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Multi-valued version of SCC, SKC, KSC, and CSC conditions in Ptolemy metric spaces

SJ Hosseini Ghoncheh and A Razani*

*Correspondence: razani@ipm.ir
Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran

Abstract

In this paper, multi-valued version of SCC, SKC, KSC, and CSC conditions in Ptolemy metric space are presented. Then the existence of a fixed point for these mappings in a Ptolemy metric space are proved. Finally, some examples are presented.

MSC: 47H10

Keywords: $CAT(0)$ spaces; fixed point; C condition; Ptolemy metric space

1 Introduction

The definition of a Ptolemy metric space is introduced by Schoenberg [1, 2]. In order to define it, we need to recall the definition of a Ptolemy inequality as follows.

Definition 1.1 [1] Let (X,d) be a metric space, the inequality

$$d(x,y)d(z,p) \leq d(x,z)d(y,p) + d(x,p)d(y,z)$$

is called a Ptolemy inequality, where $x,y,z,p \in X$.

Now, the definition of Ptolemy metric space is as follows.

Definition 1.2 [1] A Ptolemy metric space is a metric space where the Ptolemy inequality holds.

Schoenberg proved that every pre-Hilbert space is Ptolemaic and each linear quasi-normed Ptolemaic space is a pre-Hilbert space (see [1] and [3]). Moreover, Burckley *et al.* [4] proved that $CAT(0)$ spaces are Ptolemy metric spaces. They presented an example to show the converse is not true. Espinola and Nicolae in [5] proved a geodesic Ptolemy space with a uniformly continuous midpoint map is reflexive. With respect to this, they proved some fixed point theorems.

In 2008, Suzuki [6] introduced the C condition.

Definition 1.3 Let T be a mapping on a subset K of a metric space X, then T is said to satisfy C condition if

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \quad \text{implies} \quad d(Tx,Ty) \leq d(x,y),$$

for all $x,y \in K$.

Karapınar and Taş [7] presented some new definitions which are modifications of Suzuki's C condition as follows.

Definition 1.4 Let T be a mapping on a subset K of a metric space X.

(i) T is said to satisfy the SCC condition if

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \quad \text{implies} \quad d(Tx,Ty) \leq M(x,y),$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(Ty,y), d(Tx,y), d(x,Ty)\} \quad \text{for all } x,y \in K.$$

(ii) T is said to satisfy the SKC condition if

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \quad \text{implies} \quad d(Tx,Ty) \leq N(x,y),$$

where

$$N(x,y) = \max\left\{d(x,y), \frac{1}{2}\{d(x,Tx)+d(Ty,y)\}, \frac{1}{2}\{d(Tx,y)+d(x,Ty)\}\right\} \quad \text{for all } x,y \in K.$$

(iii) T is said to satisfy the KSC condition if

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \quad \text{implies} \quad d(Tx,Ty) \leq \frac{1}{2}\{d(x,Tx)+d(Ty,y)\}.$$

(iv) T is said to satisfy the CSC condition if

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \quad \text{implies} \quad d(Tx,Ty) \leq \frac{1}{2}\{d(Tx,y)+d(x,Ty)\}.$$

It is clear that every nonexpansive mapping satisfies the SKC condition [7, Proposition 9]. There exist mappings which do not satisfy the C condition, but they satisfy the SCC condition as the following example shows.

Example 1.5 [8] Define a mapping T on $[0,3]$ with $d(x,y) = |x-y|$ by

$$T(x) = \begin{cases} 0 & \text{if } x \neq 3, \\ 2 & \text{if } x = 3. \end{cases}$$

Karapınar and Taş [7] proved some fixed point theorems as follows.

