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Properties for certain subclasses of analytic functions with nonzero coefficients

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Abstract
In the present paper, we obtain some mapping and inclusion properties for subclasses of analytic functions by using a linear operator defined by the Gaussian hypergeometric function.
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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (a_n \neq 0), \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. We also denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

A function $f \in \mathcal{A}$ is said to be in the class $\Re^t(A,B,\alpha)$ if

$$\left|\frac{f'(z) - 1}{t(A-B) - B(f'(z) - 1)}\right| < \alpha \quad (z \in \mathbb{U}), \tag{1.2}$$

where A and B are complex numbers with $A \neq B$, $t \in \mathbb{C} \setminus \{0\}$, and α is a positive real number.

In particular, for some real numbers A and B with $-1 \leq B < A \leq 1$ and $\alpha = 1$ without any restriction of the coefficients a_n ($n \in \mathbb{N} = \{1, 2, \ldots\}$) the class $\Re^t(A,B,\alpha)$ was introduced by Dixit and Pal [1]. Moreover, by giving specific values t, A, B, and α in (1.2), we obtain subclasses studied by various researchers in earlier works (see [2–6]).

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{UST}(\alpha)$ if

$$\Re\left\{\frac{f(z) - f(\xi)}{(z-\xi)f'(z)}\right\} > \alpha \quad (z \times \xi \in \mathbb{U} \times \mathbb{U}; 0 \leq \alpha < 1).$$

Furthermore, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{UCV}(\alpha)$ if

$$\Re\left\{1 + \frac{(z-\xi)f''(z)}{f'(z)}\right\} > \alpha \quad (z \times \xi \in \mathbb{U} \times \mathbb{U}; 0 \leq \alpha < 1).$$

The classes $\mathcal{UST}(0) \equiv \mathcal{UST}$ and $\mathcal{UCV}(0) \equiv \mathcal{UCV}$ are introduced by Goodman [7, 8] (they are called the classes of uniformly starlike and uniformly convex functions, respectively). The classes of uniformly starlike and uniformly convex functions have been extensively studied by Ma and Minda [9] and Rønning [10].

Let us consider the Gaussian hypergeometric function $F(a,b;c;z)$ defined by

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} z^n \quad (a,b,c \in \mathbb{C}; c \neq 0,-1,-2,\ldots; z \in \mathbb{U}),$$

where $(\nu)_n$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the gamma function) by

$$\begin{aligned}(\nu)_n &:= \frac{\Gamma(\nu+n)}{\Gamma(\nu)} \\ &= \begin{cases} 1 & \text{if } n = 0 \text{ and } \nu \in \mathbb{C} \setminus \{0\}, \\ \nu(\nu+1)\cdots(\nu+n-1) & \text{if } n \in \mathbb{N} \text{ and } \nu \in \mathbb{C}. \end{cases}\end{aligned}$$

We note that $F(a,b;c;z) = F(b,a;c;z)$ and

$$F(a,b;c;1) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \quad \bigl(\Re\{c-a-b\} > 0\bigr).$$

We also recall (see [11, 12]) that the function $F(a,b;c;z)$ is bounded if $\Re\{c-a-b\} > 0$, and it has a pole at $z = 1$ if $\Re\{c-a-b\} \leq 0$. Moreover, univalence, starlikeness and convexity properties of $zF(a,b;c;z)$ have been extensively studied by Ponnusamy and Vuorinen [13] and Ruscheweyh and Singh [14].

For $f \in \mathcal{A}$, we define the operator $I_{a,b;c}f$ by

$$I_{a,b;c}f(z) = zF(a,b;c;z) * f(z), \tag{1.3}$$

where $*$ denotes the usual Hadamard product (or convolution) of power series. For a special case of the operator $I_{a,b;c}f$, we can refer to the paper by Swaminathan [15] and the references cited therein.

In this paper, we obtain a necessary condition for the class $\Re^t(A,B,\alpha)$ and sufficient conditions for the classes $\Re^t(A,B,\alpha)$, $\mathcal{UST}(\alpha)$, and $\mathcal{UCV}(\alpha)$, respectively. Moreover, we find a condition for univalency of the operator $I_{a,b;c}f$ defined by (1.3).

