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On ϵ-solutions for robust fractional optimization problems

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Abstract

We consider ϵ-solutions (approximate solutions) for a fractional optimization problem in the face of data uncertainty. Using robust optimization approach (worst-case approach), we establish optimality theorems and duality theorems for ϵ-solutions for the fractional optimization problem. Moreover, we give an example illustrating our duality theorems.

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1 Introduction

A robust fractional optimization problem is to optimize an objective fractional function over the constrained set defined by functions with data uncertainty.

To get the ϵ-solution (approximate solution), many authors have established ϵ-optimality conditions and ϵ-duality theorems for several kinds of optimization problems [1–7]. Especially, Lee and Lee [8] gave an ϵ-duality theorems for a convex semidefinite optimization problem with conic constraints. Also, they [9] established optimality theorems and duality theorems for ϵ-solutions for convex optimization problems with uncertainty data.

In [10–15], many authors have treated fractional programming problems in the absence of data uncertainty. Recently, many authors have studied robust optimization problems [9, 16–21]. Very recently, Jeyakumar and Li [22] established duality theorems for a fractional programming problem in the face of data uncertainty via robust optimization.

The purpose of the paper is to extend the ϵ-optimality theorems and ϵ-duality theorems in [9] to fractional optimization problems with uncertainty data.

Consider the following standard form of fractional programming problem with a geometric constraint set:

$$\text{(FP)}\quad \min \frac{f(x)}{g(x)} \quad \text{s.t.}\ h_i(x) \leqq 0, i = 1,\dots,m, \quad x \in C,$$

where $f, h_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1,\dots,m$, are convex functions, C is a closed convex cone of $\mathbb{R}^n$, and $g : \mathbb{R}^n \to \mathbb{R}$ is a concave function such that, for any $x \in C$, $f(x) \geqq 0$ and $g(x) > 0$.

The fractional programming problem (FP) in the face of data uncertainty in the constraints can be captured by the problem:

$$\begin{aligned}
\text{(UFP)} \quad & \min \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(x,u)}{g(x,v)} \\
& \text{s.t. } h_i(x,w_i) \leqq 0, i = 1,\dots,m, \\
& \quad x \in C,
\end{aligned}$$

where $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, $h_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $f(\cdot,u)$ and $h_i(\cdot,w_i)$ are convex, and $g : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, $g(\cdot,v)$ is concave, and $u \in \mathbb{R}^p$, $v \in \mathbb{R}^p$, and $w_i \in \mathbb{R}^q$ are uncertain parameters which belong to the convex and compact uncertainty sets $\mathcal{U} \subset \mathbb{R}^p$, $\mathcal{V} \subset \mathbb{R}^p$, and $\mathcal{W}_i \subset \mathbb{R}^q$, $i = 1,\dots,m$, respectively.

We study ϵ-optimality theorems and ϵ-duality theorems for the uncertain fractional programming model problem (UFP) by examining its robust (worst-case) counterpart [18]:

$$\begin{aligned}
\text{(RFP)} \quad & \min \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(x,u)}{g(x,v)} \\
& \text{s.t. } h_i(x,w_i) \leqq 0, \forall w_i \in \mathcal{W}_i, i = 1,\dots,m, \\
& \quad x \in C.
\end{aligned}$$

Clearly, $A := \{x \in C \mid h_i(x,w_i) \leqq 0, \forall w_i \in \mathcal{W}_i, i = 1,\dots,m\}$ is a feasible set of (RFP).

Let $\epsilon \geqq 0$. Then $\bar{x}$ is called an ϵ-solution of (RFP) if, for any $x \in A$,

$$\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(x,u)}{g(x,v)} \geqq \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon.$$

Using the parametric approach, we transform the problem (RFP) into the robust non-fractional convex optimization problem $(\text{RNCP})_r$ with a parametric $r \in \mathbb{R}_+$:

$$\begin{aligned}
(\text{RNCP})_r \quad & \min \max_{u\in\mathcal{U}} f(x,u) - r \min_{v\in\mathcal{V}} g(x,v) \\
& \text{s.t. } h_i(x,w_i) \leqq 0, \forall w_i \in \mathcal{W}_i, \ i = 1,\dots,m, \\
& \quad x \in C.
\end{aligned}$$

Let $\epsilon \geqq 0$. Then $\bar{x}$ is called an ϵ-solution of $(\text{RNCP})_r$ if, for any $x \in A$,

$$\max_{u\in\mathcal{U}} f(x,u) - r \min_{v\in\mathcal{V}} g(x,v) \geqq \max_{u\in\mathcal{U}} f(\bar{x},u) - r \min_{v\in\mathcal{V}} g(\bar{x},v) - \epsilon.$$

In this paper, we consider ϵ-solutions for (RFP), and we establish optimality theorems and duality theorems for ϵ-solutions for the robust fractional optimization problem. Moreover, we give an example for our duality theorems.

2 Preliminaries

Let us first recall some notation and preliminary results which will be used throughout this paper. $\mathbb{R}^n$ denotes the Euclidean space with dimension n. The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$ and is defined by $\mathbb{R}^n_+ := \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_i \geq 0\}$. We say the set A is convex whenever $\mu a_1 + (1-\mu)a_2 \in A$ for all $\mu \in [0,1]$, $a_1, a_2 \in A$. A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be convex if, for all $\mu \in [0,1]$,

$$f\big((1-\mu)x + \mu y\big) \leq (1-\mu)f(x) + \mu f(y)$$

for all $x, y \in \mathbb{R}^n$. The function f is said to be concave whenever $-f$ is convex. Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. The subdifferential of g at $a \in \operatorname{dom} g$ is defined by

$$\partial g(a) := \big\{v \in \mathbb{R}^n \mid g(x) \geqq g(a) + \langle v, x-a\rangle \forall x \in \operatorname{dom} g\big\},$$

where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathbb{R}^n$ and $\operatorname{dom} g := \{x \in \mathbb{R}^n : g(x) < +\infty\}$. Let $\epsilon \geqq 0$. Then the ϵ-subdifferential of g at $a \in \operatorname{dom} g$ is defined by

$$\partial_\epsilon g(a) := \big\{v \in \mathbb{R}^n \mid g(x) \geqq g(a) + \langle v, x-a\rangle - \epsilon \forall x \in \operatorname{dom} g\big\}.$$

The function f is said to be proper if $f(x) > -\infty$ for all $x \in \mathbb{R}^n$. We say f is a lower semicontinuous function if $\liminf_{y\to x} f(y) \geqq f(x)$ for all $x \in \mathbb{R}^n$. As usual, for any proper convex function g on $\mathbb{R}^n$, its conjugate function $g^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined, for any $x^* \in \mathbb{R}^n$, by $g^*(x^*) = \sup\{\langle x^*, x\rangle - g(x) | x \in \mathbb{R}^n\}$. The epigraph of a function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $\operatorname{epi} g$, is defined by $\operatorname{epi} g = \{(x,r) \in \mathbb{R}^n \times \mathbb{R} \mid g(x) \leqq r\}$. We denote the convex hull of a subset A of $\mathbb{R}^n$ by $\operatorname{co} A$, and denote the closure of the set A by $\operatorname{cl} A$. Let C be a closed convex set in $\mathbb{R}^n$ and $x \in C$. Then the normal cone $N_C(x)$ to C at x is defined by

$$N_C(x) = \big\{v \in \mathbb{R}^n \mid \langle v, y-x\rangle \leqq 0, \text{ for all } y \in C\big\},$$

and we let $\epsilon \geqq 0$, then the ϵ-normal cone $N_C^\epsilon(x)$ to C at x is defined by

$$N_C^\epsilon(x) = \big\{v \in \mathbb{R}^n \mid \langle v, y-x\rangle \leqq \epsilon, \text{ for all } y \in C\big\}.$$

When C is a closed convex cone in $\mathbb{R}^n$, we denote $N_C(0)$ by C^* and call it the negative dual cone of C.

Proposition 2.1 [23] *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let δ_C be the indicator function with respect to a closed convex subset C of $\mathbb{R}^n$, that is, $\delta_C(x) = 0$ if $x \in C$, and $\delta_C(x) = +\infty$ if $x \notin C$. Let $\epsilon \geqq 0$. Then*

$$\partial_\epsilon(f + \delta_C)(\bar{x}) = \bigcup_{\substack{\epsilon_0 \geqq 0, \epsilon_1 \geqq 0 \\ \epsilon_0 + \epsilon_1 = \epsilon}} \big\{\partial_{\epsilon_0} f(\bar{x}) + \partial_{\epsilon_1}\delta_C(\bar{x})\big\}.$$

Proposition 2.2 [24, 25] *If $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function and if $a \in \operatorname{dom} f := \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$, then*

$$\operatorname{epi} f^* = \bigcup_{\epsilon \geqq 0} \big\{\big(v, \langle v, a\rangle + \epsilon - f(a)\big) \mid v \in \partial_\epsilon f(a)\big\}.$$

Proposition 2.3 [26] *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then*

$$\operatorname{epi}(f+g)^* = \operatorname{epi} f^* + \operatorname{epi} g^*.$$

Moreover, if $f, g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *are proper lower semicontinuous convex functions, and if* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, *then*

$$\operatorname{epi}(f+g)^* = \operatorname{cl}(\operatorname{epi} f^* + \operatorname{epi} g^*).$$

Proposition 2.4 [22, 26] *Let* $h_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i \in I$ (*where* I *is an arbitrary index set*), *be a proper lower semicontinuous convex function. Suppose that there exists* $x_0 \in \mathbb{R}^n$ *such that* $\sup_{i \in I} h_i(x_0) < +\infty$. *Then*

$$\operatorname{epi}\Bigl(\sup_{i \in I} h_i\Bigr)^* = \operatorname{cl}\Bigl(\operatorname{co} \bigcup_{i \in I} \operatorname{epi} h_i^*\Bigr).$$

Proposition 2.5 [23] *Let* $h_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i = 1, \ldots, m$, *be proper lower semicontinuous convex functions. Let* $\epsilon \geqq 0$. *If* $\bigcup_{i=1}^m ri \operatorname{dom} h_i \neq \emptyset$, *where* $ri \operatorname{dom} h_i$ *is the relative interior of* $\operatorname{dom} h_i$, *then for all* $x \in \bigcup_{i=1}^n \operatorname{dom} h_i$,

$$\partial_\epsilon \Biggl(\sum_{i=1}^m h\Biggr)(x) = \bigcup \Biggl\{ \sum_{i=1}^m \partial_{\epsilon_i} h_i(x) \Biggm| \epsilon_i \geqq 0, i = 1, \ldots, m, \sum_{i=1}^m \epsilon_i = \epsilon \Biggr\}.$$

Proposition 2.6 [9] *Let* $h_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \ldots, m$, *be continuous functions such that, for all* $w_i \in \mathbb{R}^q$, $h_i(\cdot, w_i)$ *is a convex function and let* C *be a closed convex cone of* $\mathbb{R}^n$. *Suppose that each* $\mathcal{W}_i$, $i = 1, \ldots, m$, *is compact and convex, and there exists* $x_0 \in C$ *such that* $h_i(x_0, w_i) < 0$, *for all* $w_i \in \mathcal{W}_i$, $i = 1, \ldots, m$. *Then*

$$\bigcup_{w_i \in \mathcal{W}_i, \lambda_i \geqq 0} \operatorname{epi}\Biggl(\sum_{i=1}^m \lambda_i h_i(\cdot, w_i)\Biggr)^* + C^* \times \mathbb{R}_+$$

is closed.

