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General iterative algorithms for mixed equilibrium problems, a general system of generalized equilibria and fixed point problems

Hui-Ying Hu[1], Lu-Chuan Ceng[1*] and Yan-Lai Song[2]

*Correspondence:
zenglc@hotmail.com
[1]Department of Mathematics, Shanghai Normal University, Shanghai, 200234, China
Full list of author information is available at the end of the article

Abstract
In this paper, we introduce and analyze a general iterative algorithm for finding a common solution of a finite family of mixed equilibrium problems, a general system of generalized equilibria and a fixed point problem of nonexpansive mappings in a real Hilbert space. Under some appropriate conditions, we derive the strong convergence of the sequence generated by the proposed algorithm to a common solution, which also solves some optimization problem. The result presented in this paper improves and extends some corresponding ones in the earlier and recent literature.
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1 Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let C be a nonempty, closed and convex subset of H, and let $T: C \to C$ be a nonlinear mapping. Throughout this paper, we use $F(T)$ to denote the fixed point set of T. A mapping $T: C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

Let $F: C \times C \to R$ be a real-valued bifunction and $\varphi: C \to R$ be a real-valued function, where R is a set of real numbers. The so-called mixed equilibrium problem (MEP) is to find $x \in C$ such that

$$F(x,y) + \varphi(y) - \varphi(x) \ge 0, \quad \forall y \in C, \tag{1.2}$$

which was considered and studied in [1, 2]. The set of solutions of MEP (1.2) is denoted by $\mathrm{MEP}(F,\varphi)$. In particular, whenever $\varphi \equiv 0$, MEP (1.2) reduces to the equilibrium problem (EP) of finding $x \in C$ such that

$$F(x,y) \ge 0, \quad \forall y \in C,$$

which was considered and studied in [3–7]. The set of solutions of the EP is denoted by $EP(F)$. Given a mapping $A : C \to H$, let $F(x,y) = \langle Ax, y-x\rangle$ for all $x, y \in C$. Then $z \in EP(F)$ if and only if $\langle Ax, y-x\rangle \geq 0$ for all $y \in C$. Numerous problems in physics, optimization and economics reduce to finding a solution of the EP.

Throughout this paper, assume that $F : C \times C \to R$ is a bifunction satisfying conditions (A1)-(A4) and that $\varphi : C \to R$ is a lower semicontinuous and convex function with restriction (B1) or (B2), where

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$, for any $x, y \in C$;

(A3) F is upper hemicontinuous, *i.e.*, for each $x, y, z \in C$,

$$\limsup_{t\to 0} F\bigl(tz + (1-t)x, y\bigr) \leq F(x,y);$$

(A4) $F(x,\cdot)$ is convex and lower semicontinuous for each $x \in C$;

(B1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

(B2) C is a bounded set.

The mappings $\{T_n\}_{n=1}^{\infty}$ are said to be an infinite family of nonexpansive self-mappings on C if

$$\|T_n x - T_n y\| \leq \|x - y\|, \quad \forall x, y \in C, n \geq 1, \tag{1.3}$$

and denoted by $F(T_n)$ is a fixed point set of T_n, *i.e.*, $F(T_n) = \{x \in H : T_n x = x\}$. Finding an optimal point in the intersection $\bigcap_{n=1}^{\infty} F(T_n)$ of fixed point sets of mappings T_n, $n \geq 1$, is a matter of interest in various branches of sciences.

Recently, many authors considered some iterative methods for finding a common element of the set of solutions of MEP (1.2) and the set of fixed points of nonexpansive mappings; see, *e.g.*, [2, 8, 9] and the references therein.

A mapping $A : C \to H$ is said to be:

(i) Monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

(ii) Strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C.$$

In such a case, A is said to be η-strongly monotone.

(iii) Inverse-strongly monotone if there exists a constant $\zeta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \zeta\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

In such a case, A is said to be ζ-inverse-strongly monotone.