Theorem 1.6 *Let T be a mapping on a closed subset K of a metric space X. Assume T satisfies the SKC, KSC, SCC or CSC condition, then $F(T)$ is closed. Moreover, if X is strictly convex and K is convex, then $F(T)$ is also convex.*

Theorem 1.7 *Let T be a mapping on a closed subset K of a metric space X which satisfying the SKC, KSC, SCC or CSC condition, then $d(x,Ty) \leq 5d(Tx,x) + d(x,y)$ holds for $x,y \in K$.*

Hosseini Ghoncheh and Razani [8] proved some fixed point theorems for the *SCC*, *SKC*, *KSC*, and *CSC* conditions in a single-valued version in Ptolemy metric space. In this paper, the notation of *SCC*, *SKC*, *KSC*, and *CSC* conditions are generalized for multi-valued mappings and some new fixed point theorems are obtained in Ptolemy metric spaces.

Let X be a metric space and $\{x_n\}$ be a bounded sequence in X. For $x \in X$, let

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ in K is given by

$$r(K, \{x_n\}) = \inf_{x\in K} r(x, \{x_n\}),$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ in K is the set

$$A(K, \{x_n\}) = \{x \in K : r(x, \{x_n\})\} = r(K, \{x_n\}).$$

Definition 1.8 [9] A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to be Δ-convergent to $x \in X$, if x is the unique asymptotic center of every subsequence of $\{x_n\}$.

Lemma 1.9

(i) *Every bounded sequence in X has a Δ-convergent subsequence* [10, p.3690].

(ii) *If C is a closed convex subset of X and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C* [11, Proposition 2.1].

(iii) *If C is a closed convex subset of X and if $f : C \to X$ is a nonexpansive mapping, then the conditions, $\{x_n\}\Delta$-converges to x and $d(x_n, f(x_n)) \to 0$, imply $x \in C$ and $f(x) = x$* [10, Proposition 3.7].

Lemma 1.10 [12] *If $\{x_n\}$ is a bounded sequence in X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

The next lemma and theorem play main roles for obtaining a fixed point in the Ptolemy metric spaces.

Lemma 1.11 [13] *Let $\{z_n\}$ and $\{w_n\}$ be bounded sequences in metric space K and $\lambda \in (0, 1)$. Suppose $z_{n+1} = \lambda w_n + (1 - \lambda)z_n$ and $d(w_{n+1}, w_n) \leq d(z_{n+1}, z_n)$ for all $n \in \mathbb{N}$. Then* $\limsup_{n\to\infty} d(w_n, z_n) = 0$.

Theorem 1.12 [5] *Let X be a complete geodesic Ptolemy space with a uniformly continuous midpoint map, $\{x_n\} \subseteq X$ a bounded sequence and $K \subseteq X$ nonempty closed and convex. Then $\{x_n\}$ has a unique asymptotic center in K.*

2 Main results

Let X be complete geodesic Ptolemy space and $P(X)$ denote the class of all subsets of X. Denote

$$P_f(X) = \{A \subset X : A \neq \emptyset \text{ has property } f\}.$$

Thus $P_{bd}, P_{cl}, P_{cv}, P_{cp}, P_{cl,bd}, P_{cp,cv}$ denote the classes of bounded, closed, convex, compact, closed bounded, and compact convex subsets of X, respectively. Also $T : K \to P_f(X)$ is called a multi-valued mapping on X. A point $u \in X$ is called a fixed point of T if $u \in Tu$.

Definition 2.1 [14] Let K be a subset of a $CAT(0)$ space X. A map $T : X \to P(X)$ is said to satisfy the C condition if for each $x \in K$, $u_x \in Tx$, and $y \in K$

$$\frac{1}{2}d(x, u_x) \leq d(x, y)$$

there exists a $u_y \in Ty$ such that

$$d(u_x, u_y) \leq d(x, y).$$

Espinola and Nicolae [5] used the C condition as follows.

Theorem 2.2 *Let X be a complete geodesic Ptolemy space with a uniformly continuous midpoint map, and K a nonempty bounded, closed, and convex subset of X. Suppose $T : K \to P_{cp}(K)$ is a multi-valued mapping satisfying the C condition, then $F(T) \neq \emptyset$.*

Now, we extend the *SCC*, *SKC*, *KSC*, and *CSC* conditions to multi-valued versions.