Proposition 2.7 [9] *Let* $h_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \ldots, m$, *be continuous functions and let* C *be a closed convex cone of* $\mathbb{R}^n$. *Suppose that each* $\mathcal{W}_i \subseteq \mathbb{R}^q$, $i = 1, \ldots, m$, *is convex, for all* $w_i \in \mathbb{R}^q$, $h_i(\cdot, w_i)$ *is a convex function, and, for each* $x \in \mathbb{R}^n$, $h_i(x, \cdot)$ *is concave on* $\mathcal{W}_i$. *Then*

$$\bigcup_{w_i \in \mathcal{W}_i, \lambda_i \geqq 0} \operatorname{epi}\Biggl(\sum_{i=1}^m \lambda_i h_i(\cdot, w_i)\Biggr)^* + C^* \times \mathbb{R}_+$$

is convex.

Now we give the following relation between the ϵ-solutions of (RFP) and $(\mathrm{RNCP})_{\bar{r}}$.

Lemma 2.1 *Let* $\bar{x} \in A$ *and let* $\epsilon \geqq 0$. *If* $\max_{(u,v) \in \mathcal{U} \times \mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon \geqq 0$, *then the following statements are equivalent*:

(i) $\bar{x}$ *is an* ϵ*-solution of* (RFP);

(ii) $\bar{x}$ *is an* $\bar{\epsilon}$*-solution of* $(\mathrm{RNCP})_{\bar{r}}$, *where* $\bar{r} = \max_{(u,v) \in \mathcal{U} \times \mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$ *and* $\bar{\epsilon} = \epsilon \min_{v \in \mathcal{V}} g(\bar{x}, v)$.

Proof (⇒) Let $\bar{x} \in A$ be an ϵ-solution of (RFP). Then for any $x \in A$, $\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(x,u)}{g(x,v)} \geqq \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$. Put $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$ and $\bar{\epsilon} = \epsilon \min_{v\in\mathcal{V}} g(\bar{x},v)$. Then we have, for any $x \in A$, $\max_{u\in\mathcal{U}} f(x,u) - \min_{v\in\mathcal{V}} \bar{r}g(x,v) \geqq 0$. Since $\max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \epsilon \min_{v\in\mathcal{V}} g(\bar{x},v) = 0$, for any $x \in A$,

$$\begin{aligned}\max_{u\in\mathcal{U}} f(x,u) - \bar{r}\min_{v\in\mathcal{V}} g(x,v) &\geqq \max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \epsilon \min_{v\in\mathcal{V}} g(\bar{x},v) \\ &= \max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \bar{\epsilon}.\end{aligned}$$

Hence $\bar{x}$ is an $\bar{\epsilon}$-solution of $(\mathrm{RNCP})_{\bar{r}}$.

(⇐) Let $\bar{x} \in A$ be an $\bar{\epsilon}$-solution of $(\mathrm{RNCP})_{\bar{r}}$. Then for any $x \in A$, $\max_{u\in\mathcal{U}} f(x,u) - \bar{r}\min_{v\in\mathcal{V}} g(x,v) \geqq \max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \bar{\epsilon}$. Since $\max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \epsilon \min_{v\in\mathcal{V}} g(\bar{x},v) = 0$, for any $x \in A$, $\max_{u\in\mathcal{U}} f(x,u) - \bar{r}\min_{v\in\mathcal{V}} g(x,v) \geqq 0$. So, we have $\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(x,u)}{g(x,v)} \geqq \bar{r}$. Since $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$,

$$\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(x,u)}{g(x,v)} \geqq \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon.$$

Hence $\bar{x}$ is an ϵ-solution of (RFP). □

3 ϵ-Optimality theorems

In this section, we establish ϵ-optimality theorems for ϵ-solutions for the robust fractional optimization problem.

Now we give the following lemma which is the robust version of Farkas lemma for non-fractional convex functions.

Lemma 3.1 *Let $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ and $h_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \ldots, m$, be functions such that, for any $u \in \mathcal{U}$, $f(\cdot,u)$ and, for each $w_i \in \mathcal{W}_i$, $h_i(\cdot,w_i)$ are convex functions, and, for any $x \in \mathbb{R}^n$, $f(x,\cdot)$ is concave function. Let $g : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ be a function such that, for any $v \in \mathcal{V}$, $g(\cdot,v)$ is a concave function, and, for all $x \in \mathbb{R}$, $g(x,\cdot)$ is a convex function. Let $\mathcal{U} \subset \mathbb{R}^p$, $\mathcal{V} \subset \mathbb{R}^p$, and $\mathcal{W}_i \subset \mathbb{R}^q$, $i = 1, \ldots, m$ be convex and compact sets. Let $r \geqq 0$ and let C be a closed convex cone of $\mathbb{R}^n$. Assume that $A := \{x \in C \mid h_i(x,w_i) \leqq 0, \forall w_i \in \mathcal{W}_i, i = 1, \ldots, m\} \neq \emptyset$. Then the following statements are equivalent*:

(i) $\{x \in C \mid h_i(x,w_i) \leqq 0, \forall w_i \in \mathcal{W}_i, i = 1, \ldots, m\} \subseteq \{x \in \mathbb{R}^n \mid \max_{u\in\mathcal{U}} f(x,u) - r\min_{v\in\mathcal{V}} g(x,v) \geqq 0\}$;

(ii) *there exist $\bar{u} \in \mathcal{U}$ and $\bar{v} \in \mathcal{V}$ such that*

$$\bigl\{x \in C \mid h_i(x,w_i) \leqq 0, \forall w_i \in \mathcal{W}_i, i = 1, \ldots, m\bigr\} \subseteq \bigl\{x \in \mathbb{R}^n \mid f(x,\bar{u}) - rg(x,\bar{v}) \geqq 0\bigr\};$$

(iii)

$$\begin{aligned}(0,0) \in {}& \bigcup_{u\in\mathcal{U}} \operatorname{epi}\bigl(f(\cdot,u)\bigr)^* + \bigcup_{v\in\mathcal{V}} \operatorname{epi}\bigl(-rg(\cdot,v)\bigr)^* \\ &+ \operatorname{cl}\operatorname{co}\left(\bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0} \operatorname{epi}\left(\sum_{i=1}^m \lambda_i h_i(\cdot,w_i)\right)^* + C^* \times \mathbb{R}_+\right);\end{aligned}$$

(iv)

$$\begin{aligned}(0,0) \in\ & \operatorname{epi}\Big(\max_{u\in\mathcal{U}} f(\cdot,u)\Big)^* + \operatorname{epi}\Big(-r\min_{v\in\mathcal{V}} g(\cdot,v)\Big)^* \\ & + \operatorname{cl}\operatorname{co}\Bigg(\bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0} \operatorname{epi}\Bigg(\sum_{i=1}^m \lambda_i h_i(\cdot,w_i)\Bigg)^* + C^*\times\mathbb{R}_+\Bigg).\end{aligned}$$

Proof Let $D := \{x\in\mathbb{R}^n \mid h_i(x,w_i)\leqq 0, \forall w_i\in\mathcal{W}_i, i=1,\dots,m\}$. Then $A = C\cap D$. We will prove that $\operatorname{epi}\delta_A^* = \operatorname{cl}\operatorname{co}(\bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0}\operatorname{epi}(\sum_{i=1}^m\lambda_i h_i(\cdot,w_i))^* + C^*\times\mathbb{R}_+)$. For any $x\in\mathbb{R}^n$,

$$\delta_A(x) = \delta_C(x) + \delta_D(x) \quad\text{and}\quad \delta_D(x) = \sup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0}\sum_{i=1}^m \lambda_i h_i(x,w_i).$$

Thus, by Propositions 2.3 and 2.4, we have

$$\begin{aligned}\operatorname{epi}\delta_A^* &= \operatorname{epi}(\delta_D+\delta_C)^* = \operatorname{cl}\big(\operatorname{epi}\delta_D^* + \operatorname{epi}\delta_C^*\big) \\ &= \operatorname{cl}\Bigg(\operatorname{epi}\Bigg(\sup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0}\sum_{i=1}^m\lambda_i h_i(\cdot,w_i)\Bigg)^* + \operatorname{epi}\delta_C^*\Bigg) \\ &= \operatorname{cl}\Bigg(\operatorname{cl}\operatorname{co}\bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0}\operatorname{epi}\Bigg(\sum_{i=1}^m\lambda_i h_i(\cdot,w_i)\Bigg)^* + \operatorname{epi}\delta_C^*\Bigg) \\ &= \operatorname{cl}\operatorname{co}\Bigg(\bigcup_{w_i\in\mathcal{W}_i,\ \lambda_i\geqq 0}\operatorname{epi}\Bigg(\sum_{i=1}^m\lambda_i h_i(\cdot,w_i)\Bigg)^* + C^*\times\mathbb{R}_+\Bigg).\end{aligned}$$

[(i) ⇔ (iv)] Now we assume that the statement (iv) holds. Then, by Proposition 2.3, the statement (iv) is equivalent to

$$\begin{aligned}(0,0) &\in \operatorname{epi}\Big(\max_{u\in\mathcal{U}} f(\cdot,u)\Big)^* + \operatorname{epi}\Big(-r\min_{v\in\mathcal{V}} g(\cdot,v)\Big)^* + \operatorname{epi}\delta_A^* \\ &= \operatorname{epi}\Big(\max_{u\in\mathcal{U}} f(\cdot,u) - r\min_{v\in\mathcal{V}} g(\cdot,v) + \delta_A\Big)^*.\end{aligned}$$

Equivalently, by definition of epigraph of $(\max_{u\in\mathcal{U}} f(\cdot,u) - r\min_{v\in\mathcal{V}} g(\cdot,v) + \delta_A)^*$,

$$\Big(\max_{u\in\mathcal{U}} f(\cdot,u) - r\min_{v\in\mathcal{V}} g(\cdot,v) + \delta_A\Big)^*(0) \leqq 0.$$

From the definition of a conjugate function, for any $x\in\mathbb{R}^n$,

$$\Big(\max_{u\in\mathcal{U}} f(\cdot,u) - r\min_{v\in\mathcal{V}} g(\cdot,v) + \delta_A\Big)(x) \geqq 0.$$

It is equivalent to the statement that, for any $x\in A$,

$$\max_{u\in\mathcal{U}} f(x,u) - r\min_{v\in\mathcal{V}} g(x,v) \geqq 0.$$

[(ii) ⇔ (iii)] Now we assume that the statement (iii) holds. Then the statement (iii) is equivalent to

$$(0,0) \in \bigcup_{u\in\mathcal{U}} \operatorname{epi}\bigl(f(\cdot,u)\bigr)^* + \bigcup_{v\in\mathcal{V}} \operatorname{epi}\bigl(-rg(\cdot,v)\bigr)^* + \operatorname{epi}\delta_A^*.$$

