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On new generalizations of Smarzewski's fixed point theorem

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Abstract
In this work, we prove some new generalizations of Smarzewski's fixed point theorem and some new fixed point theorems which are original and quite different from the well-known results in the literature.
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Keywords: strictly convex Banach space; uniformly convex Banach space; uniformly convex in every direction (*UCED*); λ-firmly nonexpansive mapping; asymptotic radius; asymptotic center; (*UAC*)-property; reactive firmly nonexpansive mapping; Smarzewski's fixed point theorem

1 Introduction and preliminaries

Let $(X, \|\cdot\|)$ be a normed space with its zero vector θ. We use $B(X)$ and $S(X)$ to denote respectively the *closed unit ball* and *unit sphere* centered at θ with radius 1, that is,

$$B(X) = \{x \in X : \|x\| \leq 1\}$$

and

$$S(X) = \{x \in X : \|x\| = 1\}.$$

The notion of uniformly convex (*UC*, for short) Banach space was introduced by Clarkson [1], and the research of geometric properties of the Banach space started from 1936. The function $\delta_X : [0,2] \to [0,1]$, defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B(X), \|x-y\| \geq \varepsilon\right\} \quad \text{for } \varepsilon \in [0,2],$$

is called the *modulus of convexity of* X. The normed space X is called *uniformly convex* if $\delta_X(\varepsilon) > 0$ for every $\varepsilon \in (0,2]$. It is well known that a uniformly convex Banach space is reflexive and all Hilbert spaces and Banach spaces ℓ^p and L^p $(1 < p < \infty)$ all are uniformly convex; see, *e.g.*, [2–7] for more details. The normed space X is said to be *strictly convex* if $\|x + y\| < 2$ whenever $x, y \in S(X)$ with $\|x - y\| > 0$. It is obvious that a Banach space X is strictly convex if and only if $\delta_X(2) = 1$. It is well known that the strict convexity of a normed space X can be characterized by the properties: for any nonzero vectors $x, y \in X$, if

$\|x+y\| = \|x\| + \|y\|$, then $y = cx$ for some real $c > 0$. For each $\varepsilon > 0$, the *modulus of convexity of X in the direction $z \in S(X)$* is defined by

$$\delta_X(\varepsilon, z) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B(X), x - y = \lambda z, |\lambda| \geq \varepsilon\right\}.$$

Clearly, $\delta_X(\varepsilon) = \inf\{\delta_X(\varepsilon, z) : z \in S(X)\}$. The Banach space X is called *uniformly convex in every direction* (*UCED*, for short) if for any $z \in S(X)$ and $\varepsilon > 0$, $\delta(\varepsilon, z) > 0$. Some characterizations of *UCED* Banach spaces were proved by Day *et al.* [7]; see also [4].

Fact 1.1 (see, *e.g.*, [2–4, 7])
(a) Every *UC* Banach space is *UCED*.
(b) Every *UCED* Banach space is strictly convex.

Let $(X, \|\cdot\|)$ be a Banach space and K be a given nonempty closed subset of X. For $x \in X$ and a bounded sequence $\{x_n\} \subset X$, define the *asymptotic radius of* $\{x_n\}$ *at* x as the number

$$r\bigl(x, \{x_n\}\bigr) = \limsup_{n\to\infty} \|x - x_n\|.$$

The *asymptotic radius of* $\{x_n\}$ *with respect to* K is defined by

$$r\bigl(K, \{x_n\}\bigr) = \inf\bigl\{r\bigl(x, \{x_n\}\bigr) : x \in K\bigr\},$$

and the set

$$A\bigl(K, \{x_n\}\bigr) = \bigl\{x \in K : r\bigl(x, \{x_n\}\bigr) = r\bigl(K, \{x_n\}\bigr)\bigr\}$$

is called the *asymptotic center of* $\{x_n\}$ *with respect to* K. For any bounded sequence $\{x_n\}$ in X, $r(x, \{x_n\})$ is easily seen to be a nonnegative, continuous and convex functional of $x \in X$. Moreover, if K is a nonempty convex subset of X, then $A(K, \{x_n\})$ is also convex.

Fact 1.2 [8, Lemma 2.2] Every bounded sequence in a UCED Banach space X has a unique asymptotic center with respect to any nonempty weakly compact convex subset of X.

