, x_n),$$

which is a contradiction. So we have $\max\{d(x_n, x_{n-1}), d(x_{n+1}, x_n)\} = d(x_n, x_{n-1})$. From (3.8), we get

$$d(x_{n+1}, x_n) < \psi\big(d(x_n, x_{n-1})\big)$$

for all $n \in \mathbb{N}$. Inductively, for each $n \in \mathbb{N}$, we obtain

$$d(x_{n+1}, x_n) < \psi^n\big(d(x_1, x_0)\big). \tag{3.9}$$

Now we show that $\{x_n\}$ is a Cauchy sequence. Take $\varepsilon > 0$ and $N(\varepsilon) \in \mathbb{N}$ in such a way that $\sum_{n \geq N(\varepsilon)} \psi^n(d(x_1, x_0)) \leq \varepsilon$. Let $n, m \in \mathbb{N}$ with $m > n > N(\varepsilon)$. Due to the triangle inequality, we have

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \psi^k\big(d(x_1, x_0)\big) \\ &\leq \sum_{n \geq N(\varepsilon)} \psi^n\big(d(x_1, x_0)\big) < \varepsilon. \end{aligned} \tag{3.10}$$

Hence, we conclude that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d). Thus, there exists $u \in X$ such that $\lim_{n \to \infty} x_n = u$. Since T is continuous,

$$u = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} Tx_n = T\Big(\lim_{n \to \infty} x_n\Big) = Tu,$$

that is, u is a fixed point of T. This completes the proof. □

Corollary 16 *Let (X,d) be a complete metric space and $T : X \to X$ be an (α,ψ)-Meir-Keeler contractive mapping. Suppose that*

(i) *T is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*;
(iii) *T is continuous.*

Then there exists $u \in X$ such that $Tu = u$.

We obtain the following fixed point result without any continuity assumption for the mapping T.

Theorem 17 *Let (X,d) be a complete metric space and $T : X \to X$ be a generalized-(α,ψ)-Meir-Keeler contractive mapping such that ψ is continuous. Suppose that*

(i) *T is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*;
(iii′) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$ then $\alpha(x_n, x) \geq 1$ for all n.*

Then there exists $u \in X$ such that $Tu = u$.

Proof Following the proof of Theorem 15, we obtain the sequence $\{x_n\}$ in X defined by $x_{n+1} = Tx_n$ for all $n \geq 0$ and which converges for some $u \in X$. From (3.7) and condition (iii), we have $\alpha(x_n, u) \geq 1$ for all $n \in \mathbb{N}$. Next, we suppose that $d(u, Tu) \neq 0$. Applying Remark 12, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
d(u, Tu) &\leq d(Tx_n, u) + d(Tx_n, Tu) \\
&\leq d(x_{n+1}, u) + \alpha(x_n, u) d(Tx_n, Tu) \\
&< d(x_{n+1}, u) + \psi\bigl(M(x_n, u)\bigr) \\
&= d(x_{n+1}, u) + \psi\Bigl(\max\Bigl\{d(x_n, u), d(Tx_n, x_n), d(Tu, u), \\
&\quad \frac{1}{2}\bigl[d(Tx_n, u) + d(x_n, Tu)\bigr]\Bigr\}\Bigr) \\
&= d(x_{n+1}, u) + \psi\Bigl(\max\Bigl\{d(x_n, u), d(x_{n+1}, x_n), d(Tu, u), \\
&\quad \frac{1}{2}\bigl[d(x_{n+1}, u) + d(x_n, Tu)\bigr]\Bigr\}\Bigr).
\end{aligned}$$

Letting $n \to \infty$ in the above equality and keeping the continuity of ψ in mind, we get

$$d(u, Tu) \leq \psi\bigl(d(u, Tu)\bigr) < d(u, Tu),$$

which is a contradiction. Thus, we have $d(u, Tu) = 0$, that is, $u = Tu$. Therefore, u is a fixed point of T. This completes the proof. □

In the next corollary, we can omit the continuity hypothesis at every point of ψ.

Corollary 18 *Let (X,d) be a complete metric space and $T : X \to X$ be an (α,ψ)-Meir-Keeler contractive mapping. Suppose that*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$ then $\alpha(x_n, x) \geq 1$ for all n.*

Then there exists $u \in X$ such that $Tu = u$.

Proof Since the proof of the existence of a fixed point is straightforward by following the same lines as in the proof of Theorem 17, in order to avoid repetition, the details are omitted. □

Now, we present the following example in support of our main result.