It means that there exist $\bar{u} \in \mathcal{U}$ and $\bar{v} \in \mathcal{V}$ such that

$$(0,0) \in \operatorname{epi}\bigl(f(\cdot,\bar{u}) - rg(\cdot,\bar{v}) + \delta_A\bigr)^*.$$

It is equivalent to the statement that there exist $\bar{u} \in \mathcal{U}$ and $\bar{v} \in \mathcal{V}$ such that

$$\bigl(f(\cdot,\bar{u}) - rg(\cdot,\bar{v}) + \delta_A\bigr)^*(0) \leqq 0.$$

From the definition of a conjugate function, there exist $\bar{u} \in \mathcal{U}$ and $\bar{v} \in \mathcal{V}$ such that, for any $x \in \mathbb{R}^n$,

$$\bigl(f(\cdot,\bar{u}) - rg(\cdot,\bar{v}) + \delta_A\bigr)(x) \geqq 0.$$

It means that there exist $\bar{u} \in \mathcal{U}$ and $\bar{v} \in \mathcal{V}$ such that, for any $x \in A$,

$$f(x,\bar{u}) - rg(x,\bar{v}) \geqq 0.$$

[(iii) ⇔ (iv)] To get the desired result, it suffices to show that

$$\bigcup_{u\in\mathcal{U}} \operatorname{epi}\bigl(f(\cdot,u)\bigr)^* = \operatorname{epi}\Bigl(\max_{u\in\mathcal{U}} f(\cdot,u)\Bigr)^*, \tag{1}$$

$$\bigcup_{v\in\mathcal{V}} \operatorname{epi}\bigl(-rg(\cdot,v)\bigr)^* = \operatorname{epi}\Bigl(-r\min_{v\in\mathcal{V}} g(\cdot,v)\Bigr)^*. \tag{2}$$

By Proposition 2.4, $\operatorname{epi}(\max_{u\in\mathcal{U}} f(\cdot,u))^* = \operatorname{cl}\operatorname{co}\bigcup_{u\in\mathcal{U}} \operatorname{epi}(f(\cdot,u))^*$. Let $(z_1,\alpha_1),(z_2,\alpha_2) \in \bigcup_{u\in\mathcal{U}} \operatorname{epi}(f(\cdot,u))^*$ and let $\mu \in [0,1]$. Then there exist $u_1,u_2 \in \mathcal{U}$ such that $(z_1,\alpha_1) \in \operatorname{epi}(f(\cdot,u_1))^*$ and $(z_2,\alpha_2) \in \operatorname{epi}(f(\cdot,u_2))^*$, that is, $(f(\cdot,u_1))^*(z_1) \leqq \alpha_1$ and $(f(\cdot,u_2))^*(z_2) \leqq \alpha_2$. Using the definition of a conjugate function, we have, for all $x \in \mathbb{R}^n$,

$$\langle z_1,x\rangle - f(x,u_1) \leqq \alpha_1 \quad \text{and} \quad \langle z_2,x\rangle - f(x,u_2) \leqq \alpha_2. \tag{3}$$

Since, for all $x \in \mathbb{R}^n$, $f(x,\cdot)$ is concave, we have $f(x,\mu u_1 + (1-\mu)u_2) \geqq \mu f(x,u_1) + (1-\mu)f(x,u_2)$, *i.e.*,

$$-f\bigl(x,\mu u_1 + (1-\mu)u_2\bigr) \leqq -\mu f(x,u_1) - (1-\mu)f(x,u_2). \tag{4}$$

So, from (3) and (4), we have, for all $x \in \mathbb{R}^n$,

$$\bigl\langle \mu z_1 + (1-\mu)z_2, x\bigr\rangle - f\bigl(x,\mu u_1 + (1-\mu)u_2\bigr) \leqq \mu\alpha_1 + (1-\mu)\alpha_2,$$

and so $(f(\cdot,\mu u_1+(1-\mu)u_2))^*(\mu z_1+(1-\mu)z_2) \leqq \mu\alpha_1+(1-\mu)\alpha_2$. Hence, we have

$$\bigl(\mu z_1+(1-\mu)z_2,\mu\alpha_1+(1-\mu)\alpha_2\bigr) \in \operatorname{epi}\bigl(f\bigl(\cdot,\mu u_1+(1-\mu)u_2\bigr)\bigr)^*.$$

So, $\bigcup_{u\in\mathcal{U}} \operatorname{epi}(f(\cdot,u))^*$ is convex.

Now we show that $\bigcup_{u\in\mathcal{U}} \operatorname{epi}(f(\cdot,u))^*$ is closed. Let

$$(z_n,\alpha_n) \in \bigcup_{u\in\mathcal{U}} \operatorname{epi}\bigl(f(\cdot,u)\bigr)^*$$

with $(z_n,\alpha_n) \to (z^*,\alpha^*)$ as $n\to\infty$. Then there exists $u_n \in \mathcal{U}$ such that $(f(\cdot,u_n))^*(z_n) \leqq \alpha_n$. Since $\mathcal{U}$ is compact, we may assume that $u_n \to u^* \in \mathcal{U}$ as $n\to\infty$. So, for each $x\in\mathbb{R}^n$,

$$\langle z_n,x\rangle - f(x,u_n) \leqq \alpha_n.$$

Since, for all $x\in\mathbb{R}^n$, $f(x,\cdot)$ is concave, $f(x,\cdot)$ is continuous. Passing to the limit as $n\to\infty$, we get, for each $x\in\mathbb{R}^n$, $\langle z^*,x\rangle - f(x,u^*) \leqq \alpha^*$. Hence, we have

$$\bigl(z^*,\alpha^*\bigr) \in \operatorname{epi}\bigl(f\bigl(\cdot,u^*\bigr)\bigr)^*.$$

So, $\bigcup_{u\in\mathcal{U}} \operatorname{epi}(f(\cdot,u))^*$ is closed. Thus, (1) holds.

Moreover, since, for all $x\in\mathbb{R}^n$, $g(x,\cdot)$ is convex and $r \geqq 0$, for all $x\in\mathbb{R}^n$, $-rg(x,\cdot)$ is concave. So, similarly, we can prove that (2) holds. □

Remark 3.1 Using convex-concave minimax theorem (Corollary 37.3.2 in [27]), we can prove that the statement (i) in Lemma 3.1 is equivalent to the statement (ii) in Lemma 3.1.

Remark 3.2 From proving in Lemma 3.1 that the statement (i) is equivalent to the statement (iv), we see that we can prove the equivalent relation without the assumptions that, for all $x\in\mathbb{R}^n$, $f(x,\cdot)$, and $g(x,\cdot)$ are concave and convex, respectively.

From Lemmas 2.1 and 3.1, we can get the following theorem.

Theorem 3.1 *Let $f:\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}$ and $h_i:\mathbb{R}^n\times\mathbb{R}^q\to\mathbb{R}$, $i=1,\ldots,m$, be functions such that, for any $u\in\mathcal{U}$, $f(\cdot,u)$, and, for each $w_i\in\mathcal{W}_i$, $h_i(\cdot,w_i)$ are convex functions, and, for any $x\in\mathbb{R}^n$, $f(x,\cdot)$ is concave function. Let $g:\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}$ be a function such that, for any $v\in\mathcal{V}$, $g(\cdot,v)$ is a concave function, and, for all $x\in\mathbb{R}^n$, $g(x,\cdot)$ is a convex function. Let $\mathcal{U}\subset\mathbb{R}^p$, $\mathcal{V}\subset\mathbb{R}^p$, and $\mathcal{W}_i\subset\mathbb{R}^q$, $i=1,\ldots,m$. Let $\bar{x}\in A$ and let $\bar{r}=\max_{(u,v)\in\mathcal{U}\times\mathcal{V}}\frac{f(\bar{x},u)}{g(\bar{x},v)}-\epsilon$. Suppose that $\bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0}\operatorname{epi}(\sum_{i=1}^m\lambda_i h_i(\cdot,w_i))^* + C^*\times\mathbb{R}_+$ is closed and convex. Then the following statements are equivalent*:

(i) *$\bar{x}$ is an ϵ-solution of* (RFP);

(ii) *There exist $\bar{u}\in\mathcal{U}$, $\bar{v}\in\mathcal{V}$, $\bar{w}_i\in\mathcal{W}_i$, and $\bar{\lambda}_i\geqq 0$, $i=1,\ldots,m$ such that, for any $x\in C$,*

$$f(x,\bar{u}) - \bar{r}g(x,\bar{v}) + \sum_{i=1}^m \bar{\lambda}_i h_i(x,\bar{w}_i) \geqq 0.$$

Proof (⇒) Let $\bar{x}$ be an ϵ-solution of (RFP). Then, by Lemma 2.1, equivalently, $\bar{x}$ is an $\bar{\epsilon}$-solution of $(\text{RNCP})_{\bar{r}}$, where $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$ and $\bar{\epsilon} = \epsilon \min_{v\in\mathcal{V}} g(\bar{x},v)$, that is, for any $x \in A$, $\max_{u\in\mathcal{U}} f(x,u) - \bar{r}\min_{v\in\mathcal{V}} g(x,v) \geqq \max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \epsilon\min_{v\in\mathcal{V}} g(\bar{x},v)$. Since $\max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \epsilon\min_{v\in\mathcal{V}} g(\bar{x},v) = 0$, we have $A \subseteq \{x \in C \mid \max_{u\in\mathcal{U}} f(x,u) - \bar{r}\min_{v\in\mathcal{V}} g(x,v) \geqq 0\}$. Then, by Lemma 3.1, we have

$$\begin{aligned}(0,0) \in {} & \bigcup_{u\in\mathcal{U}} \operatorname{epi}\bigl(f(\cdot,u)\bigr)^* + \bigcup_{v\in\mathcal{V}} \operatorname{epi}\bigl(-\bar{r}g(\cdot,v)\bigr)^* \\ & + \operatorname{cl}\operatorname{co}\Biggl(\bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0} \operatorname{epi}\Biggl(\sum_{i=1}^m \lambda_i h_i(\cdot,w_i)\Biggr)^* + C^*\times\mathbb{R}_+\Biggr).\end{aligned}$$

Moreover, by assumption,

$$(0,0) \in \bigcup_{u\in\mathcal{U}} \operatorname{epi}\bigl(f(\cdot,u)\bigr)^* + \bigcup_{v\in\mathcal{V}} \operatorname{epi}\bigl(-\bar{r}g(\cdot,v)\bigr)^* + \bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0} \operatorname{epi}\Biggl(\sum_{i=1}^m \lambda_i h_i(\cdot,w_i)\Biggr)^* + C^*\times\mathbb{R}_+.$$

So, there exist $\bar{u} \in \mathcal{U}$, $\bar{v} \in \mathcal{V}$, $\bar{w}_i \in \mathcal{W}_i$, and $\bar{\lambda}_i \geqq 0$, $i = 1,\ldots,m$ such that

$$(0,0) \in \operatorname{epi}\bigl(f(\cdot,\bar{u})\bigr)^* + \operatorname{epi}\bigl(-\bar{r}g(\cdot,\bar{v})\bigr)^* + \operatorname{epi}\Biggl(\sum_{i=1}^m \bar{\lambda}_i h_i(\cdot,\bar{w}_i)\Biggr)^* + C^*\times\mathbb{R}_+.$$