Definition 1.1 A normed space $(X, \|\cdot\|)$ is said to have the (*UAC*)*-property* if every bounded sequence in X has a unique asymptotic center with respect to any nonempty weakly compact convex subset of X.

According to Facts 1.1 and 1.2, it is easy to know that Hilbert spaces, *UC* Banach spaces and *UCED* Banach spaces all have the (*UAC*)-property.

Let C be a nonempty subset of a normed space $(X, \|\cdot\|)$ and $T : C \to X$ be a mapping. T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

The concept of firmly nonexpansive mappings was introduced by Bruck [9]. Let $\lambda \in (0, 1)$. The mapping T is said to be λ*-firmly nonexpansive* [9] if

$$\|Tx - Ty\| \leq \bigl\|(1-\lambda)(x - y) + \lambda(Tx - Ty)\bigr\| \quad \text{for all } x, y \in C.$$

It is obvious that every λ-firmly nonexpansive mapping is nonexpansive, but the converse is not true. The following example shows that there exists a nonexpansive mapping which is not a λ-firmly nonexpansive mapping for some $\lambda \in (0,1)$.

Example A Let $X = \mathbb{R}$ with the absolute-value norm $|\cdot|$ and $C = [2,10]$. Let $T : C \to X$ be defined by $Tx = -x$. Then T is a nonexpansive mapping. For $x = 6$, $y = 4$ and $\lambda = \frac{1}{2}$, we have

$$\left|T(x) - T(y)\right| = 2 > 0 = \left|(1-\lambda)(x-y) + \lambda\big(T(x) - T(y)\big)\right|,$$

which deduces that T is not a $\frac{1}{2}$-firmly nonexpansive mapping. In fact, T is not λ-firmly nonexpansive for all $\lambda \in (0,1)$.

In 1965, Browder [10], Kirk [11] and Göhde [12] proved respectively that every nonexpansive mapping T from a nonempty weakly compact convex subset K of a uniformly convex Banach space X into itself has a fixed point. It is know that the convexity of sets and mappings plays an important role in fixed point theory and the union of convex sets does not ensure that it is convex. In 1991, Smarzewski [13] proved the following interesting theorem.

Theorem 1.1 (Smarzewski [13]) *Let X be a uniformly convex Banach space and $C = \bigcup_{k=1}^{n} C_k$ be a finite union of nonempty weakly compact convex subsets C_k of X. If $T : C \to C$ is a λ-firmly nonexpansive mapping for some $\lambda \in (0,1)$, then T has a fixed point in C.*

Smarzewski's fixed point theorem (*i.e.*, Theorem 1.1) is not always true if T is merely nonexpansive, even in $X = \mathbb{R}$.

Example B [13] Let $X = \mathbb{R}$ with the absolute-value norm $|\cdot|$ and $C = [-2,-1] \cup [-2,-1]$. Then the mapping $T : C \to C$ defined by $Tx = -x$ is nonexpansive and fixed point free.

In this paper, in order to promote Smarzewski's fixed point theorem, we first introduce the concept of reactive firmly nonexpansive mappings.

Definition 1.2 Let C be a nonempty subset of a normed space $(X, \|\cdot\|)$ and $\varphi : C \times C \to [0,1)$ be a function. A mapping $T : C \to X$ is said to be *reactive firmly nonexpansive with respect to* φ if

$$\|Tx - Ty\| \le \left\|\big(1 - \varphi(x,y)\big)(x-y) + \varphi(x,y)(Tx - Ty)\right\| \quad \text{for all } x, y \in C.$$

Remark 1.1

(a) Every reactive firmly nonexpansive mapping is nonexpansive.

(b) It is obvious that any λ-firmly nonexpansive mapping is reactive firmly nonexpansive with respect to the function φ defined by $\varphi(s,t) = \lambda$ for all $(s,t) \in C \times C$.

Example C Let $X = \mathbb{R}$ with the absolute-value norm $|\cdot|$ and $C = [2,5] \cup [10,20]$. Let $T : C \to X$ be defined by

$$Tx := \begin{cases} -\frac{3}{5}x, & \text{if } x \in [2,5], \\ -\frac{1}{2}x, & \text{if } x \in [10,20]. \end{cases}$$

Then the following statements hold.

(a) T is a nonexpansive mapping.

(b) T is not $\frac{1}{3}$-firmly nonexpansive.