Example 19 Let $X = \mathbb{R}$ with the usual metric $d(x,y) = |x - y|$. Then $d_2((x,y)(u,v)) = d(x,u) + d(y,v)$ becomes a metric on X^2. Define $T_F : X^2 \to X^2$ as follows: $T_F(x,y) = (F(x,y), F(y,x))$ where $F : X^2 \to X$ is defined by

$$F(x,y) = \begin{cases} \frac{x-5y}{7} & \text{if } x, y \in [0,\infty), \\ 0 & \text{otherwise,} \end{cases}$$

for all $(x,y) \in X^2$. Let us take

$$\alpha\big((x,y),(u,v)\big) = \begin{cases} 1 & \text{if } (x,y),(u,v) \in [0,\infty) \times [0,\infty), \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that T_F is an (α,ψ)-Meir-Keeler contractive mapping with $\psi(t) = \frac{t}{2}$. Indeed, for all $(x,y),(u,v) \in X^2$, we have

$$\varepsilon \leq \psi\big(d_2\big((x,y),(u,v)\big)\big) = \frac{1}{2}|x-u| + \frac{1}{2}|y-v| < \varepsilon + \delta(\varepsilon).$$

On the other hand, by elementary calculations and taking into account that T_F is α-admissible, we get

$$\begin{aligned} d_2\big(T_F\big((x,y)\big), T_F\big((u,v)\big)\big) &= \alpha\big((x,y),(u,v)\big) d_2\big(T_F\big((x,y)\big), T_F\big((u,v)\big)\big) \\ &= \frac{5}{7}|x-u| + \frac{5}{7}|y-v| \\ &\leq \frac{5}{7} 2\big(\varepsilon + \delta(\varepsilon)\big) < \varepsilon, \end{aligned}$$

with $\frac{1}{4}\delta(\varepsilon) < \varepsilon$.

Remark 20 Note that the main theorem of [11] is not applicable to this example. Hence, our result is stronger than the main result of [11]. Indeed, we take $x = u$ in the statement of the mentioned theorem, we get

$$\varepsilon \leq \psi\big(d_2\big((x,y),(u,v)\big)\big) = \frac{1}{2}|y-v| < \varepsilon + \delta(\varepsilon).$$

On the other hand, by elementary calculations, we get

$$d\big(F\big((x,y),F(u,v)\big)\big)=\frac{5}{7}|y-v|.$$

Thus, we get

$$2\varepsilon\leq|y-v|<\frac{7}{5}\varepsilon<2\varepsilon,$$

which is a contradiction.

4 Applications

4.1 Fixed point results on an ordinary metric space

We have the following fixed point results in ordinary metric space.

Theorem 21 *Let (X,d) be a complete metric space and $T:X\to X$ be continuous mapping and there exists $\psi\in\Psi$ satisfying the following condition*:

for each $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that

$$\varepsilon\leq\psi\big(M(x,y)\big)<\varepsilon+\delta(\varepsilon)\quad\textit{implies}\quad d(Tx,Ty)<\varepsilon,\tag{4.1}$$

where

$$M(x,y)=\max\left\{d(x,y),d(Tx,x),d(Ty,y),\frac{1}{2}\big[d(Tx,y)+d(x,Ty)\big]\right\}.$$

Then there exists $u\in X$ such that $Tu=u$.

Proof Consider the mapping $\alpha:X\times X\to[0,\infty)$ defined by

$$\alpha(x,y)=1\quad\text{for all } x,y\in X.$$

From the definition of α, it easy to see that T is α-admissible and also it is a generalized-(α,ψ)-Meir-Keeler contractive mapping. Moreover, all the hypotheses of Theorem 15 (or Theorem 17) are satisfied and so the existence of the fixed point of T follows from Theorem 15 (or Theorem 17). □

Taking $\psi(t)=kt$, where $k\in(0,1)$, we get the following result.

Corollary 22 *Let (X,d) be a complete metric space and $T:X\to X$ be continuous mapping satisfying the following condition*:

for each $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that

$$\varepsilon\leq kM(x,y)<\varepsilon+\delta(\varepsilon)\quad\textit{implies}\quad d(Tx,Ty)<\varepsilon\tag{4.2}$$

for all $x,y\in X$, where $k\in(0,1)$ and

$$M(x,y)=\max\left\{d(x,y),d(Tx,x),d(Ty,y),\frac{1}{2}\big[d(Tx,y)+d(x,Ty)\big]\right\}.$$

Then there exists $u\in X$ such that $Tu=u$.