Then there exist $s \in \mathbb{R}^n$, $\eta \geqq 0$, $t \in \mathbb{R}^n$, $\mu \geqq 0$, $z_i \in \mathbb{R}^n$, $\rho_i \geqq 0$, $i = 1,\ldots,m$, $c^* \in C^*$, and $\gamma \in \mathbb{R}_+$ such that

$$\begin{aligned}(0,0) = {} & \bigl(s, \bigl(f(\cdot,\bar{u})\bigr)^*(s) + \eta\bigr) + \bigl(t, \bigl(-\bar{r}g(\cdot,\bar{v})\bigr)^*(t) + \mu\bigr) \\ & + \sum_{i=1}^m \bigl(z_i, \bigl(\bar{\lambda}_i h_i(\cdot,\bar{w}_i)\bigr)^*(z_i) + \rho_i\bigr) + \bigl(c^*,\gamma\bigr).\end{aligned}$$

So, $0 = s + t + \sum_{i=1}^m z_i + c^*$ and $0 = (f(\cdot,\bar{u}))^*(s) + \eta + (-\bar{r}g(\cdot,\bar{v}))^*(t) + \mu + \sum_{i=1}^m ((\bar{\lambda}_i h_i(\cdot,\bar{w}_i))^*(z_i) + \rho_i) + \gamma$. Hence, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}& -\Biggl\langle \sum_{i=1}^m z_i, x\Biggr\rangle - \langle c^*, x\rangle - f(x,\bar{u}) - \bigl(-\bar{r}g(x,\bar{v})\bigr) \\ & \quad = \langle s,x\rangle + \langle t,x\rangle - f(x,\bar{u}) - \bigl(-\bar{r}g(x,\bar{v})\bigr) \\ & \quad \leqq \bigl(f(\cdot,\bar{u})\bigr)^*(s) + \bigl(-\bar{r}g(\cdot,v)\bigr)^*(t) \\ & \quad = -\eta - \mu - \sum_{i=1}^m \bigl(\bigl(\bar{\lambda}_i h_i(\cdot,\bar{w}_i)\bigr)^*(z_i) + \rho_i\bigr) - \gamma. \end{aligned} \tag{5}$$

Since $\eta \geqq 0$, $\mu \geqq 0$, $\rho_i \geqq 0$, $i = 1,\ldots,m$, and $c^* \in C^*$, from (5), for any $x \in C$,

$$\begin{aligned}0 \leqq {} & \Biggl\langle \sum_{i=1}^m z_i, x\Biggr\rangle + \langle c^*, x\rangle + f(x,\bar{u}) + \bigl(-\bar{r}g(x,\bar{v})\bigr) - \eta - \mu \\ & - \sum_{i=1}^m \bigl(\bar{\lambda}_i h_i(\cdot,\bar{w}_1)\bigr)^*(z_i) - \sum_{i=1}^m \bar{\lambda}_i\rho_i - \gamma\end{aligned}$$

$$\leqq \left\langle \sum_{i=1}^{m} z_i, x \right\rangle + f(x,\bar{u}) - \bar{r}g(x,\bar{v}) - \sum_{i=1}^{m} \big(\bar{\lambda}_i h_i(\cdot,\bar{w}_i)\big)^*(z_i)$$

$$\leqq f(x,\bar{u}) - \bar{r}g(x,\bar{v}) + \sum_{i=1}^{m} \big(\bar{\lambda}_i h_i(x,\bar{w}_i)\big).$$

($\Leftarrow$) Suppose that there exist $\bar{u} \in \mathcal{U}$, $\bar{v} \in \mathcal{V}$, $\bar{w}_i \in \mathcal{W}_i$, and $\bar{\lambda}_i \geqq 0$, $i = 1,\ldots,m$, such that, for any $x \in C$,

$$f(x,\bar{u}) - \bar{r}g(x,\bar{v}) + \sum_{i=1}^{m} \bar{\lambda}_i h_i(x,\bar{w}_i) \geqq 0. \tag{6}$$

Since $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$, we have $\max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \epsilon \min_{v\in\mathcal{V}} g(\bar{x},v) = 0$. So, from (6), we have, for any $x \in A$,

$$\begin{aligned}
\max_{u\in\mathcal{U}} f(x,u) - \bar{r}\min_{v\in\mathcal{V}} g(x,v) &\geqq \max_{u\in\mathcal{U}} f(x,u) - \bar{r}\min_{v\in\mathcal{V}} g(x,v) + \sum_{i=1}^{m} \bar{\lambda}_i h_i(x,\bar{w}_i) \\
&\geqq f(x,\bar{u}) - \bar{r}g(x,\bar{v}) + \sum_{i=1}^{m} \bar{\lambda}_i h_i(x,\bar{w}_i) \\
&\geqq 0 \\
&= \max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \epsilon \min_{v\in\mathcal{V}} g(\bar{x},v).
\end{aligned}$$

Hence, for any $x \in A$, $\max_{u\in\mathcal{U}} f(x,u) - \bar{r}\min_{v\in\mathcal{V}} g(x,v) \geqq \max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \epsilon \min_{v\in\mathcal{V}} g(\bar{x},v)$. It means that $\bar{x}$ is an $\bar{\epsilon}$-solution of $(\mathrm{RNCP})_{\bar{r}}$. Thus, by Lemma 2.1, $\bar{x}$ is an ϵ-solution of (RFP). □

Using Remark 3.2 and Lemmas 2.1 and 3.1, we can obtain the following characterization of an ϵ-solution for (RFP).

Theorem 3.2 (ϵ-Optimality theorem) *Let* $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ *and* $h_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1,\ldots,m$, *be functions such that, for any* $u \in \mathcal{U}$, $f(\cdot,u)$, *and, for each* $w_i \in \mathcal{W}_i$, $h_i(\cdot,w_i)$ *are convex functions. Let* $g : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ *be a function such that, for any* $v \in \mathcal{V}$, $g(\cdot,v)$ *is a concave function. Let* $\mathcal{U} \subset \mathbb{R}^p$, $\mathcal{V} \subset \mathbb{R}^p$, *and* $\mathcal{W}_i \subset \mathbb{R}^q$, $i = 1,\ldots,m$. *Let* $\bar{x} \in A$ *and let* $\epsilon \geqq 0$. *Let* $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$. *If* $\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} < \epsilon$, *then* $\bar{x}$ *is an* ϵ*-solution of* (RFP). *If* $\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} \geqq \epsilon$ *and*

$$\bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0} \mathrm{epi}\left(\sum_{i=1}^{m} \lambda_i h_i(\cdot,w_i)\right)^* + C^* \times \mathbb{R}_+$$

is closed and convex, then the following statements are equivalent:

(i) $\bar{x}$ *is an* ϵ*-solution of* (RFP);

(ii) *there exist* $\bar{w}_i \in \mathcal{W}_i$ *and* $\bar{\lambda}_i \geqq 0$, $i = 1, \ldots, m$, $\epsilon_0^1 \geqq 0$, $\epsilon_0^2 \geqq 0$, *and* $\epsilon_i \geqq 0$, $i = 1, \ldots, m+1$ *such that*

$$0 \in \partial_{\epsilon_0^1}\Big(\max_{u\in\mathcal{U}} f(\cdot,u)\Big)(\bar{x}) + \partial_{\epsilon_0^2}\Big(-\bar{r}\min_{v\in\mathcal{V}} g(\cdot,v)\Big)(\bar{x}) + \sum_{i=1}^{m} \partial_{\epsilon_i}\big(\bar{\lambda}_i h_i(\cdot,\bar{w}_i)\big)(\bar{x}) + N_C^{\epsilon_{m+1}}(\bar{x}), \tag{7}$$

$$\max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) = \epsilon \min_{v\in\mathcal{V}} g(\bar{x},v) \quad \textit{and} \tag{8}$$

$$\epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1}\epsilon_i - \epsilon\min_{v\in\mathcal{V}} g(\bar{x},v) = \sum_{i=1}^{m}\bar{\lambda}_i h_i(\bar{x},\bar{w}_i). \tag{9}$$

Proof [(i) ⇒ (ii)] We assume that $\bar{x}$ is an ϵ-solution of (RFP). Then, by Lemma 2.1, $\bar{x}$ is an $\bar{\epsilon}$-solution of $(\mathrm{RNCP})_{\bar{r}}$, where $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$ and $\bar{\epsilon} = \epsilon\min_{v\in\mathcal{V}} g(\bar{x},v)$, that is, for any $x \in A$, $\max_{u\in\mathcal{U}} f(x,u) - \bar{r}\min_{v\in\mathcal{V}} g(x,v) \geqq \max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \epsilon\min_{v\in\mathcal{V}} g(\bar{x},v)$. Since $\max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \epsilon\min_{v\in\mathcal{V}} g(\bar{x},v) = 0$, we have $A \subseteq \{x \in C \mid \max_{u\in\mathcal{U}} f(x,u) - \bar{r}\min_{v\in\mathcal{V}} g(x,v) \geqq 0\}$. By Lemma 3.1,

$$(0,0) \in \operatorname{epi}\Big(\max_{u\in\mathcal{U}} f(\cdot,u)\Big)^* + \operatorname{epi}\Big(-\bar{r}\min_{v\in\mathcal{V}} g(\cdot,v)\Big)^* + \operatorname{cl}\operatorname{co}\left(\bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0} \operatorname{epi}\left(\sum_{i=1}^{m}\lambda_i h_i(\cdot,w_i)\right)^* + C^* \times \mathbb{R}_+\right).$$

By assumption,

$$(0,0) \in \operatorname{epi}\Big(\max_{u\in\mathcal{U}} f(\cdot,u)\Big)^* + \operatorname{epi}\Big(-\bar{r}\min_{v\in\mathcal{V}} g(\cdot,v)\Big)^* + \bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0} \operatorname{epi}\left(\sum_{i=1}^{m}\lambda_i h_i(\cdot,w_i)\right)^* + C^* \times \mathbb{R}_+.$$

So, there exist $\bar{w}_i \in \mathcal{W}_i$ and $\bar{\lambda}_i \geqq 0$, $i = 1, \ldots, m$, such that

$$(0,0) \in \operatorname{epi}\Big(\max_{u\in\mathcal{U}} f(\cdot,u)\Big)^* + \operatorname{epi}\Big(-\bar{r}\min_{v\in\mathcal{V}} g(\cdot,v)\Big)^* + \operatorname{epi}\left(\sum_{i=1}^{m}\bar{\lambda}_i h_i(\cdot,\bar{w}_i)\right)^* + \operatorname{epi}\delta_C^*.$$