(c) Define $\varphi : C \times C \to [0,1)$ by

$$\varphi(s,t) := \begin{cases} \frac{1}{10}, & \text{if } s,t \in [2,5], \\ \frac{1}{3}, & \text{otherwise.} \end{cases}$$

Then T is reactive firmly nonexpansive with respect to φ.

Proof Obviously, statement (a) holds. To see (b), let $x = 4$ and $y = 3$. Since

$$\left|T(x) - T(y)\right| = \frac{3}{5} > \frac{7}{15} = \left|\left(1 - \frac{1}{3}\right)(x - y) + \frac{1}{3}(Tx - Ty)\right|,$$

we show that T is not $\frac{1}{3}$-firmly nonexpansive. Finally, we prove (c). We consider the following four possible cases to verify

$$|Tx - Ty| \le \left|\bigl(1 - \varphi(x,y)\bigr)(x - y) + \varphi(x,y)(Tx - Ty)\right| \tag{1.1}$$

for all $x, y \in C$.

Case 1. If $x, y \in [2,5]$, then

$$\left|T(x) - T(y)\right| = \frac{3}{5}|x - y|$$

and

$$\begin{aligned} \left|\bigl(1 - \varphi(x,y)\bigr)(x - y) + \varphi(x,y)(Tx - Ty)\right| &= \left|\left(1 - \frac{1}{10}\right)(x - y) + \frac{1}{10}(Tx - Ty)\right| \\ &= \frac{21}{25}|x - y|. \end{aligned}$$

So (1.1) holds for all $x, y \in [2,5]$.

Case 2. If $x \in [2,5]$ and $y \in [10,20]$, then

$$\left|T(x) - T(y)\right| = \frac{1}{2}y - \frac{3}{5}x$$

and

$$\begin{aligned} \left|\bigl(1 - \varphi(x,y)\bigr)(x - y) + \varphi(x,y)(Tx - Ty)\right| &= \left|\left(1 - \frac{1}{3}\right)(x - y) + \frac{1}{3}(Tx - Ty)\right| \\ &= \frac{1}{2}y - \frac{7}{15}x. \end{aligned}$$

Since

$$\left(\frac{1}{2}y - \frac{7}{15}x\right) - \left(\frac{1}{2}y - \frac{3}{5}x\right) = \frac{2}{15}x > 0,$$

we prove that (1.1) holds for all $x \in [2,5]$ and $y \in [10,20]$.

Case 3. If $x \in [10,20]$ and $y \in [2,5]$, then

$$\left|T(x) - T(y)\right| = \frac{1}{2}x - \frac{3}{5}y$$

and

$$\left|\bigl(1-\varphi(x,y)\bigr)(x-y) + \varphi(x,y)(Tx - Ty)\right| = \frac{1}{2}x - \frac{7}{15}y.$$

Since

$$\left(\frac{1}{2}x - \frac{7}{15}y\right) - \left(\frac{1}{2}x - \frac{3}{5}y\right) = \frac{2}{15}y > 0,$$

we prove that (1.1) holds for all $x \in [10,20]$ and $y \in [2,5]$.

Case 4. If $x, y \in [10,20]$, then

$$\left|T(x) - T(y)\right| = \frac{1}{2}|x-y|$$

and

$$\begin{aligned}\left|\bigl(1-\varphi(x,y)\bigr)(x-y) + \varphi(x,y)(Tx - Ty)\right| &= \left|\left(1-\frac{1}{3}\right)(x-y) + \frac{1}{3}(Tx-Ty)\right| \\ &= \frac{1}{2}|x-y|.\end{aligned}$$

So (1.1) holds for all $x, y \in [10,20]$.

By Cases 1-4, we verify that inequality (1.1) holds for all $x, y \in C$. Hence T is reactive firmly nonexpansive with respect to φ and (c) is proved. □

In this paper, we establish some generalizations of Smarzewski's fixed point theorem for reactive firmly nonexpansive mappings and some new fixed point theorems which are original and quite different from the well-known results in the literature.

2 New generalizations of Smarzewski's fixed point theorem and applications to fixed point theory

In this section, we first establish a new fixed point theorem for reactive firmly nonexpansive mappings which is generalized Smarzewski's fixed point theorem. We assume $0 < \varphi(s,t) < 1$ for all $(s,t) \in C \times C$ in the following main theorem.