2 Main result

Theorem 1 *Let a function f of the form* (1.1) *be in the class* $\Re^t(A,B,\alpha)$ *with* $a_n = |a_n|e^{i\frac{(3n+1)\pi}{2}}$ $(n \in \mathbb{N} \setminus \{1\})$, *then*

$$\sum_{n=2}^{\infty} n\bigl(1-\alpha|B|\bigr)|a_n| \leq \alpha|t||A-B|. \tag{2.1}$$

Proof From the definition of $\Re^t(A,B,\alpha)$, we have

$$\bigl|f'(z)-1\bigr| < \alpha\bigl|t(A-B) - B\bigl(f'(z)-1\bigr)\bigr| \quad (z \in \mathbb{U}),$$

and so

$$\left|\sum_{n=2}^{\infty} n a_n z^{n-1}\right| < \alpha \left| t(A-B) - B \sum_{n=2}^{\infty} n a_n z^{n-1}\right|. \tag{2.2}$$

If we take $z = re^{i\frac{\pi}{2}}$, then we see that

$$a_n z^{n-1} = |a_n| r^{n-1} \quad (0 \leq r < 1). \tag{2.3}$$

Then, by using (2.3) to (2.2), we have

$$\begin{aligned}\sum_{n=2}^{\infty} n|a_n| r^{n-1} &< \alpha \left| t(A-B) - B \sum_{n=2}^{\infty} n|a_n| r^{n-1}\right| \\ &< \alpha \left| t(A-B)\right| + \alpha |B| \sum_{n=2}^{\infty} n|a_n| r^{n-1},\end{aligned}$$

or, equivalently,

$$\sum_{n=2}^{\infty} n\bigl(1 - \alpha|B|\bigr)|a_n| r^{n-1} < \alpha |t||A-B|. \tag{2.4}$$

Letting $r \to 1^-$ in (2.4), we have the inequality (2.1). Thus we complete the proof of Theorem 1. □

Theorem 2 *Let a function f of the form* (1.1) *be in the class $\mathcal{A}$. If*

$$\sum_{n=2}^{\infty} n\bigl(1 + \alpha|B|\bigr)|a_n| \leq \alpha |t||A-B|, \tag{2.5}$$

where A and B are complex numbers with $A \neq B$, $t \in \mathbb{C} \setminus \{0\}$, and α is a positive real number, then $f \in \Re^t(A,B,\alpha)$. The result is sharp for the function defined by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{\alpha t(A-B)\epsilon}{n^2(n-1)(1+\alpha|B|)} z^n$$
$$\bigl(A, B \in \mathbb{C}; A \neq B; t \in \mathbb{C} \setminus \{0\}; |\epsilon| = 1; z \in \mathbb{U}\bigr).$$

Proof In view of the definition of $\Re^t(A,B,\alpha)$, it suffices to prove that

$$\left|f'(z) - 1\right| < \alpha \left| t(A-B) - B\bigl(f'(z) - 1\bigr)\right| \quad (z \in \mathbb{U}). \tag{2.6}$$

From (2.6), we have

$$\left|\sum_{n=2}^{\infty} n a_n z^{n-1}\right| < \alpha \left| t(A-B) - B \sum_{n=2}^{\infty} n a_n z^{n-1}\right|.$$

Hence it is sufficient to show that

$$\sum_{n=2}^{\infty} n|a_n|r^{n-1} < \alpha\left(|t||A-B| - |B|\sum_{n=2}^{\infty} n|a_n|r^{n-1}\right),$$

which is equivalent to the relation

$$\sum_{n=2}^{\infty} n\bigl(1+\alpha|B|\bigr)|a_n|r^{n-1} < \alpha|t||A-B|. \tag{2.7}$$