By Proposition 2.2, we obtain

$$\begin{aligned}(0,0) \in{}& \bigcup_{\epsilon_0^1\geqq 0}\Big\{\Big(\xi_0^1, \langle\xi_0^1,\bar{x}\rangle + \epsilon_0^1 - \max_{u\in\mathcal{U}} f(\bar{x},u)\Big) \mid \xi_0^1 \in \partial_{\epsilon_0^1}\Big(\max_{u\in\mathcal{U}} f(\cdot,u)\Big)(\bar{x})\Big\} \\ &+ \bigcup_{\epsilon_0^2\geqq 0}\Big\{\Big(\xi_0^2, \langle\xi_0^2,\bar{x}\rangle + \epsilon_0^2 + \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v)\Big) \mid \xi_0^2 \in \partial_{\epsilon_0^2}\Big(-\bar{r}\min_{v\in\mathcal{V}} g(\cdot,v)\Big)(\bar{x})\Big\} \\ &+ \bigcup_{\epsilon^*\geqq 0}\left\{\left(\xi^*, \langle\xi^*,\bar{x}\rangle + \epsilon^* - \sum_{i=1}^{m}\bar{\lambda}_i h_i(\bar{x},\bar{w}_i)\right) \,\middle|\, \xi^* \in \partial_{\epsilon^*}\left(\sum_{i=1}^{m}\bar{\lambda}_i h_i(\cdot,\bar{w}_i)\right)(\bar{x})\right\} \\ &+ \bigcup_{\epsilon_{m+1}\geqq 0}\big\{\big(\xi_{m+1}, \langle\xi_{m+1},\bar{x}\rangle + \epsilon_{m+1} - \delta_C(\bar{x})\big) \mid \xi_{m+1} \in \partial_{\epsilon_{m+1}}\delta_C(\bar{x})\big\}.\end{aligned}$$

So, there exist $\bar{\xi}_0^1 \in \partial_{\epsilon_0^1}(\max_{u\in\mathcal{U}} f(\cdot,u))(\bar{x})$, $\bar{\xi}_0^2 \in \partial_{\epsilon_0^2}(-\bar{r}\min_{v\in\mathcal{V}} g(\cdot,v))(\bar{x})$, $\bar{\xi}^* \in \partial_{\epsilon^*}(\sum_{i=1}^m \bar{\lambda}_i h_i(\cdot, \bar{w}_i))(\bar{x})$, $\bar{\xi}_{m+1} \in \partial_{\epsilon_{m+1}}\delta_C(\bar{x})$, $\epsilon_0^1 \geqq 0$, $\epsilon_0^2 \geqq 0$, $\epsilon^* \geqq 0$, and $\epsilon_{m+1} \geqq 0$ such that

$$0 = \bar{\xi}_0^1 + \bar{\xi}_0^2 + \bar{\xi}^* + \bar{\xi}_{m+1} \quad \text{and}$$

$$\epsilon_0^1 + \epsilon_0^2 + \epsilon^* + \epsilon_{m+1} = \max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) + \sum_{i=1}^m \bar{\lambda}_i h_i(\bar{x},\bar{w}_i).$$

By Proposition 2.5, there exist $\bar{\xi}_0^1 \in \partial_{\epsilon_0^1}(\max_{u\in\mathcal{U}} f(\cdot,u))(\bar{x})$, $\bar{\xi}_0^2 \in \partial_{\epsilon_0^2}(-\bar{r}\min_{v\in\mathcal{V}} g(\cdot,v))(\bar{x})$, $\bar{\xi}_i \in \partial_{\epsilon_i}(\bar{\lambda}_i h_i(\cdot,\bar{w}_i))(\bar{x})$, $\bar{\xi}_{m+1} \in \partial_{\epsilon_{m+1}}\delta_C(\bar{x})$, $\epsilon_0^1 \geqq 0$, $\epsilon_0^2 \geqq 0$, $\epsilon_i \geqq 0$, $i = 1,\dots,m$, and $\epsilon_{m+1} \geqq 0$ such that

$$\begin{aligned} 0 \in{}& \partial_{\epsilon_0^1}\Bigl(\max_{u\in\mathcal{U}} f(\cdot,u)\Bigr)(\bar{x}) + \partial_{\epsilon_0^2}\Bigl(-\bar{r}\min_{v\in\mathcal{V}} g(\cdot,v)\Bigr)(\bar{x}) \\ &+ \sum_{i=1}^m \partial_{\epsilon_i}\bigl(\bar{\lambda}_i h_i(\cdot,\bar{w}_i)\bigr)(\bar{x}) + N_C^{\epsilon_{m+1}}(\bar{x}) \quad \text{and} \\ \epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1}\epsilon_i ={}& \max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) + \sum_{i=1}^m \bar{\lambda}_i h_i(\bar{x},\bar{w}_i). \end{aligned} \tag{10}$$

Since $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$,

$$\max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \epsilon\min_{v\in\mathcal{V}} g(\bar{x},v) = 0. \tag{11}$$

So, (8) holds, and so, from (10) and (11), we have

$$\begin{aligned} 0 \in{}& \partial_{\epsilon_0^1}\Bigl(\max_{u\in\mathcal{U}} f(\cdot,u)\Bigr)(\bar{x}) + \partial_{\epsilon_0^2}\Bigl(-\bar{r}\min_{v\in\mathcal{V}} g(\cdot,v)\Bigr)(\bar{x}) \\ &+ \sum_{i=1}^m \partial_{\epsilon_i}\bigl(\bar{\lambda}_i h_i(\cdot,\bar{w}_i)\bigr)(\bar{x}) + N_C^{\epsilon_{m+1}}(\bar{x}) \quad \text{and} \\ \epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1}\epsilon_i - \epsilon\min_{v\in\mathcal{V}} g(\bar{x},v) ={}& \sum_{i=1}^m \bar{\lambda}_i h_i(\bar{x},\bar{w}_i). \end{aligned}$$

Thus the conditions (7) and (9) hold.

[(ii) ⇒ (i)] Taking account of the converse of the process for proving (i) ⇒ (ii), we can easily check that the statement (ii) ⇒ (i) holds. □

If for all $x \in \mathbb{R}^n$, $f(x,\cdot)$ is concave, and, for all $x \in \mathbb{R}$, $g(x,\cdot)$ is convex, then using Lemmas 2.1 and 3.1, we can obtain the following characterization of an ϵ-solution for (RFP).

Theorem 3.3 (ϵ-Optimality theorem) *Let $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ and $h_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1,\dots,m$, be functions such that, for any $u \in \mathcal{U}$, $f(\cdot,u)$, and, for each $w_i \in \mathcal{W}_i$, $h_i(\cdot,w_i)$ are convex functions, and, for all $x \in \mathbb{R}^n$, $f(x,\cdot)$ is concave function. Let $g : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ be a function such that, for any $v \in \mathcal{V}$, $g(\cdot,v)$ is a concave function, and, for all $x \in \mathbb{R}$, $g(x,\cdot)$ is a convex function. Let $\mathcal{U} \subset \mathbb{R}^p$, $\mathcal{V} \subset \mathbb{R}^p$, and $\mathcal{W}_i \subset \mathbb{R}^q$, $i = 1,\dots,m$. Let $\bar{x} \in A$ and let $\epsilon \geqq 0$. Let $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$. If $\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} < \epsilon$, then $\bar{x}$ is an ϵ-solution of* (RFP). *If*

$\max_{(u,v)\in\mathcal{U}\times\mathcal{V}}\frac{f(\bar{x},u)}{g(\bar{x},v)}\geqq\epsilon$ *and*

$$\bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0}\operatorname{epi}\left(\sum_{i=1}^m\lambda_i h_i(\cdot,w_i)\right)^*+C^*\times\mathbb{R}_+$$

is closed and convex, then the following statements are equivalent:

(i) $\bar{x}$ *is an* ϵ*-solution of* (RFP);

(ii) *there exist* $\bar{u}\in\mathcal{U}$, $\bar{v}\in\mathcal{V}$, $\bar{w}_i\in\mathcal{W}_i$, $\bar{\lambda}_i\geqq 0$, $i=1,\dots,m$, $\epsilon_0^1\geqq 0$, $\epsilon_0^2\geqq 0$, *and* $\epsilon_i\geqq 0$, $i=1,\dots,m+1$ *such that*

$$0\in\partial_{\epsilon_0^1}\big(f(\cdot,\bar{u})\big)(\bar{x})+\partial_{\epsilon_0^2}\big(-\bar{r}g(\cdot,\bar{v})\big)(\bar{x})+\sum_{i=1}^m\partial_{\epsilon_i}\big(\bar{\lambda}_i h_i(\cdot,\bar{w}_i)\big)(\bar{x})+N_C^{\epsilon_{m+1}}(\bar{x}),\tag{12}$$

$$\max_{u\in\mathcal{U}}f(\bar{x},u)-\min_{v\in\mathcal{V}}\bar{r}g(\bar{x},v)=\epsilon\min_{v\in\mathcal{V}}g(\bar{x},v)\quad\textit{and}\tag{13}$$

$$\epsilon_0^1+\epsilon_0^2+\sum_{i=1}^{m+1}\epsilon_i-\epsilon\min_{v\in\mathcal{V}}g(\bar{x},v)\leqq\sum_{i=1}^m\bar{\lambda}_i h_i(\bar{x},\bar{w}_i).\tag{14}$$

Proof [(i) $\Rightarrow$ (ii)] Let $\bar{x}$ be an ϵ-solution of (RFP). Then, by Lemma 2.1, $\bar{x}$ is an $\bar{\epsilon}$-solution of $(\text{RNCP})_{\bar{r}}$, where $\bar{r}=\max_{(u,v)\in\mathcal{U}\times\mathcal{V}}\frac{f(\bar{x},u)}{g(\bar{x},v)}-\epsilon$ and $\bar{\epsilon}=\epsilon\min_{v\in\mathcal{V}}g(\bar{x},v)$, that is, for any $x\in A$, $\max_{u\in\mathcal{U}}f(x,u)-\bar{r}\min_{v\in\mathcal{V}}g(x,v)\geqq\max_{u\in\mathcal{U}}f(\bar{x},u)-\bar{r}\min_{v\in\mathcal{V}}g(\bar{x},v)-\epsilon\min_{v\in\mathcal{V}}g(\bar{x},v)$. Since $\max_{u\in\mathcal{U}}f(\bar{x},u)-\bar{r}\min_{v\in\mathcal{V}}g(\bar{x},v)-\epsilon\min_{v\in\mathcal{V}}g(\bar{x},v)=0$, we have $A\subseteq\{x\in C\mid\max_{u\in\mathcal{U}}f(x,u)-\bar{r}\min_{v\in\mathcal{V}}g(x,v)\geqq 0\}$. By Lemma 3.1,

$$\begin{aligned}(0,0)\in{}&\bigcup_{u\in\mathcal{U}}\operatorname{epi}\big(f(\cdot,u)\big)^*+\bigcup_{v\in\mathcal{V}}\operatorname{epi}\big(-\bar{r}g(\cdot,v)\big)^*\\&+\operatorname{cl}\operatorname{co}\left(\bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0}\operatorname{epi}\left(\sum_{i=1}^m\lambda_i h_i(\cdot,w_i)\right)^*+C^*\times\mathbb{R}_+\right).\end{aligned}$$