Definition 2.3 Let K be a subset of a geodesic Ptolemy space X. A map $T : X \to P(X)$ is said to satisfy conditions (i) *SCC*, (ii) *SKC*, (iii) *KSC*, (iv) *CSC* if for each $x \in K$, $u_x \in Tx$, and $y \in K$

$$\frac{1}{2}d(x, u_x) \leq d(x, y)$$

there exists a $u_y \in Ty$ such that

(i) $d(u_x, u_y) \leq M'(x, y)$, where

$$M'(x, y) = \max\bigl\{d(x, y), d(x, u_x), d(u_y, y), d(u_x, y), d(x, u_y)\bigr\},$$

(ii) $d(u_x, u_y) \leq N'(x, y)$, where

$$N'(x, y) = \max\left\{d(x, y), \frac{1}{2}\bigl\{d(x, u_x) + d(u_y, y)\bigr\}, \frac{1}{2}\bigl\{d(u_x, y) + d(x, u_y)\bigr\}\right\},$$

(iii) $d(u_x, u_y) \leq \frac{1}{2}\{d(x, u_x) + d(u_y, y)\}$,

(iv) $d(u_x, u_y) \leq \frac{1}{2}\{d(u_x, y) + d(x, u_y)\}$.

Remark 2.4 Notice that any *KSC* or *CSC* map is a *SKC* map.

Lemma 2.5 *Let X be a complete geodesic Ptolemy space, and K a nonempty closed subset of X. Suppose $T : K \to P_{cp}(K)$ is a multi-valued mapping satisfying the SKC condition, then for every $x, y \in K$, $u_x \in T(x)$ and $u_{xx} \in T(u_x)$ the following hold:*

(i) $d(u_x, u_{xx}) \leq d(x, u_x)$,

(ii) *either* $\frac{1}{2}d(x,u_x) \leq d(x,y)$ *or* $\frac{1}{2}d(u_x,u_{xx}) \leq d(u_x,y)$,
(iii) *either* $d(u_x,u_y) \leq N'(x,y)$ *or* $d(u_y,u_{xx}) \leq N'(u_x,y)$,

where

$$N'(u_x,y) = \max\left\{d(u_x,y), \frac{1}{2}\{d(u_{xx},u_x) + d(u_y,y)\}, \frac{1}{2}\{d(u_{xx},y) + d(u_y,u_x)\}\right\}.$$

Proof The first statement follows from the *SKC* condition. Indeed, we always have

$$\frac{1}{2}d(x,u_x) \leq d(x,u_x),$$

which yields

$$d(u_x,u_{xx}) \leq N'(x,u_x), \tag{2.1}$$

where

$$\begin{aligned} N'(x,u_x) &= \max\left\{d(x,u_x), \frac{1}{2}\{d(u_x,x) + d(u_{xx},u_x)\}, \frac{1}{2}\{d(u_x,u_x) + d(u_{xx},x)\}\right\} \\ &= \max\left\{d(x,u_x), \frac{1}{2}\{d(u_x,x) + d(u_{xx},u_x)\}, \frac{1}{2}d(u_{xx},x)\right\}. \end{aligned}$$

If $N'(x,u_x) = d(x,u_x)$ we are done. If $N'(x,u_x) = \frac{1}{2}\{d(u_x,x) + d(u_{xx},u_x)\}$ then (2.1) turns into

$$d(u_x,u_{xx}) \leq N'(x,u_x) = \frac{1}{2}\{d(u_x,x) + d(u_{xx},u_x)\}. \tag{2.2}$$

By simplifying (2.2), one can get (i). For the case $N'(x,u_x) = \frac{1}{2}d(u_{xx},x)$ (2.1) turns into

$$d(u_x,u_{xx}) \leq N'(x,u_x) = \frac{1}{2}d(u_{xx},x) \leq \frac{1}{2}\{d(u_x,x) + d(u_{xx},u_x)\},$$

which implies (i). It is clear that (iii) is a consequence of (ii). To prove (ii), assume the contrary, that is,

$$\frac{1}{2}d(u_x,x) > d(x,y) \quad \text{and} \quad \frac{1}{2}d(u_{xx},u_x) > d(u_x,y)$$

hold for all $x,y \in K$. Thus by triangle inequality and (i), we have

$$\begin{aligned} d(x,u_x) &\leq d(x,y) + d(y,u_x) \\ &< \frac{1}{2}\{d(u_x,x) + d(u_{xx},u_x)\} \\ &\leq \frac{1}{2}d(u_x,x) + \frac{1}{2}d(u_x,x) = d(x,u_x). \end{aligned}$$