Letting $r \to 1^-$ in (2.7), we have

$$\sum_{n=2}^{\infty} n\bigl(1+\alpha|B|\bigr)|a_n| \leq \alpha|t||A-B|.$$

Therefore, by the assumption (2.5), we prove that $f \in \Re^t(A,B,\alpha)$. □

Theorem 3 *Let a function f of the form* (1.1) *be in the class $\mathcal{A}$. If*

$$\sum_{n=2}^{\infty} \bigl((3-\alpha)n-2\bigr)|a_n| \leq 1-\alpha \quad (0 \leq \alpha < 1), \tag{2.8}$$

then $f \in \mathcal{UST}(\alpha)$. The result is sharp for the function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{(1-\alpha)\epsilon}{n(n-1)((3-\alpha)n-2)} z^n \quad \bigl(0 \leq \alpha < 1; |\epsilon| = 1\bigr).$$

Proof It suffices to show that

$$\left|\frac{f(z)-f(\xi)}{(z-\xi)f'(z)} - 1\right| \leq 1-\alpha \quad \bigl(0 \leq \alpha < 1; (z,\xi) \in \mathbb{U}\times\mathbb{U}\bigr).$$

We have

$$\begin{aligned}
&\left|\frac{f(z)-f(\xi)}{(z-\xi)f'(z)} - 1\right| \\
&\quad = \left|\frac{\sum_{n=2}^{\infty} a_n(\xi^{n-1} + z\xi^{n-2} + \cdots + z^{n-1}) - \sum_{n=2}^{\infty} na_n z^{n-1}}{1 + \sum_{n=2}^{\infty} na_n z^{n-1}}\right| \\
&\quad \leq \frac{\sum_{n=2}^{\infty} 2(n-1)|a_n|}{1 - \sum_{n=2}^{\infty} n|a_n|},
\end{aligned}$$

which is bounded by $1-\alpha$ if the assumption (2.8) is satisfied. Thus we complete the proof of Theorem 3. □

Theorem 4 *Let a function f of the form* (1.1) *be in the class $\mathcal{A}$. If*

$$\sum_{n=2}^{\infty} n(2n-1-\alpha)|a_n| \leq 1-\alpha \quad (0 \leq \alpha < 1), \tag{2.9}$$

then $f \in \mathcal{UCV}(\alpha)$. *The result is sharp for the function*

$$f(z) = z + \sum_{n=2}^{\infty} \frac{(1-\alpha)\epsilon}{n^2(n-1)(2n-1-\alpha)} z^n \quad \bigl(0 \le \alpha < 1; |\epsilon| = 1; z \in \mathbb{U}\bigr).$$

Proof It suffices to show that

$$\left| \frac{(z-\varphi)f''(z)}{f'(z)} \right| < 1-\alpha \quad \bigl(0 \le \alpha < 1; (z,\varphi) \in \mathbb{U} \times \mathbb{U}\bigr).$$

We have

$$\left| \frac{(z-\varphi)f''(z)}{f'(z)} \right| = \left| \frac{(z-\varphi)\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}}{1-\sum_{n=2}^{\infty} n a_n z^{n-1}} \right| \le \frac{2\sum_{n=1}^{\infty} n(n-1)|a_n|}{1-\sum_{n=2}^{\infty} n|a_n|},$$

which is bounded by $1-\alpha$ if the assumption (2.9) is satisfied. Thus we complete the proof of Theorem 4. □

Theorem 5 *Let* $a, b \in \mathbb{C} \setminus \{0\}$ *and* $c > |a| + |b|$. *If* $f \in \Re^t(A,B,\alpha)$ *with* $a_n = |a_n| e^{i\frac{(3n+1)\pi}{2}}$, $0 < |B| < 1$, *and*

$$\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)} \le \frac{1-\alpha|B|}{1+\alpha|B|} + 1,$$

then $I_{a,b;c}f \in \Re^t(A,B,\alpha)$, *where the operator* $I_{a,b;c}f$ *is defined by* (1.3).