By assumption,

$$(0,0)\in\bigcup_{u\in\mathcal{U}}\operatorname{epi}\big(f(\cdot,u)\big)^*+\bigcup_{v\in\mathcal{V}}\operatorname{epi}\big(-\bar{r}g(\cdot,v)\big)^*+\bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0}\operatorname{epi}\left(\sum_{i=1}^m\lambda_i h_i(\cdot,w_i)\right)^*+C^*\times\mathbb{R}_+.$$

Since $C^*\times\mathbb{R}_+=\operatorname{epi}\delta_C^*$, there exist $\bar{u}\in\mathcal{U}$, $\bar{v}\in\mathcal{V}$, $\bar{w}_i\in\mathcal{W}_i$, and $\bar{\lambda}_i\geqq 0$, $i=1,\dots,m$, such that

$$(0,0)\in\operatorname{epi}\big(f(\cdot,\bar{u})\big)^*+\operatorname{epi}\big(-\bar{r}g(\cdot,\bar{v})\big)^*+\operatorname{epi}\left(\sum_{i=1}^m\bar{\lambda}_i h_i(\cdot,\bar{w}_i)\right)^*+\operatorname{epi}\delta_C^*.$$

By Proposition 2.2, we obtain

$$\begin{aligned}(0,0)\in{}&\bigcup_{\epsilon_0^1\geqq 0}\big\{\big(\xi_0^1,\langle\xi_0^1,\bar{x}\rangle+\epsilon_0^1-f(\bar{x},\bar{u})\big)\mid\xi_0^1\in\partial_{\epsilon_0^1}\big(f(\cdot,\bar{u})\big)(\bar{x})\big\}\\&+\bigcup_{\epsilon_0^2\geqq 0}\big\{\big(\xi_0^2,\langle\xi_0^2,\bar{x}\rangle+\epsilon_0^2+\bar{r}g(\bar{x},\bar{v})\big)\mid\xi_0^2\in\partial_{\epsilon_0^2}\big(-\bar{r}g(\cdot,\bar{v})\big)(\bar{x})\big\}\end{aligned}$$

$$+ \bigcup_{\epsilon^* \geqq 0} \left\{ \left(\xi^*, \langle \xi^*, \bar{x} \rangle + \epsilon^* - \sum_{i=1}^{m} \bar{\lambda}_i h_i(\bar{x}, \bar{w}_i) \right) \,\middle|\, \xi^* \in \partial_{\epsilon^*} \left(\sum_{i=1}^{m} \bar{\lambda}_i h_i(\cdot, \bar{w}_i) \right)(\bar{x}) \right\}$$
$$+ \bigcup_{\epsilon_{m+1} \geqq 0} \left\{ \left(\xi_{m+1}, \langle \xi_{m+1}, \bar{x} \rangle + \epsilon_{m+1} - \delta_C(\bar{x}) \right) \mid \xi_{m+1} \in \partial_{\epsilon_{m+1}} \delta_C(\bar{x}) \right\}.$$

So, there exist $\bar{\xi}_0^1 \in \partial_{\epsilon_0^1}(f(\cdot, \bar{u}))(\bar{x})$, $\bar{\xi}_0^2 \in \partial_{\epsilon_0^2}(-\bar{r}g(\cdot, \bar{v}))(\bar{x})$, $\bar{\xi}^* \in \partial_{\epsilon^*}(\sum_{i=1}^m \bar{\lambda}_i h_i(\cdot, \bar{w}_i))(\bar{x})$, $\bar{\xi}_{m+1} \in \partial_{\epsilon_{m+1}} \delta_C(\bar{x})$, $\epsilon_0^1 \geqq 0$, $\epsilon_0^2 \geqq 0$, $\epsilon^* \geqq 0$, and $\epsilon_{m+1} \geqq 0$ such that

$$0 = \bar{\xi}_0^1 + \bar{\xi}_0^2 + \bar{\xi}^* + \bar{\xi}_{m+1} \quad \text{and} \quad \epsilon_0^1 + \epsilon_0^2 + \epsilon^* + \epsilon_{m+1} = f(\bar{x}, \bar{u}) - \bar{r}g(\bar{x}, \bar{v}) + \sum_{i=1}^{m} \bar{\lambda}_i h_i(\bar{x}, \bar{w}_i).$$

By Proposition 2.5, there exist $\bar{\xi}_0^1 \in \partial_{\epsilon_0^1}(f(\cdot, \bar{u}))(\bar{x})$, $\bar{\xi}_0^2 \in \partial_{\epsilon_0^2}(-\bar{r}g(\cdot, \bar{v}))(\bar{x})$, $\bar{\xi}_i \in \partial_{\epsilon_i}(\bar{\lambda}_i h_i(\cdot, \bar{w}_i))(\bar{x})$, $\bar{\xi}_{m+1} \in \partial_{\epsilon_{m+1}} \delta_C(\bar{x})$, $\epsilon_0^1 \geqq 0$, $\epsilon_0^2 \geqq 0$, $\epsilon_i \geqq 0$, $i = 1, \ldots, m$, and $\epsilon_{m+1} \geqq 0$ such that

$$\begin{aligned} &0 \in \partial_{\epsilon_0^1}\big(f(\cdot, \bar{u})\big)(\bar{x}) + \partial_{\epsilon_0^2}\big(-\bar{r}g(\cdot, \bar{v})\big)(\bar{x}) \\ &\quad + \sum_{i=1}^{m} \partial_{\epsilon_i}\big(\bar{\lambda}_i h_i(\cdot, \bar{w}_i)\big)(\bar{x}) + N_C^{\epsilon_{m+1}}(\bar{x}) \quad \text{and} \\ &\epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1} \epsilon_i = f(\bar{x}, \bar{u}) - \bar{r}g(\bar{x}, \bar{v}) + \sum_{i=1}^{m} \bar{\lambda}_i h_i(\bar{x}, \bar{w}_i). \end{aligned} \tag{15}$$

Since $\bar{r} = \max_{(u,v) \in \mathcal{U} \times \mathcal{V}} \frac{f(\bar{x}, u)}{g(\bar{x}, v)} - \epsilon$, we have $\max_{u \in \mathcal{U}} f(\bar{x}, u) - \bar{r} \min_{v \in \mathcal{V}} g(\bar{x}, v) = \epsilon \min_{v \in \mathcal{V}} g(\bar{x}, v)$. So, we have

$$f(\bar{x}, \bar{u}) - \bar{r}g(\bar{x}, \bar{v}) \leqq \max_{u \in \mathcal{U}} f(\bar{x}, u) - \bar{r} \min_{v \in \mathcal{V}} g(\bar{x}, v) = \epsilon \min_{v \in \mathcal{V}} g(\bar{x}, v). \tag{16}$$

So, the condition (13) holds. Also, from (15) and (16), we have

$$\begin{aligned} &0 \in \partial_{\epsilon_0^1}\Big(\max_{u \in \mathcal{U}} f(\cdot, u)\Big)(\bar{x}) + \partial_{\epsilon_0^2}\Big(-\bar{r} \min_{v \in \mathcal{V}} g(\cdot, v)\Big)(\bar{x}) \\ &\quad + \sum_{i=1}^{m} \partial_{\epsilon_i}\big(\bar{\lambda}_i h_i(\cdot, \bar{w}_i)\big)(\bar{x}) + N_C^{\epsilon_{m+1}}(\bar{x}) \quad \text{and} \\ &\epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1} \epsilon_i - \epsilon \min_{v \in \mathcal{V}} g(\bar{x}, v) \leqq \sum_{i=1}^{m} \bar{\lambda}_i h_i(\bar{x}, \bar{w}_i). \end{aligned}$$

Consequently, the conditions (12) and (14) hold.

[(ii) ⇒ (i)] Taking account of the converse of the process for proving (i) ⇒ (ii), we can easily check that the statement (ii) ⇒ (i) holds. □

Remark 3.3 Assume that $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ are functions such that, for all $x \in \mathbb{R}^n$, $f(x, \cdot)$, and $g(x, \cdot)$ are concave and convex, respectively. Then we know that Theorem 3.2 is equivalent to Theorem 3.3 from Lemma 3.1, immediately.

4 ϵ-Duality theorems

Following the approach in [13], we formulate a dual problem (RFD) for (RFP) as follows:

$$\begin{aligned}
\text{(RFD)} \quad & \max r \\
& \text{s.t. } 0 \in \partial_{\epsilon_0^1}\Big(\max_{u\in\mathcal{U}} f(\cdot,u)\Big)(x) + \partial_{\epsilon_0^2}\Big(-r\min_{v\in\mathcal{V}} g(\cdot,v)\Big)(x) \\
& \qquad + \sum_{i=1}^{m} \partial_{\epsilon_i}\big(\lambda_i h_i(\cdot,w_i)\big)(x) + N_C^{\epsilon_{m+1}}(x), \\
& \max_{u\in\mathcal{U}} f(x,u) - r\min_{v\in\mathcal{V}} g(x,v) \geqq \epsilon \min_{v\in\mathcal{V}} g(x,v), \\
& \epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1}\epsilon_i - \epsilon\min_{v\in\mathcal{V}} g(x,v) \leqq \sum_{i=1}^{m}\lambda_i h_i(x,w_i), \\
& r \geqq 0, w_i \in \mathcal{W}_i, \lambda_i \geqq 0, i = 1,\ldots,m, \\
& \epsilon_0^1 \geqq 0, \epsilon_0^2 \geqq 0, \epsilon_i \geqq 0, i = 1,\ldots,m+1.
\end{aligned}$$

Clearly,

$$\begin{aligned}
F := \Bigg\{ (x,w,\lambda,r) \,\Bigg|\, & 0 \in \partial_{\epsilon_0^1}\Big(\max_{u\in\mathcal{U}} f(\cdot,u)\Big)(x) + \partial_{\epsilon_0^2}\Big(-r\min_{v\in\mathcal{V}} g(\cdot,v)\Big)(x) \\
& + \sum_{i=1}^{m} \partial_{\epsilon_i}\big(\lambda_i h_i(\cdot,w_i)\big)(x) + N_{\mathbb{R}_+}^{\epsilon_2}(x), \max_{u\in\mathcal{U}} f(x,u) - r\min_{v\in\mathcal{V}} g(x,v) \geqq \epsilon g(x,v), \\
& \epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1}\epsilon_i - \epsilon\min_{v\in\mathcal{V}} g(x,v) \leqq \sum_{i=1}^{m}\lambda_i h_i(x,w_i), \\
& r \geqq 0, w_i \in \mathcal{W}_i, \lambda_i \geqq 0, \epsilon_0^1 \geqq 0, \epsilon_0^2 \geqq 0, \epsilon_i \geqq 0, i = 1,\ldots,m, \epsilon_{m+1} \geqq 0 \Bigg\}
\end{aligned}$$

is the feasible set of (RFD).

Let $\epsilon \geqq 0$. Then $(\bar{x},\bar{w},\bar{\lambda},\bar{r})$ is called an ϵ-solution of (RFD) if, for any $(y,w,\lambda,r) \in F$, $\bar{r} \geqq r - \epsilon$.