Theorem 2.1 *Let X be a strictly convex Banach space with its zero vector θ and $C = \bigcup_{k=1}^{n} C_k$ be a finite union of nonempty weakly compact convex subsets C_k of X. Let $\varphi : C \times C \to (0,1)$ be a function and $T : C \to C$ be a mapping. Suppose that*

(a) *X has the (UAC)-property,*

(b) *T is reactive firmly nonexpansive with respect to φ.*

Then T has a fixed point in C.

Proof Let $z \in C$ be given. Since C is T-invariant and C is bounded, the sequence $\{T^j z\}_{j=1}^{\infty} \subset C$ is bounded. Define the functional $r : X \to [0,\infty)$, the asymptotic radius of $\{T^j z\}$ at $x \in X$, by

$$r(x) := r(x, \{T^j z\}) = \limsup_{j\to\infty} \|x - T^j z\|.$$

For each $1 \leq k \leq n$, let the number

$$r(C_k) := r(C_k, \{T^j z\}) = \inf\{r(x) : x \in C_k\}$$

and the set

$$A(C_k) := A(C_k, \{T^j z\}) = \{x \in C_k : r(x) = r(C_k)\}$$

be respectively the asymptotic radius and the asymptotic center of the sequence $\{T^j z\}$ with respect to C_k. Since X has the (UAC)-property, let $x_k \in C_k$ be the unique asymptotic center of $\{T^j z\}$ with respect to C_k for $1 \leq k \leq n$. So

$$r(x_k) = r(C_k) = \inf\{r(x) : x \in C_k\} \quad \text{for each } 1 \leq k \leq n.$$

For any k and j, since T is nonexpansive, we have

$$\|Tx_k - T^j z\| \leq \|x_k - T^{j-1} z\|,$$

which implies

$$r(Tx_k) = \limsup_{j\to\infty} \|Tx_k - T^j z\| \leq \limsup_{j\to\infty} \|x_k - T^{j-1} z\| = r(x_k) \quad \text{for all } k. \tag{2.1}$$

Let

$$m = \min\{r(x_k) : 1 \leq k \leq n\}.$$

Clearly, $m < \infty$. For arbitrary $x \in C = \bigcup_{k=1}^n C_k$, $x \in C_{k_x}$ for some k_x, $1 \leq k_x \leq n$. Thus we have

$$r(x) \geq r(C_k) = r(x_k) \geq m. \tag{2.2}$$

Taking the infimum for x over C yields $r(C, \{T^j z\}) \geq m$. Conversely, suppose $r(x_{k_0}) = m$ for some k_0, $1 \leq k_0 \leq n$. Then

$$m = r(x_{k_0}) \geq \inf_{x\in C} r(x) = r(C, \{T^j z\}). \tag{2.3}$$

By (2.2) and (2.3), we prove

$$r(C, \{T^j z\}) = m. \tag{2.4}$$

Put

$$L = \{x_k : r(x_k) = m\}.$$

It is obvious that $\sharp(L) \leq n$, where $\sharp(L)$ is the cardinal number of L. We claim that for $u \in C$ with $r(u) = m$ if and only if $u \in L$. Indeed, it suffices to show that if $u \in C$ with $r(u) = m$, then $u \in L$. Let $u \in C = \bigcup_{k=1}^{n} C_k$. Then there is some k_u, $1 \leq k_u \leq n$, such that $u \in C_{k_u}$. So we obtain

$$r(u) = m \leq r(x_{k_u}) \leq r(u),$$

and hence

$$r(u) = m = r(x_{k_u}).$$

By the uniqueness of asymptotic center x_{k_u}, we have $u = x_{k_u}$, which means that $u \in L$.

Next, we will prove $TL \subset L$ (*i.e.*, L is T-invariant). For any $u \in L$, since $r(u) = m$, from (2.1) and (2.4) it follows that

$$m = r(u) \geq r(Tu) \geq \inf_{x \in C} r(x) = r\left(C, \left\{T^j z\right\}\right) = m,$$

and hence $r(Tu) = m$. According to our claim, we get $Tu \in L$, which completes the assertion. Now, we show that T has a fixed point in L. Suppose to the contrary that T has no fixed point in L, that is, $Tu \neq u$ for all $u \in L$. Then $Tx_k \notin C_k$ for all $x_k \in L$. Indeed, if $Tx_k \in C_k$ for some $x_k \in L$, since