□

Theorem 2.6 *Let X be a complete geodesic Ptolemy space, K a nonempty closed subset of X. Suppose $T : K \to P_{cp}(K)$ is a multi-valued mapping satisfying SKC condition, then $d(x,u_y) \leq 7d(u_x,x) + d(x,y)$ for all $x,y \in K$, $u_x \in Tx$, and $u_y \in Ty$.*

Proof The proof is based on Lemma 2.5; it is proved that

$$d(u_x,u_y) \leq N'(x,y) \quad \text{or} \quad d(u_y,u_{xx}) \leq N'(u_x,y)$$

holds, where

$$N'(u_x,y) = \max\left\{d(u_x,y), \frac{1}{2}\{d(u_{xx},u_x)+d(u_y,y)\}, \frac{1}{2}\{d(u_{xx},y)+d(u_y,u_x)\}\right\}.$$

Consider the first case. If $N'(x,y) = d(x,y)$, then we have

$$d(x,u_y) \leq d(x,u_x)+d(u_x,u_y) \leq d(x,u_x)+d(x,y).$$

For $N'(x,y) = \frac{1}{2}\{d(u_x,x)+d(u_y,y)\}$ one can observe

$$\begin{aligned} d(x,u_y) &\leq d(x,u_x)+d(u_x,u_y) \\ &\leq d(x,u_x)+\frac{1}{2}\{d(u_x,x)+d(u_y,y)\} \\ &\leq \frac{3}{2}d(u_x,x)+\frac{1}{2}d(u_y,y) \\ &\leq \frac{3}{2}d(u_x,x)+\frac{1}{2}\{d(u_y,x)+d(x,y)\}. \end{aligned}$$

Thus,

$$\frac{1}{2}d(x,u_y) \leq \frac{3}{2}d(x,u_x)+\frac{1}{2}d(x,y) \quad \text{if and only if} \quad d(x,u_y) \leq 3d(x,u_x)+d(x,y).$$

For $N'(x,y) = \frac{1}{2}\{d(u_x,y)+d(u_y,x)\}$ one can obtain

$$\begin{aligned} d(x,u_y) &\leq d(x,u_x)+d(u_x,u_y) \\ &\leq d(x,u_x)+\frac{1}{2}\{d(u_x,y)+d(u_y,x)\} \\ &\leq d(u_x,x)+\frac{1}{2}\{d(u_x,x)+d(x,y)\}+\frac{1}{2}d(u_y,x). \end{aligned}$$

Thus

$$\frac{1}{2}d(x,u_y) \leq \frac{3}{2}d(x,u_x)+\frac{1}{2}d(x,y) \quad \text{if and only if} \quad d(x,u_y) \leq 3d(x,u_x)+d(x,y).$$

Take the second case into account. For $N'(u_x,y) = d(u_x,y)$

$$\begin{aligned} d(x,u_y) &\leq d(x,u_x)+d(u_x,u_{xx})+d(u_{xx},u_y) \\ &\leq d(x,u_x)+(u_x,x)+d(u_x,y) \\ &= 2d(x,u_x)+d(u_x,y) \\ &\leq 2d(x,u_x)+d(u_x,x)+d(x,y) \\ &= 3d(x,u_x)+d(x,y). \end{aligned}$$