Let $A : C \to H$ be a nonlinear mapping. The classical variational inequality problem (VIP) is to find $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

We use $\mathrm{VI}(C, A)$ to denote the set of solutions to VIP (1.4). One can easily see that VIP (1.4) is equivalent to a fixed point problem, the origin of which can be traced back to Lions and Stampacchia [10]. That is, $u \in C$ is a solution to VIP (1.4) if and only if u is a fixed point of the mapping $P_C(I - \lambda A)$, where $\lambda > 0$ is a constant. Variational inequality theory has been studied quite extensively and has emerged as an important tool in the study of a wide class of obstacle, unilateral, free, moving, equilibrium problems. Not only are the existence and uniqueness of solutions important topics in the study of VIP (1.4), but also how to actually find a solution of VIP (1.4) is important. Up to now, there have been many iterative algorithms in the literature for finding approximate solutions of VIP (1.4) and its extended versions; see, *e.g.*, [3, 11–14].

Recently, Ceng and Yao [8] introduced and studied the general system of generalized equilibria (GSEP) as follows: Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta_1, \Theta_2 : C \times C \to R$ be two bifunctions, $B_1, B_2 : C \to H$ be two nonlinear mappings. Consider the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \Theta_1(x^*, x) + \langle B_1 y^*, x - x^* \rangle + \frac{1}{\mu_1}\langle x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \Theta_2(y^*, y) + \langle B_2 x^*, y - y^* \rangle + \frac{1}{\mu_2}\langle y^* - x^*, y - y^* \rangle \geq 0, & \forall y \in C, \end{cases} \tag{1.5}$$

where $\mu_1 > 0$, $\mu_2 > 0$ are two constants. In particular, whenever $\Theta_1 = \Theta_2 = 0$, GSEP (1.5) reduces to the following general system of variational inequalities (GSVI): find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall y \in C, \end{cases} \tag{1.6}$$

where μ_1 and μ_2 are two positive constants. GSVI (1.6) is considered and studied in [8, 15, 16]. In particular, whenever $B_1 = B_2 = A$ and $x^* = y^*$, GSVI (1.6) reduces to VIP (1.4).

In order to prove our main results in the following sections, we need the following lemmas and propositions.

Proposition 1.1 *For given $x \in H$ and $z \in C$:*

(i) $z \in P_C x \Leftrightarrow \langle x - z, y - z \rangle \leq 0$, $\forall y \in C$;
(ii) $z \in P_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$, $\forall y \in C$;
(iii) $\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2$, $\forall x, y \in H$.

Consequently, P_C is a firmly nonexpansive mapping of H onto C and hence nonexpansive and monotone.

Given a positive number $r > 0$. Let $T_r^{(\Theta,\varphi)} : H \to C$ be the solution set of the auxiliary mixed equilibrium problem, that is, for each $x \in H$,

$$T_r^{(\Theta,\varphi)}(x) := \left\{ y \in C : \Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle y - x, z - y \rangle \geq 0, \forall z \in C \right\}.$$

Proposition 1.2 (see [2, 8]) *Let C be a nonempty closed subset of a real Hilbert space H. Let $\Theta : C \times C \to R$ be a bifunction satisfying conditions* (A1)-(A4), *and let $\varphi : C \to R$ be a lower semicontinuous and convex function with restriction* (B1) *or* (B2). *Then the following hold*:

(a) *for each $x \in H$, $T_r^{(\Theta,\varphi)} \neq \emptyset$;*
(b) *$T_r^{(\Theta,\varphi)}$ is single-valued;*
(c) *$T_r^{(\Theta,\varphi)}$ is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\left\| T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y \right\|^2 \le \left\langle T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y, x - y \right\rangle;$$

(d) *for all $s, t > 0$ and $x \in H$,*

$$\left\| T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x \right\|^2 \le \frac{s-t}{s} \left\langle T_s^{(\Theta,\varphi)}x - x, T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x \right\rangle;$$

(e) $F(T_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta, \varphi)$;
(f) $\mathrm{MEP}(\Theta, \varphi)$ *is closed and convex.*

Remark 1.1 It is easy to see from conclusions (c) and (d) in Proposition 1.2 that

$$\left\| T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y \right\| \le \|x - y\|, \quad \forall r > 0, x, y \in H$$

and

$$\left\| T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x \right\| \le \frac{|s-t|}{s} \left\| T_s^{(\Theta,\varphi)}x - x \right\|, \quad \forall s, t > 0, x \in H.$$

Remark 1.2 If $\varphi = 0$, then $T_r^{(\Theta,\varphi)}$ is rewritten as T_r^{Θ}.