$$m = r(x_k) \geq r(Tx_k) \geq \inf_{x \in C_k} r(x) = r(x_k) = m,$$

and by the uniqueness of asymptotic center x_k, we obtain $Tx_k = x_k$. This means that x_k is a fixed point for T, contradicting our assumption, hence $Tx_k \notin C_k$ for all $x_k \in L$. For $TL \subset L$, there exists $w \in L$ such that $T^\alpha w = w$ for some $1 \leq \alpha \leq \sharp(L)$. Since

$$0 \neq \|w - Tw\| = \left\|T^\alpha w - T^{\alpha+1} w\right\| \leq \left\|T^{\alpha-1} w - T^\alpha w\right\| \leq \cdots \leq \left\|Tw - T^2 w\right\| \leq \|w - Tw\|,$$

it follows that

$$0 < \xi := \|w - Tw\| = \left\|Tw - T^2 w\right\| = \cdots = \left\|T^{\alpha-1} w - T^\alpha w\right\| = \left\|T^\alpha w - T^{\alpha+1} w\right\|. \tag{2.5}$$

Since T is reactive firmly nonexpansive with respect to φ, we have

$$\begin{aligned}
\xi &= \left\|T^i w - T^{i+1} w\right\| \\
&\leq \left\|\left(1 - \varphi\left(T^{i-1} w, T^i w\right)\right)\left(T^{i-1} w - T^i w\right) + \varphi\left(T^{i-1} w, T^i w\right)\left(T^i w - T^{i+1} w\right)\right\| \\
&\leq \left(1 - \varphi\left(T^{i-1} w, T^i w\right)\right)\left\|T^{i-1} w - T^i w\right\| + \varphi\left(T^{i-1} w, T^i w\right)\left\|T^i w - T^{i+1} w\right\| \\
&= \xi
\end{aligned}$$

for $1 \leq i \leq \alpha$, where $T^0 = I$ (the identity mapping). Therefore, in view of strict convexity of the norm, there is $t_i > 0$ such that

$$T^{i-1} w - T^i w = t_i\left(T^i w - T^{i+1} w\right) \quad \text{for } 1 \leq i \leq \alpha. \tag{2.6}$$

From (2.5) and (2.6), we have

$$\xi = \left\| T^{i-1}w - T^{i}w \right\| = t_i \left\| T^{i}w - T^{i+1}w \right\| = t_i \xi,$$

which implies that $t_i = 1$ for all i. So, by (2.6) again, we get

$$\theta \neq v := w - Tw = Tw - T^2 w = \cdots = T^{\alpha-1}w - T^{\alpha}w.$$

Since

$$T^{\alpha}w = T^{\alpha-1}w - v = \left(T^{\alpha-2}w - v\right) - v = T^{\alpha-2}w - 2v = \cdots = w - \alpha v,$$

we obtain

$$w = T^{\alpha}w = w - \alpha v,$$

which implies $v = \theta$, contradicting the fact that $v \neq \theta$. Therefore T must have a fixed point in $L \subset C$ and this completes the proof. □

The following results are immediate consequences of Theorem 2.1.

Corollary 2.1 *Let X be a strictly convex Banach space with its zero vector θ and $C = \bigcup_{k=1}^{n} C_k$ be a finite union of nonempty weakly compact convex subsets C_k of X. Let $T : C \to C$ be a mapping. Suppose that*
(a) *X has the (UAC)-property,*
(b) *T is λ-firmly nonexpansive for some $\lambda \in (0,1)$.*
Then T has a fixed point in C.

Corollary 2.2 *Let $C = \bigcup_{k=1}^{n} C_k$ be a finite union of nonempty weakly compact convex subsets C_k of a UCED Banach space X and $\varphi : C \times C \to (0,1)$ be a function. If $T : C \to C$ is T is reactive firmly nonexpansive with respect to φ, then T has a fixed point in C.*

Corollary 2.3 [8, Theorem 2.8] *Let $C = \bigcup_{k=1}^{n} C_k$ be a finite union of nonempty weakly compact convex subsets C_k of a UCED Banach space X. If $T : C \to C$ is λ-firmly nonexpansive for some $\lambda \in (0,1)$, then T has a fixed point in C.*

Remark 2.1 Theorem 2.1 and Corollaries 2.1 and 2.2 all generalize and improve Smarzewski's fixed point theorem, [8, Theorem 2.8] and [14, Theorems 2.8, 2.9].