If $N'(u_x, y) = \frac{1}{2}\{d(u_{xx}, u_x) + d(u_y, y)\}$ then

$$\begin{aligned} d(x, u_y) &\le d(x, u_x) + d(u_x, u_{xx}) + d(u_{xx}, u_y) \\ &\le 2d(x, u_x) + \frac{1}{2}\bigl\{d(u_{xx}, u_x) + d(u_y, y)\bigr\} \\ &\le \frac{5}{2}d(x, u_x) + \frac{1}{2}d(u_y, y) \\ &\le \frac{5}{2}d(x, u_x) + \frac{1}{2}\bigl\{d(u_y, x) + d(x, y)\bigr\}; \end{aligned}$$

then

$$\frac{1}{2}d(x, u_y) \le \frac{5}{2}d(x, u_x) + \frac{1}{2}d(x, y) \quad \text{if and only if} \quad d(x, u_y) \le 5d(x, u_x) + d(x, y).$$

For the last case, $N'(u_x, y) = \frac{1}{2}\{d(u_{xx}, y) + d(u_y, u_x)\}$ and we have

$$\begin{aligned} d(x, u_y) &\le d(x, u_x) + d(u_x, u_{xx}) + d(u_{xx}, u_y) \\ &\le 2d(x, u_x) + \frac{1}{2}\bigl\{d(u_{xx}, y) + d(u_y, u_x)\bigr\} \\ &\le 2d(x, u_x) + \frac{1}{2}\bigl\{d(u_{xx}, u_x) + d(u_x, x) + d(x, y)\bigr\} + \frac{1}{2}\bigl\{d(u_y, x) + d(x, u_x)\bigr\} \\ &\le \frac{7}{2}d(x, u_x) + \frac{1}{2}d(u_y, x) + \frac{1}{2}d(x, y). \end{aligned}$$

Thus

$$\frac{1}{2}d(x, u_y) \le \frac{7}{2}d(x, u_x) + \frac{1}{2}d(x, y) \quad \text{if and only if} \quad d(x, u_y) \le 7d(x, u_x) + d(x, y).$$

Hence, the result follows from all the above cases. □

Corollary 2.7 *Let X be a complete geodesic Ptolemy space, K a nonempty closed subset of X. Suppose $T : K \to P_{cp}(K)$ is a multi-valued mapping satisfying SCC condition, then $d(x, u_y) \le 7d(u_x, x) + d(x, y)$ for all $x, y \in K$, $u_x \in Tx$, and $u_y \in Ty$.*

Theorem 2.8 *Let X be a complete geodesic Ptolemy space with a uniformly continuous midpoint map, and K a nonempty, bounded, closed, and convex subset of X. Suppose $T : K \to P_{cp}(K)$ is a multi-valued mapping satisfying the SKC condition and x_n is a sequence in K with $\lim_{n\to\infty} d(x_n, u_{x_n}) = 0$, where $u_{x_n} \in Tx_n$, then $F(T) \neq \emptyset$.*

Proof By Theorem 1.12, x_n has unique asymptotic center denoted by x. Let $n \in \mathbb{N}$. Applying Theorem 2.6 for x_n, x, and u_{x_n}, respectively, it follows that there exists $u_{z_n} \in Tx$ such that $d(x_n, u_{z_n}) \le 7d(x_n, u_{x_n}) + d(x_n, x)$.

Let $u_{z_{n_k}}$ be a subsequence of u_{z_n} that converges to some $u_z \in Tx$, then

$$\begin{aligned} d(x_{n_k}, u_z) &\le d(x_{n_k}, u_{z_{n_k}}) + d(u_{z_{n_k}}, u_z) \\ &\le 7d(x_{n_k}, u_{x_{n_k}}) + d(x_{n_k}, x) + d(u_{z_{n_k}}, u_z), \end{aligned}$$

taking the superior limit as $k \to \infty$ and knowing that the asymptotic center of $\{x_{n_k}\}$ is precisely x. Thus we obtain $x = u_z \in Tx$. Hence the proof is complete. □

By the same idea of [4, p.6] we construct a function $T : X \to P(X)$, which is SKC and has a fixed point.

Example 2.9 Consider the space

$$X = \{(0,0), (0,1), (1,1), (1,2)\}$$

with l^∞ metric,

$$d\big((x_1,y_1),(x_2,y_2)\big) = \max\big\{|x_1 - x_2|, |y_1 - y_2|\big\}.$$

X is a geodesic Ptolemy space, but it is not a $CAT(0)$ space (see [4]).