Ceng and Yao [8] transformed GSEP (1.5) into a fixed point problem in the following way.

Lemma 1.1 (see [8]) *Let C be a nonempty closed convex subset of H. Let $\Theta_1, \Theta_2 : C \times C \to R$ be two bifunctions satisfying conditions* (A1)-(A4), *and let the mappings $B_1, B_2 : C \to H$ be ζ_1-inverse strongly monotone and ζ_2-inverse strongly monotone, respectively. Let $\mu_1 \in (0, 2\zeta_1)$ and $\mu_2 \in (0, 2\zeta_2)$, respectively. Then, for given $x^*, y^* \in C$, (x^*, y^*) is a solution of GSEP* (1.5) *if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = T_{\mu_1}^{\Theta_1}(I - \mu_1 B_1) T_{\mu_2}^{\Theta_2}(I - \mu_2 B_2)x, \quad \forall x \in C,$$

where $y^* = T_{\mu_2}^{\Theta_2}(I - \mu_2 B_2)x^*$.

Lemma 1.2 (see [8]) *For given $x^*, y^* \in C$, (x^*, y^*) is a solution of GSVI* (1.6) *if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by*

$$Gx = P_C(I - \mu_1 B_1) P_C(I - \mu_2 B_2)x, \quad \forall x \in C,$$

where $y^* = P_C(x^* - \mu_2 B_2 x^*)$ and P_C is the projection of H onto C.

Remark 1.3 If $\Theta_1, \Theta_2 : C \times C \to R$ are two bifunctions satisfying (A1)-(A4), the mappings $B_1, B_2 : C \to H$ are ζ_1-inverse strongly monotone and ζ_2-inverse strongly monotone, respectively, then $G : C \to C$ is a nonexpansive mapping provided $\mu_1 \in (0, 2\zeta_1)$ and $\mu_2 \in (0, 2\zeta_2)$.

Throughout this paper, the set of fixed points of the mapping G is denoted by Γ.

On the other hand, Moudafi [1] introduced the viscosity approximation method for nonexpansive mappings (see also [17] for further developments in both Hilbert spaces and Banach spaces).

A mapping $f : C \to C$ is called α-contractive if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in C.$$

Let f be a contraction on C. Starting with an arbitrary initial $x_1 \in C$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T x_n, \quad n \geq 0, \tag{1.7}$$

where T is a nonexpansive mapping of C into itself and $\{\alpha_n\}$ is a sequence in $(0,1)$. It is proved [1, 17] that under certain appropriate conditions imposed on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.6) converges strongly to the unique solution $x^* \in F(T)$ to the VIP

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad x \in F(T).$$

A linear bounded operator A is said to be $\bar{\gamma}$-strongly positive on H if there exists a constant $\bar{\gamma} \in (0,1)$ such that

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2, \quad \forall x \in H. \tag{1.8}$$

The typical problem is to minimize a quadratic function on a real Hilbert space H,

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - \langle x, u \rangle, \tag{1.9}$$

where C is a nonempty closed convex subset of H, u is a given point in H and A is a strongly positive bounded linear operator on H.

In 2006, Marino and Xu [18] introduced and considered the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n A) T x_n, \quad \forall n \geq 0, \tag{1.10}$$

where A is a strongly positive bounded linear operator on a real Hilbert space H, f is a contraction on H. They proved that the above sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0, \quad \forall x \in F(T),$$

which is the optimality condition for the minimization problem

$$\min_{x\in C}\frac{1}{2}\langle Ax,x\rangle - h(x),$$

where h is a potential function for γf (*i.e.*, $h'(x) = \gamma f(x)$ for all $x \in H$).

In 2007, Takahashi and Takahashi [5] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of the EP and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Let $S : C \to H$ be a nonexpansive mapping. Starting with arbitrary initial $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ recursively by

$$\begin{cases} F(u_n,y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n, & \forall n \geq 1. \end{cases} \tag{1.11}$$

They proved that under appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* \in F(S) \cap \mathrm{EP}(F)$, where $x^* = P_{F(S)\cap \mathrm{EP}(F)}f(x^*)$.