When $\epsilon = 0$, $\max_{u\in\mathcal{U}} f(x,u) = f(x)$, $\min_{v\in\mathcal{V}} g(x,v) = g(x)$, and $h_i(x,w_i) = h_i(x)$, $i = 1,\ldots,m$, (RFP) becomes (FP), and (RFD) collapses to the Mond-Weir type dual problem (FD) for (FP) as follows [28]:

$$\begin{aligned}
\text{(FD)} \quad & \max r \\
& \text{s.t. } 0 \in \partial f(x) + \partial(-rg)(x) + \sum_{i=1}^{m}\partial\lambda_i h_i(x) + N_C(x), \\
& f(x) - rg(x) \geqq 0, \lambda_i h_i(x) \geqq 0, \\
& r \geqq 0, \lambda_i \geqq 0, i = 1,\ldots,m.
\end{aligned}$$

Now, we prove ϵ-weak and ϵ-strong duality theorems which hold between (RFP) and (RFD).

Theorem 4.1 (ϵ-Weak duality theorem) *For any feasible x of* (RFP) *and any feasible (y, w, λ, r) of* (RFD),

$$\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(x,u)}{g(x,v)} \geqq r - \epsilon.$$

Proof Let x and (y, w, λ, r) be feasible solutions of (RFP) and (RFD), respectively. Then there exist $\bar{\xi}_0^1 \in \partial_{\epsilon_0^1}(\max_{u\in\mathcal{U}} f(\cdot, u))(y)$, $\bar{\xi}_0^2 \in \partial_{\epsilon_0^2}(-r\min_{v\in\mathcal{V}} g(\cdot, v))(y)$, $\bar{\xi}_i \in \partial_{\epsilon_i}(\lambda_i h_i(\cdot, w_i))(y)$, $\bar{\xi}_{m+1} \in N_C^{\epsilon_{m+1}}(y)$, $\epsilon_0^1 \geqq 0$, $\epsilon_0^2 \geqq 0$, $\epsilon_i \geqq 0$, $i = 1, \ldots, m$, and $\epsilon_{m+1} \geqq 0$ such that

$$\bar{\xi}_0^1 + \bar{\xi}_0^2 + \sum_{i=1}^{m+1} \bar{\xi}_i = 0, \qquad \max_{u\in\mathcal{U}} f(y,u) - r \min_{v\in\mathcal{V}} g(y,v) \geqq \epsilon \min_{v\in\mathcal{V}} g(y,v) \quad \text{and}$$

$$\epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1} \epsilon_i - \epsilon \min_{v\in\mathcal{V}} g(y,v) \leqq \sum_{i=1}^{m} \lambda_i h_i(y, w_i).$$

Thus, we have

$$\begin{aligned}
&\max_{u\in\mathcal{U}} f(x,u) - r\min_{v\in\mathcal{V}} g(x,v) + \epsilon \min_{v\in\mathcal{V}} g(x,v) \\
&\quad \geqq \max_{u\in\mathcal{U}} f(y,u) - r\min_{v\in\mathcal{V}} g(y,v) + \langle \bar{\xi}_0^1 + \bar{\xi}_0^2, x - y\rangle - \epsilon_0^1 - \epsilon_0^2 + \epsilon \min_{v\in\mathcal{V}} g(x,v) \\
&\quad = \max_{u\in\mathcal{U}} f(y,u) - r\min_{v\in\mathcal{V}} g(y,v) - \left\langle \sum_{i=1}^{m+1} \bar{\xi}_i, x - y \right\rangle - \epsilon_0^1 - \epsilon_0^2 + \epsilon \min_{v\in\mathcal{V}} g(x,v) \\
&\quad \geqq \max_{u\in\mathcal{U}} f(y,u) - r\min_{v\in\mathcal{V}} g(y,v) + \sum_{i=1}^{m} \lambda_i h_i(y,w_i) - \sum_{i=1}^{m} \lambda_i h_i(x,w_i) - \epsilon_0^1 - \epsilon_0^2 - \sum_{i=1}^{m+1} \epsilon_i \\
&\qquad + \epsilon \min_{v\in\mathcal{V}} g(x,v) \\
&\quad \geqq \max_{u\in\mathcal{U}} f(y,u) - r\min_{v\in\mathcal{V}} g(y,v) + \sum_{i=1}^{m} \lambda_i h_i(y,w_i) - \epsilon_0^1 - \epsilon_0^2 - \sum_{i=1}^{m+1} \epsilon_i \\
&\quad \geqq \max_{u\in\mathcal{U}} f(y,u) - r\min_{v\in\mathcal{V}} g(y,v) - \epsilon \min_{v\in\mathcal{V}} g(y,v) \\
&\quad \geqq 0.
\end{aligned}$$

Hence, we have $\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(x,u)}{g(x,v)} \geqq r - \epsilon$. □

Theorem 4.2 (ϵ-Strong duality theorem) *Suppose that*

$$\bigcup_{w_i\in\mathcal{W}_i, \lambda_i\geqq 0} \operatorname{epi}\left(\sum_{i=1}^{m} \lambda_i g_i(\cdot, w_i)\right)^* + C^* \times \mathbb{R}_+$$

is closed. If $\bar{x}$ is an ϵ-solution of (RFP) *and $\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon \geqq 0$, then there exist $\bar{w} \in \mathbb{R}^q$, $\bar{\lambda} \in \mathbb{R}_+^m$, and $\bar{r} \in \mathbb{R}_+$ such that $(\bar{x}, \bar{w}, \bar{\lambda}, \bar{r})$ is a 2ϵ-solution of* (RFD).

Proof Let $\bar{x} \in A$ be an ϵ-solution of (RFP). Let $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)}$. Then, by Theorem 3.2, there exist $\bar{w}_i \in \mathcal{W}_i$, $\bar{\lambda}_i \geqq 0$, $\epsilon_0^1 \geqq 0$, $\epsilon_0^2 \geqq 0$, $\epsilon_i \geqq 0$, $i = 1, \ldots, m$, and ϵ_{m+1} such that

$$0 \in \partial_{\epsilon_0^1}\Big(\max_{u\in\mathcal{U}} f(\cdot,u)\Big)(\bar{x}) + \partial_{\epsilon_0^2}\Big(-\bar{r}\min_{v\in\mathcal{V}} g(\cdot,v)\Big)(\bar{x}) + \sum_{i=1}^{m} \partial_{\epsilon_i}\big(\bar{\lambda}_i h_i(\cdot,\bar{w}_i)\big)(\bar{x}) + N_C^{\epsilon_{m+1}}(\bar{x}),$$

$$\max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) = \epsilon \min_{v\in\mathcal{V}} g(\bar{x},v) \quad \text{and}$$

$$\epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1}\epsilon_i - \epsilon\min_{v\in\mathcal{V}} g(\bar{x},v) = \sum_{i=1}^{m}\bar{\lambda}_i h_i(\bar{x},\bar{w}_i).$$

So, $(\bar{x},\bar{w},\bar{\lambda},\bar{r})$ is a feasible solution of (RFD). For any feasible (y,u,v,w,λ,v) of (RFD), it follows from Theorem 4.1 (ϵ-weak duality theorem) that

$$\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon \geqq r - \epsilon - \epsilon = r - 2\epsilon.$$

Thus $(\bar{x},\bar{w},\bar{\lambda},\bar{r})$ is a 2ϵ-solution of (RFD). □

Remark 4.1 Using the optimality conditions of Theorem 3.2, robust fractional dual problem (RFD) for a robust fractional problem (RFP) in the convex constraint functions with uncertainty is formulated. However, when we formulated the dual problem using optimality condition in Theorem 3.3, we could not know whether ϵ-weak duality theorem is established, or not. It is an open question.

Now we give an example illustrating our duality theorems.

Example 4.1 Consider the following fractional programming problem with uncertainty:

$$\begin{aligned}
\text{(RFP)} \quad & \min \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{ux+1}{vx+2} \\
& \text{s.t. } 2w_1 x - 3 \leqq 0, w_1 \in [1,2], \\
& \quad x \in \mathbb{R}_+,
\end{aligned}$$

where $\mathcal{U} = [1,2]$ and $\mathcal{V} = [1,2]$.

Now we transform the problem (RFP) into the robust non-fractional convex optimization problem $(\text{RNCP})_r$ with a parametric $r \in \mathbb{R}_+$:

$$\begin{aligned}
(\text{RNCP})_r \quad & \min \max_{u\in[1,2]}(ux+1) - r\min_{v\in[1,2]}(vx+2) \\
& \text{s.t. } 2w_1 x - 3 \leqq 0, w_1 \in [1,2], \\
& \quad x \in \mathbb{R}_+.
\end{aligned}$$

Let $f(x,u) = ux+1$, $g(x,v) = vx+2$, $h_1(x,w_1) = -2w_1 x - 3$, and $\epsilon \in [0, \frac{9}{22}]$. Then $A := \{x \in \mathbb{R} \mid 0 \leqq x \leqq \frac{3}{4}\}$ is the set of all robust feasible solutions of (RFP) and $\bar{A} := \{x \in \mathbb{R} \mid 0 \leqq x \leqq \frac{4\epsilon}{3-2\epsilon}\}$ is the set of all ϵ-solutions of (RFP). Let $F := \{(y,w_1,\lambda_1,r) \mid 0 \in \partial_{\epsilon_0^1}(\max_{u\in\mathcal{U}} f(\cdot,u))(y) + \partial_{\epsilon_0^2}(-r\min_{v\in\mathcal{V}} g(\cdot,v))(y) + \partial_{\epsilon_1}(\lambda_1 h_1(\cdot,w_1))(y) + N_{\mathbb{R}_+}^{\epsilon_2}(x), \max_{u\in\mathcal{U}} f(y,u) - r\min_{v\in\mathcal{V}} g(y,v) \geqq$

$\epsilon \min_{v\in\mathcal{V}} g(y,v), \epsilon_0^1+\epsilon_0^2+\epsilon_1+\epsilon_2-\epsilon \min_{v\in\mathcal{V}} g(y,v) \leqq \lambda_1 h_1(y,w_1), r\geqq 0, w_1\in[1,2], \lambda_1\geqq 0, \epsilon_0^1\geqq 0, \epsilon_0^2\geqq 0, \epsilon_1\geqq 0, \epsilon_2\geqq 0\}$. Then we formulate a dual problem (RFD) for (RFP) as follows:

$$\text{(RFD)}\quad \max r \quad \text{s.t.}\ (y,w_1,\lambda_1,r)\in F.$$

Then F is the set of all robust feasible solutions of (RFD).