Define a mapping T on X by

$$T(x,y) = \begin{cases} \{(1,1),(0,0)\} & \text{if } (x,y) \neq (0,0), \\ \{(0,1)\} & \text{if } (x,y) = (0,0). \end{cases}$$

T satisfies the SKC condition. Suppose $x = (0,0)$ and $y = (1,1)$, thus $Tx = \{(0,1)\}$, then $u_x = (0,1)$, so

$$\frac{1}{2}d(x,u_x) = \frac{1}{2}d\big((0,0),(0,1)\big) = \frac{1}{2} \leq d(x,y) = d\big((0,0),(1,1)\big) = 1,$$

and we can choose $u_y = (0,0)$,

$$\begin{aligned} N'\big((0,0),(1,1)\big) = \max\Big\{ & d\big((0,0),(1,1)\big), \frac{1}{2}\big[d\big((0,0),(0,1)\big) + d\big((0,0),(1,1)\big)\big], \\ & \frac{1}{2}\big[d\big((0,1),(1,1)\big) + d\big((0,0),(0,0)\big)\big]\Big\} = 1; \end{aligned}$$

thus

$$d(u_x,u_y) = d\big((0,1),(0,0)\big) = 1 \leq N'(x,y) = N'\big((0,0),(1,1)\big) = 1.$$

One can check the SKC condition holds for the other points of the space X.

Note that $(1,1) \in T(1,1)$; thus $F(T) = \{(1,1)\} \neq \emptyset$.

Corollary 2.10 *Let X be a complete geodesic Ptolemy space with a uniformly continuous midpoint map, and K a nonempty bounded, closed, and convex subset of X. Suppose $T : K \to P_{cp}(K)$ is a multi-valued mapping satisfying the condition SCC and x_n is a sequence in K with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $F(T) \neq \emptyset$.*

One can find in [15] the multi-valued version of the E_μ and C_λ conditions.

Definition 2.11 Let K be a subset of a metric space (X, d). A map $T : K \to P_{cl,bd}(X)$ is said to satisfy the E'_μ condition provided that

$$\operatorname{dist}(x, Ty) \leq \mu \operatorname{dist}(x, Tx) + d(x, y), \quad x, y \in K;$$

we say that T satisfies the E' condition whenever T satisfies E'_μ for some $\mu \geq 1$.

One can replace the metric space with a Ptolemy space in the following definition.

Definition 2.12 Let K be a subset of a metric space (X, d) and $\lambda \in (0, 1)$. A map $T : K \to P(X)$ is said to satisfy the C'_λ condition if for each $x, y \in K$,

$$\lambda \operatorname{dist}(x, Tx) \leq d(x, y)$$

implies

$$H(Tx, Ty) \leq d(x, y),$$

where $H(\cdot, \cdot)$ stands for the Hausdorff distance.

Theorem 2.13 *Let X be a complete geodesic Ptolemy space with a uniformly continuous midpoint map, and K be a nonempty bounded, closed, and convex subset of X. Suppose $T : K \to P_{cl,bd}(K)$ is a multi-valued mapping satisfying E' and C'_λ conditions, then $F(T) \neq \emptyset$.*

Proof We find an approximate fixed point for T. Take $x_0 \in K$, since $Tx_0 \neq \emptyset$ we can choose $y_0 \in Tx_0$. Define

$$x_1 = (1 - \lambda)x_0 \oplus \lambda y_0.$$

Since K is convex, $x_1 \in K$. Let $y_1 \in Tx_1$ be chosen such that

$$d(y_0, y_1) = \operatorname{dist}(y_0, Tx_1).$$

Similarly, set

$$x_2 = (1 - \lambda)x_1 \oplus \lambda y_1.$$

Again we choose $y_2 \in Tx_2$ such that

$$d(y_1, y_2) = \operatorname{dist}(y_1, Tx_2).$$

By the same argument, we get $y_2 \in K$. In this way we find a sequence $\{x_n\} \subset K$ such that