Subsequently, Plubtieng and Punpaeng [19] introduced a general iterative process for finding a common element of the set of solutions of the EP and the set of fixed points of a nonexpansive mapping in a Hilbert space.

Let $S : H \to H$ be a nonexpansive mapping. Starting with an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{cases} F(u_n,y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (1-\alpha_n A)Su_n, & \forall n \geq 1. \end{cases} \tag{1.12}$$

They proved that under appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ generated by (1.12) converges strongly to the unique solution $x^* \in F(S) \cap \mathrm{EP}(F)$ to the VIP

$$\big\langle (\gamma f - A)x^*, x - x^*\big\rangle \leq 0, \quad \forall x \in F(S) \cap \mathrm{EP}(F),$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(S)\cap \mathrm{EP}(F)}\frac{1}{2}\langle Ax,x\rangle - h(x),$$

where h is a potential function for γf (*i.e.*, $h'(x) = \gamma f(x)$ for all $x \in H$).

In 2001, Yamada [20] introduced a hybrid steepest descent method for a nonexpansive mapping T as follows:

$$x_{n+1} = Tx_n - \mu\lambda_n F(Tx_n), \quad \forall n \geq 0, \tag{1.13}$$

where F is a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa, \eta > 0$ and $0 < \mu < \frac{2\eta}{\kappa^2}$. He proved that if $\{\lambda_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.13) converges strongly to the unique solution of the variational inequality

$$\big\langle Fx^*, x - x^*\big\rangle \leq 0, \quad \forall x \in F(T).$$

In 2010, Tian [21] combined the iterative method (1.10) with Yamada's method (1.13) and considered the following general viscosity-type iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n \mu F)Tx_n, \quad \forall n \geq 0. \tag{1.14}$$

Then he proved that the sequence $\{x_n\}$ generated by (1.14) converges strongly to the unique solution of the variational inequality

$$\langle (\gamma f - \mu F)x^*, x - x^* \rangle \leq 0, \quad \forall x \in F(T).$$

Recently, Ceng *et al.* [22] introduced implicit and explicit iterative schemes for finding the fixed points of a nonexpansive mapping T on a nonempty, closed and convex subset C in a real Hilbert space H as follows:

$$x_t = P_C\bigl[t\gamma Vx_t + (1 - t\mu F)Tx_t\bigr] \tag{1.15}$$

and

$$x_{n+1} = P_C\bigl[\alpha_n \gamma Vx_n + (1 - \alpha_n \mu F)Tx_n\bigr], \quad \forall n \geq 0, \tag{1.16}$$

where V is an L-Lipschitzian mapping with constant $L \geq 0$ and F is a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa, \eta > 0$ and $0 < \mu < \frac{2\eta}{\kappa^2}$. Then they proved that the sequences generated by (1.15) and (1.16) converge strongly to the unique solution of the variational inequality

$$\langle (\gamma V - \mu F)x^*, x - x^* \rangle \leq 0, \quad \forall x \in F(T).$$

Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0,1]$. For any $n \geq 1$, define a mapping W_n of C into itself as follows:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \lambda_n T_n U_{n,n+1} + (1 - \lambda_n)I, \\ U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1 - \lambda_{n-1})I, \\ \ldots, \\ U_{n,k} = \lambda_k T_k U_{n,k+1} + (1 - \lambda_k)I, \\ U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1})I, \\ \ldots, \\ U_{n,2} = \lambda_2 T_2 U_{n,3} + (1 - \lambda_2)I, \\ W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1 - \lambda_1)I. \end{cases} \tag{1.17}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

Very recently, Chen [23] introduced and considered the following iterative scheme:

$$x_{n+1} = P_C\bigl[\alpha_n \gamma f(x_n) + (1 - \alpha_n A)W_n x_n\bigr], \quad \forall n \geq 0, \tag{1.18}$$

where A is a strongly positive bounded linear operator, f is a contraction on H, and W_n is defined as (1.17). He proved that the above sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\bigl\langle (\gamma f - A)x^*, x - x^* \bigr\rangle \leq 0, \quad \forall x \in \bigcap_{n=1}^{\infty} F(T_n),$$

which is the optimality condition for the minimization problem

$$\min_{x \in \bigcap_{n=1}^{\infty} F(T_n)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (*i.e.*, $h'(x) = \gamma f(x)$ for all $x \in H$).