4.2 Fixed point results on a metric space endowed with an arbitrary binary relation

In this section, we present fixed point theorems on a metric space endowed with an arbitrary binary relation. The following notions and definition are needed.

Let (X,d) be a metric space and $\mathcal{R}$ be a binary relation over X. Denote

$$\mathcal{S} := \mathcal{R} \cup \mathcal{R}^{-1};$$

this is the symmetric relation attached to $\mathcal{R}$. Clearly,

$$x,y \in X, \quad x\mathcal{S}y \quad \Longleftrightarrow \quad x\mathcal{R}y \quad \text{or} \quad y\mathcal{R}x.$$

Definition 23 We say that $T : X \to X$ is a comparative mapping if T maps comparable elements into comparable elements, that is,

$$x\mathcal{S}y \quad \Longrightarrow \quad (Tx)\mathcal{S}(Ty)$$

for all $x,y \in X$.

Definition 24 Let (X,d) be a metric space, $\mathcal{S}$ be a symmetric relation attached to binary relation $\mathcal{R}$ over X and $T : X \to X$. The mapping T is called a generalized-ψ-Meir-Keeler contractive mapping with respect to $\mathcal{S}$ if there exists a function $\psi \in \Psi$ satisfying the following condition:

for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for $x,y \in X$ for which $x\mathcal{S}y$,

$$\varepsilon \leq \psi\bigl(M(x,y)\bigr) < \varepsilon + \delta(\varepsilon) \quad \text{implies} \quad d(Tx,Ty) < \varepsilon, \tag{4.3}$$

where

$$M(x,y) = \max\left\{d(x,y), d(Tx,x), d(Ty,y), \frac{1}{2}\bigl[d(Tx,y) + d(x,Ty)\bigr]\right\}.$$

Theorem 25 *Let (X,d) be a complete metric space, $\mathcal{R}$ be a binary relation over X and $T : X \to X$ be a comparative generalized-(α,ψ)-Meir-Keeler contractive mapping. Suppose that*

(i) *T is comparative mapping;*
(ii) *there exists $x_0 \in X$ such that $x_0\mathcal{S}(Tx_0)$;*
(iii) *T is continuous.*

Then there exists $u \in X$ such that $Tu = u$.

Proof Consider the mapping $\alpha : X \times X \to [0,\infty)$ defined by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x\mathcal{S}y, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

From condition (ii), we get $\alpha(x_0, Tx_0) \geq 1$. It follows from T is comparative mapping that T is an α-admissible mapping. Since T is a generalized-ψ-Meir-Keeler contractive mapping with respect to $\mathcal{S}$, we have, for all $x, y \in X$,

$$\varepsilon \leq \psi\big(M(x,y)\big) < \varepsilon + \delta(\varepsilon) \quad \text{implies} \quad \alpha(x,y)d(Tx,Ty) < \varepsilon. \tag{4.5}$$

This implies that T is a generalized-(α,ψ)-Meir-Keeler contractive mapping. Now all the hypotheses of Theorem 15 are satisfied and so the existence of the fixed point of T follows from Theorem 15. □

In order to remove the continuity of T, we need the following condition:

($\mathcal{W}$) if $\{x_n\}$ is the sequence in X such that $x_n \mathcal{R} x_{n+1}$ for all $n \in \mathbb{N}$ and it converges to the point $x \in X$, then $x_n \mathcal{S} u$.

Theorem 26 *Let (X,d) be a complete metric space, $\mathcal{S}$ be a symmetric relation attached to binary relation $\mathcal{R}$ over X and $T : X \to X$ be a generalized-(α,ψ)-Meir-Keeler contractive mapping with respect to $\mathcal{S}$ such that ψ is continuous. Suppose that*

(i) *T is comparative mapping*;
(ii) *there exists $x_0 \in X$ such that $x_0 \mathcal{S}(Tx_0)$*;
(iii) *the condition $(\mathcal{W})$ holds.*

Then there exists $u \in X$ such that $Tu = u$.

Proof The result follows from Theorem 17 by considering the mapping α given by (4.4) and by observing that condition $(\mathcal{W})$ implies condition (iii′). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors participated in the design of this work and performed equally. All authors read and approved the final manuscript.

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