Now we calculate the set $F=\widetilde{A}\cup\widetilde{B}$.

$$\begin{aligned}
\widetilde{A} :=& \Big\{(0,w_1,\lambda_1,r)\mid 0\in \partial_{\epsilon_0^1}\Big(\max_{u\in\mathcal{U}} f(\cdot,u)\Big)(0) \\
&+\partial_{\epsilon_0^2}\Big(-r\min_{v\in\mathcal{V}} g(\cdot,v)\Big)(0)+\partial_{\epsilon_1}\big(\lambda_1 h_1(\cdot,w_1)\big)(0) \\
&+N_{\mathbb{R}_+}^{\epsilon_2}(0), \max_{u\in\mathcal{U}} f(0,u)-r\min_{v\in\mathcal{V}} g(0,v)\geqq \epsilon\min_{v\in\mathcal{V}} g(0,v), \epsilon_0^1+\epsilon_0^2+\epsilon_1+\epsilon_2-\epsilon\min_{v\in\mathcal{V}} g(0,v) \\
&\leqq \lambda_1 h_1(0,w_1), r\geqq 0, u\in[1,2], \lambda_1\geqq 0, \epsilon_0^1\geqq 0, \epsilon_0^2\geqq 0, \epsilon_1\geqq 0, \epsilon_2\geqq 0\Big\} \\
=& \big\{(0,w_1,\lambda_1,r)\mid 0\in\{2\}+\{-r\}+\{2\lambda_1 w_1\}+(-\infty,0], 1-2r\geqq 2\epsilon, \epsilon_0^1+\epsilon_0^2+\epsilon_1+\epsilon_2 \\
&-2\epsilon\leqq -3\lambda_1, r\geqq 0, w_1\in[1,2], \lambda_1\geqq 0, \epsilon_0^1\geqq 0, \epsilon_0^2\geqq 0, \epsilon_1\geqq 0, \epsilon_2\geqq 0\big\} \\
=& \Big\{(0,w_1,\lambda_1,r)\ \Big|\ r\leqq 2+2\lambda_1 w_1, r\leqq \frac{1-2\epsilon}{2}, \epsilon_0^1+\epsilon_0^2+\epsilon_1+\epsilon_2-2\epsilon\leqq -3\lambda_1, r\geqq 0, \\
&w_1\in[1,2], \lambda_1\geqq 0, \epsilon_0^1\geqq 0, \epsilon_0^2\geqq 0, \epsilon_1\geqq 0, \epsilon_2\geqq 0\Big\}, \\
\widetilde{B} :=& \Big\{(y,w_1,\lambda_1,r)\mid 0\in \partial_{\epsilon_0^1}\Big(\max_{u\in\mathcal{U}} f(\cdot,u)\Big)(y)+\partial_{\epsilon_0^2}\Big(-r\min_{v\in\mathcal{V}} g(\cdot,v)\Big)(y)+\partial_{\epsilon_1}\big(\lambda_1 h_1(\cdot,w_1)\big)(y) \\
&+N_{\mathbb{R}_+}^{\epsilon_2}(y), \max_{u\in\mathcal{U}} f(y,u)-r\min_{v\in\mathcal{V}} g(y,v)\geqq \epsilon\min_{v\in\mathcal{V}} g(y,v), \epsilon_0^1+\epsilon_0^2+\epsilon_1+\epsilon_2-\epsilon\min_{v\in\mathcal{V}} g(y,v) \\
&\leqq \lambda_1 h_1(y,w_1), y>0, r\geqq 0, w_1\in[1,2], \lambda_1\geqq 0, \epsilon_0^1\geqq 0, \epsilon_0^2\geqq 0, \epsilon_1\geqq 0, \epsilon_2\geqq 0\Big\} \\
=& \Big\{(y,w_1,\lambda_1,r)\ \Big|\ 0\in\{2\}+\{-r\}+\{2\lambda_1 w_1\}+\Big[-\frac{\epsilon_2}{y},0\Big], 2y+1-r(y+2)\geqq \epsilon(y+2), \\
&y>0, \epsilon_0^1+\epsilon_0^2+\epsilon_1+\epsilon_2-\epsilon(y+2)\leqq \lambda_1(2w_1 y-3), r\geqq 0, w_1\in[1,2], \lambda_1\geqq 0, \\
&\epsilon_0^1\geqq 0, \epsilon_0^2\geqq 0, \epsilon_1\geqq 0, \epsilon_2\geqq 0\Big\} \\
=& \Big\{(y,w_1,\lambda_1,r)\ \Big|\ 0\in\Big[2-r+2\lambda_1 w_1-\frac{\epsilon_2}{y}, 2-r+2\lambda_1 w_1\Big], 2y+1-r(y+2)\geqq \\
&\epsilon(y+2), \epsilon_0^1+\epsilon_0^2+\epsilon_1+\epsilon_2-\epsilon(y+2)\leqq \lambda_1(2w_1 y-3), y>0, r\geqq 0, w_1\in[1,2], \\
&\lambda_1\geqq 0, \epsilon_0^1\geqq 0, \epsilon_0^2\geqq 0, \epsilon_1\geqq 0, \epsilon_2\geqq 0\Big\}.
\end{aligned}$$

We can check for any $x\in A$ and any $(y,w_1,\lambda_1,r)\in F$,

$$\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(x,u)}{g(x,v)}\geqq r-\epsilon,$$

that is, ϵ-weak duality holds. Indeed, let $x \in A$ and $(y, w_1, \lambda_1, r) \in \widetilde{A}$ be fixed. Then

$$\begin{aligned}
&\max_{u\in[1,2]} f(x,u) - r \min_{v\in[1,2]} g(x,v) + \epsilon \min_{v\in[1,2]} g(x,v) \\
&\quad = 2x + 1 - r(x+2) + \epsilon(x+2) \\
&\quad = (2-r)x + 1 - 2r + \epsilon(x+2) \\
&\quad \geqq -2\lambda_1 w_1 x + 2\epsilon + \epsilon(x+2) \\
&\quad \geqq -3\lambda_1 + 2\epsilon + \epsilon(x+2) \\
&\quad \geqq -3\lambda_1 + \epsilon_0^1 + \epsilon_0^2 + \epsilon_1 + \epsilon_2 + 3\lambda_1 + \epsilon(x+2) \\
&\quad \geqq 0.
\end{aligned}$$

Moreover, let $x \in A$ and $(y, u, v, w_1, \lambda_1, r) \in \widetilde{B}$ be fixed.

$$\begin{aligned}
&\max_{u\in[1,2]} f(x,u) - r \min_{v\in[1,2]} g(x,v) + \epsilon \min_{v\in[1,2]} g(x,v) \\
&\quad = 2x + 1 - r(x+2) + \epsilon(x+2) \\
&\quad = 2y + 1 - r(y+2) + (2-r)(x-y) + \epsilon(x+2).
\end{aligned}$$

If $x - y \geqq 0$, then

$$\begin{aligned}
&\max_{u\in[1,2]} f(x,u) - r \min_{v\in[1,2]} g(x,v) + \epsilon \min_{v\in[1,2]} g(x,v) \\
&\quad = 2y + 1 - r(y+2) + (2-r)(x-y) + \epsilon(x+2) \\
&\quad \geqq 2y + 1 - r(y+2) - 2\lambda_1 w_1(x-y) + \epsilon(x+2) \\
&\quad \geqq 2y + 1 - r(y+2) + 2\lambda_1 w_1 y - 2\lambda_1 w_1 x + \epsilon(x+2) \\
&\quad \geqq \epsilon(y+2) + 2\lambda_1 w_1 y - 3\lambda_1 + \epsilon(x+2) \\
&\quad \geqq \epsilon_0^1 + \epsilon_0^2 + \epsilon_1 + \epsilon_2 + 3\lambda_1 - 3\lambda_1 + \epsilon(x+2) \\
&\quad \geqq 0.
\end{aligned}$$

If $x - y < 0$, then

$$\begin{aligned}
&\max_{u\in[1,2]} f(x,u) - r \min_{v\in[1,2]} g(x,v) + \epsilon \min_{v\in[1,2]} g(x,v) \\
&\quad = 2y + 1 - r(y+2) + (2-r)(x-y) + \epsilon(x+2) \\
&\quad \geqq 2y + 1 - r(y+2) + \left(-2\lambda_1 w_1 + \frac{\epsilon_2}{y}\right)(x-y) + \epsilon(x+2) \\
&\quad \geqq 2y + 1 - r(y+2) + 2\lambda_1 w_1 y - \epsilon_2 - 2\lambda_1 w_1 x + \frac{\epsilon_2}{y}x + \epsilon(x+2) \\
&\quad \geqq \epsilon(y+2) + 2\lambda_1 w_1 y - \epsilon_2 - 3\lambda_1 + \frac{\epsilon_2}{y}x + \epsilon(x+2) \\
&\quad \geqq \epsilon_0^1 + \epsilon_0^2 + \epsilon_1 + \epsilon_2 + 3\lambda_1 - \epsilon_2 - 3\lambda_1 + \frac{\epsilon_2}{y}x + \epsilon(x+2) \\
&\quad \geqq 0.
\end{aligned}$$

Let $\epsilon = \frac{1}{3}$. Then $\bar{x} \in \bar{A} := \{x \in \mathbb{R} \mid 0 \leqq x \leqq \frac{4}{7}\}$ is the set of all ϵ-solutions of (RFP) and $\frac{1}{6} \leqq \bar{r} \leqq \frac{1}{2}$.

If $\bar{x} = 0$, then $\bar{r} = \frac{1}{6}$. When $\epsilon = \frac{1}{3}$, we can calculate the set $\tilde{A}$ as follows:

$$\tilde{A} := \left\{(0, w_1, \lambda_1, r) \,\middle|\, 0 \leqq r \leqq \frac{1}{6}, 0 \leqq \lambda_1 \leqq \frac{2}{9}, w_1 \in [1,2]\right\}.$$

Let $\bar{w}_1 = 2$, $\bar{\lambda}_1 = \frac{1}{9}$. Then $(0, 2, \frac{5}{8}, \frac{1}{9}) \in \tilde{A}$. So, we have

$$\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x}, u)}{g(\bar{x}, v)} - \epsilon = \frac{1}{6} = \frac{5}{6} - 2\epsilon \geqq r - 2\epsilon.$$

Hence, $(0, 2, \frac{1}{9}, \frac{1}{6})$ is a 2ϵ-solution of (RFD). So, ϵ-strong duality holds. If $0 < \bar{x} \leqq \frac{4}{7}$, then $\frac{1}{6} < \bar{r} \leqq \frac{1}{2}$. When $\epsilon = \frac{1}{3}$, we can calculate the set $\tilde{B}$ as follows:

$$\widetilde{B} := \left\{(y, w_1, \lambda_1, r) \,\middle|\, y > 0, 2 + 2\lambda_1 w_1 - \frac{\epsilon_2}{y} \leqq r \leqq \frac{5y+1}{3(y+2)}, \epsilon_2 - \frac{1}{3}(y+2) \leqq \right.$$
$$\left. \lambda_1(2w_1 y - 3), r \geqq 0, u \in [1,2], v \in [1,2], w_1 \in [1,2], \epsilon_2 \geqq 0\right\}.$$

Let $\bar{w}_1 = 2$, $\bar{\lambda}_1 = 0$, and $\epsilon_2 = \frac{\bar{x}+2}{3}$. Then $(\bar{x}, 2, 0, \frac{5\bar{x}+1}{3(\bar{x}+2)}) \in \tilde{B}$. So, we have

$$\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x}, u)}{g(\bar{x}, v)} - \epsilon = \frac{5\bar{x}+1}{3(\bar{x}+2)} \geqq r \geqq r - 2\epsilon.$$

Hence, $(\bar{x}, 2, 0, \frac{5\bar{x}+1}{3(\bar{x}+2)})$ is a 2ϵ-solution of (RFD). So, ϵ-strong duality holds.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this paper and they read and approved the final manuscript.

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