$$x_{n+1} = (1 - \lambda)x_n \oplus \lambda y_n,$$

where $y_n \in Tx_n$ and

$$d(y_{n-1}, y_n) = \operatorname{dist}(y_{n-1}, Tx_n).$$

For every $n \in \mathbb{N}$

$$\lambda d(x_n, y_n) = d(x_n, x_{n+1}),$$

for which it follows that

$$\lambda \operatorname{dist}(x_n, Tx_n) \le \lambda d(x_n, y_n) = d(x_n, x_{n+1});$$

since T satisfies the C'_λ condition,

$$H(Tx_n, Tx_{n+1}) \le d(x_n, x_{n+1}),$$

this implies

$$\begin{aligned} d(y_{n+1}, y_n) &= \operatorname{dist}(y_n, Tx_{n+1}) \\ &\le H(Tx_n, Tx_{n+1}) \\ &\le d(x_n, x_{n+1}). \end{aligned}$$

Now, we apply Lemma 1.11 to conclude $\lim_{n\to\infty} d(x_n, y_n) = 0$, where $y_n \in Tx_n$. The bounded sequence $\{x_n\}$ is Δ-convergent, hence by passing to a subsequence $\Delta\text{-}\lim_n x_n = v \in K$. We choose $z_n \in Tv$ such that

$$d(x_n, z_n) = \operatorname{dist}(x_n, Tv).$$

Since Tv is compact, the sequence $\{z_n\}$ has a convergent subsequence $\{z_{n_k}\}$ with $\lim_{k\to\infty} z_{n_k} = w \in Tv$. Moreover, $z_n \in K$, and K is closed; then $w \in K$. By the E' condition

$$\operatorname{dist}(x_{n_k}, Tv) \le \mu \operatorname{dist}(x_{n_k}, Tx_{n_k}) + d(x_{n_k}, v) \quad \text{for some } \mu \ge 1.$$

Note that

$$\begin{aligned} d(x_{n_k}, w) &\le d(x_{n_k}, z_{n_k}) + d(z_{n_k}, w) \\ &\le \mu \operatorname{dist}(x_{n_k}, Tx_{n_k}) + d(x_{n_k}, v) + d(z_{n_k}, w); \end{aligned}$$

this implies

$$\limsup_{n\to\infty} d(x_{n_k}, w) \le \limsup_{n\to\infty} d(x_{n_k}, v).$$

Thus by the Opial property, $w = v \in Tv$. □

Example 2.14 [15] Let $X = \mathbb{R}$ and $D = [0, \frac{7}{2}]$. Define a mapping T on D with $d(x, y) = |x - y|$ by

$$T(x) = \begin{cases} [0, \frac{x}{7}] & \text{if } x \ne \frac{7}{2}, \\ \{1\} & \text{if } x = \frac{7}{2}. \end{cases}$$

First we show T satisfies the C'_λ condition. Let $x, y \in [0, \frac{7}{2})$, then

$$H(Tx, Ty) = \left|\frac{x-y}{7}\right| \leq |x-y|.$$

Let $x \in [0, \frac{5}{2}]$ and $y = \frac{7}{2}$, then

$$H(Tx, Ty) = 1 \leq \frac{7}{2} - x.$$

Let $x \in (\frac{5}{2}, \frac{7}{2})$ and $y = \frac{7}{2}$, then $\operatorname{dist}(x, Tx) = \frac{6x}{7}$, thus

$$\frac{1}{2}\operatorname{dist}(x, Tx) = \frac{6x}{14} > \frac{30}{28} > 1 > |x-y|$$

and

$$\frac{1}{2}\operatorname{dist}(y, Ty) = \frac{5}{4} > 1 > |x-y|.$$

Thus T satisfies the C'_λ condition with $\lambda = \frac{1}{2}$. Let $x, y \in D$, then

$$\operatorname{dist}(x, Ty) \leq 3d(x, Tx) + |x-y|,$$

this shows T satisfies the E' condition. Since $T(0) = \{0\}$, $0 \in F(T) \neq \emptyset$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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