Proof We want to prove from Theorem 2 that

$$T_1 := \sum_{n=2}^{\infty} n\bigl(1+\alpha|B|\bigr)|A_n| \le \alpha|t||A-B|,$$

where

$$A_n = \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n.$$

Since, from Theorem 1,

$$|a_n| \le \frac{\alpha|t||A-B|}{n(1-\alpha|B|)},$$

$$\begin{aligned}
T_1 &\le \frac{\alpha|t||A-B|(1+\alpha|B|)}{1-\alpha|B|} \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&= \frac{\alpha|t||A-B|(1+\alpha|B|)}{1-\alpha|B|} \left(\sum_{n=0}^{\infty} \frac{(|a|)_n(|b|)_n}{(c)_n(1)_n} - 1 \right) \\
&= \frac{\alpha|t||A-B|(1+\alpha|B|)}{1-\alpha|B|} \left(\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)} - 1 \right) \\
&\le \alpha|t||A-B|,
\end{aligned}$$

which completes the proof of Theorem 5. □

We introduce the following lemma which is needed for the proof of the next theorem.

Lemma 1 [16] *Let ω be regular in the unit disk $\mathbb{U}$ with $\omega(0) = 0$. Then, if $|\omega(z)|$ attains a maximum value on the circle $|z| = r$ $(0 \leq r < 1)$ at a point z_0, we can write*

$$z_0\omega'(z_0) = k\omega(z_0) \quad (k \geq 1).$$

Theorem 6 *Let a function f of the form* (1.1) *be in the class $\mathcal{A}$. Assume*

$$\left|\frac{(I_{a,b;c}f(z))' - 1}{1-\alpha}\right|^{\beta}\left|\frac{z(I_{a,b;c}f(z))''}{(I_{a,b;c}f(z))' - \alpha}\right|^{\gamma} < \frac{1}{2^{\gamma}} \quad (z \in \mathbb{U}) \tag{2.10}$$

for some real α $(0 \leq \alpha < 1)$, $\beta > 0$, and $\gamma > 0$. Then

$$\left|\left(I_{a,b;c}f(z)\right)' - 1\right| < 1 - \alpha \quad (z \in \mathbb{U}). \tag{2.11}$$

Proof Let us define ω by

$$\omega(z) = \frac{(I_{a,b;c}f(z))' - 1}{1-\alpha} \quad (z \in \mathbb{U}).$$

Then it follows that ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$. By (2.10), we have

$$\begin{aligned} \left|\omega(z)\right|^{\beta}\left|\frac{z\omega'(z)}{\omega(z)+1}\right|^{\gamma} &= \left|\omega(z)\right|^{\beta+\gamma}\left|\frac{z\omega'(z)}{\omega(z)}\frac{1}{\omega(z)+1}\right|^{\gamma} \\ &< \frac{1}{2^{\gamma}} \quad (z \in \mathbb{U}). \end{aligned} \tag{2.12}$$

Suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|}\left|\omega(z)\right| = \left|\omega(z_0)\right| = 1.$$

Then, by Lemma 1, we can put

$$\frac{z_0\omega'(z_0)}{\omega(z_0)} = k \geq 1.$$

Hence, we obtain

$$\begin{aligned} \left|\omega(z_0)\right|^{\beta}\left|\frac{z_0\omega'(z_0)}{\omega(z_0)+1}\right|^{\gamma} &= \left|\frac{z_0\omega'(z_0)}{\omega(z_0)+1}\right|^{\gamma} \\ &\geq \left(\frac{k}{2}\right)^{\gamma} \geq \frac{1}{2^{\gamma}}, \end{aligned}$$

which contradicts the condition (2.12). This shows that

$$\left|\omega(z)\right| = \left|\frac{(I_{a,b;c}f(z))' - 1}{1-\alpha}\right| < 1 \quad (z \in \mathbb{U}).$$

Thus we complete the proof of Theorem 6. □

Remark 1 The condition (2.11) in Theorem 6 implies that

$$\Re\{(I_{a,b;c}f(z))'\} > 0 \quad (z \in \mathbb{U}).$$

Therefore, by the Noshiro-Warschawski theorem [17], the operator $I_{a,b;c}f$ is univalent in $\mathbb{U}$ under the restrictions of Theorem 6.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

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