In Theorem 2.1, if $C_1 = C_2 = \cdots = C_n := C$, then we obtain the following new fixed point theorem.

Theorem 2.2 *Let X be a strictly convex Banach space with its zero vector θ and C be a nonempty weakly compact convex subset of X. Let $\varphi : C \times C \to (0,1)$ be a function and $T : C \to C$ be a mapping. Suppose that*
(a) *X has the (UAC)-property,*
(b) *T is reactive firmly nonexpansive with respect to φ.*
Then T has a fixed point in C.

The following results are immediate from Theorem 2.2.

Corollary 2.4 *Let X be a strictly convex Banach space with its zero vector θ and C be a nonempty weakly compact convex subset of X. Let $\varphi : C \times C \to (0,1)$ be a function and $T : C \to C$ be a mapping. Suppose that*
(a) *X has the (UAC)-property,*
(b) *T is λ-firmly nonexpansive for some $\lambda \in (0,1)$.*
Then T has a fixed point in C.

Corollary 2.5 *Let C be a nonempty weakly compact convex subset of a UCED Banach space X. Let $\varphi : C \times C \to (0,1)$ be a function. If $T : C \to C$ is reactive firmly nonexpansive with respect to φ, then T has a fixed point in C.*

Corollary 2.6 *Let C be a nonempty weakly compact convex subset of a UCED Banach space X. If $T : C \to C$ is λ-firmly nonexpansive for some $\lambda \in (0,1)$, then T has a fixed point in C.*

Definition 2.1 Let C be a nonempty subset of a normed space $(X, \|\cdot\|)$ and $\alpha, \beta \in (0,1)$. A mapping $T : C \to X$ is said to be (α,β)-firmly nonexpansive if

$$\|Tx - Ty\| \le \big\|(1-\beta)\big[(1-\alpha)(x-y) + \alpha(Tx - Ty)\big] + \beta(Tx - Ty)\big\| \quad \text{for all } x, y \in C.$$

Finally, by applying Theorem 2.1, we give some new fixed point theorems for (α,β)-firmly nonexpansive mappings.

Theorem 2.3 *Let X be a strictly convex Banach space with its zero vector θ and $C = \bigcup_{k=1}^{n} C_k$ be a finite union of nonempty weakly compact convex subsets C_k of X. Let $T : C \to C$ be a mapping and α, β be positive real numbers satisfying $0 < (1-\alpha)(1-\beta) < 1$. Suppose that*
(a) *X has the (UAC)-property,*
(b) *T is (α,β)-firmly nonexpansive.*
Then T has a fixed point in C.

Proof Since $\alpha, \beta > 0$ and $0 < (1-\alpha)(1-\beta) < 1$, we have $\alpha + \beta - \alpha\beta \in (0,1)$. Define $\varphi : C \times C \to (0,1)$ by

$$\varphi(s,t) := \alpha + \beta - \alpha\beta \quad \text{for } (s,t) \in C \times C.$$

Due to T is (α,β)-firmly nonexpansive, we obtain

$$\begin{aligned} \|Tx - Ty\| &\le \big\|(1-\beta)\big[(1-\alpha)(x-y) + \alpha(Tx - Ty)\big] + \beta(Tx - Ty)\big\| \\ &= \big\|\big(1 - \varphi(x,y)\big)(x-y) + \varphi(x,y)(Tx - Ty)\big\| \end{aligned}$$

for all $x, y \in C$. So, T is reactive firmly nonexpansive with respect to φ. Therefore the conclusion follows from Theorem 2.1. □

Corollary 2.7 *Let C be a nonempty weakly compact convex subset of a UCED Banach space X. Let α, β be positive real numbers satisfying* $0 < (1-\alpha)(1-\beta) < 1$. *If* $T : C \to C$ *is* (α, β)*-firmly nonexpansive, then T has a fixed point in C.*

Corollary 2.8 *Let C be a nonempty weakly compact convex subset of a uniformly convex Banach space X. Let α, β be positive real numbers satisfying* $0 < (1-\alpha)(1-\beta) < 1$. *If* $T : C \to C$ *is* (α, β)*-firmly nonexpansive, then T has a fixed point in C.*

Competing interests
The author declares that he has no competing interests.

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