More recently, Rattanaseeha [7] introduced an iterative algorithm:

$$\begin{cases} x_1 \in H, \quad \text{arbitrarily given}, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = P_C[\alpha_n \gamma f(x_n) + (1 - \alpha_n A)W_n x_n], \quad \forall n \geq 1, \end{cases} \tag{1.19}$$

where A is a strongly positive bounded linear operator, f is a contraction on H, and W_n is defined as (1.17). He proved that the above sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\bigl\langle (\gamma f - A)x^*, x - x^* \bigr\rangle \leq 0, \quad \forall x \in \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(F).$$

Nowadays, Wang *et al.* [24] introduced an iterative algorithm:

$$\begin{cases} x_1 \in H, \\ F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = P_c[\alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)W_n u_n], \quad \forall n \geq 1, \end{cases} \tag{1.20}$$

where A is a strongly positive bounded linear operator, f is an l-Lipschitz continuous mapping, $\{W_n\}$ is defined by (1.17), and $G = P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$. They proved that the above sequence $\{x_n\}$ converges strongly to $x^* \in \Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{MEP}(F, \varphi) \cap \Gamma$, where Γ is a fixed point set of the mapping $G = P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$, which is the unique solution of the VIP

$$\bigl\langle (A - \gamma f)x^*, x^* - x \bigr\rangle \leq 0, \quad \forall x \in \Omega,$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x \in \Omega} \frac{1}{2}\langle Ax, x\rangle - h(x).$$

Our concern now is the following:

Question 1 Can Theorem 3.1 of Rattanaseeha [7], Theorem 3.1 of Wang *et al.* [24] and so on be extended from one mixed equilibrium problem to the convex combination of a finite family of the mixed equilibrium problems?

Question 2 We know that GSEP (1.5) is more general than GSVI (1.6). What happens if GSVI (1.6) is replaced by GSEP (1.5)?

Question 3 We know that the η-strongly monotone and L-Lipschitz operator is more general than the strongly positive bounded linear operator. What happens if the strongly positive bounded linear operator is replaced by the η-strongly monotone and L-Lipschitz operator?

The purpose of this article is to give the affirmative answers to these questions mentioned above. Let $B_i : C \to H$ be ζ_i-inverse strongly monotone for $i = 1, 2$, B_3 be a κ-Lipschitz and η-strongly monotone operator and $f : C \to H$ be an l-Lipschitz mapping on H. Motivated by the above facts, in this paper we propose and analyze the general iterative algorithm

$$\begin{cases} y_n = \lambda_n W_n G(\sum_{m=1}^N \beta_{n,m} u_{n,m}) + (1 - \lambda_n)(\sum_{m=1}^N \beta_{n,m} u_{n,m}), \\ x_{n+1} = P_C[\alpha_n \gamma f(x_n) + (1 - \alpha_n \mu B_3) W_n G y_n], \quad \forall n \geq 1, \end{cases} \tag{1.21}$$

where $\{u_{n,m}\}$ is such that

$$F_m(u_{n,m}, y) + \varphi(y) - \varphi(u_{n,m}) + \frac{1}{r_{n,m}} \langle y - u_{n,m}, u_{n,m} - x_n \rangle \geq 0, \quad \forall y \in C,$$

for each $1 \leq m \leq N$, W_n is defined by (1.17) and $G = T_{\mu_1}^{\Theta_1}(I - \mu_1 B_1) T_{\mu_2}^{\Theta_2}(I - \mu_2 B_2)$, and $x_0 \in C$ is an arbitrary initial point, for finding a common solution of a finite family of MEP (1.2), GSEP (1.5) and the fixed point problem of an infinite family of nonexpansive self-mappings $\{T_n\}_{n=1}^\infty$ on C. It is proven that under some mild conditions imposed on parameters, the sequence $\{x_n\}$ generated by (1.20) converges strongly to $x^* \in \Omega := (\bigcap_{n=1}^\infty F(T_n)) \cap (\bigcap_{m=1}^N \mathrm{MEP}(F_m, \varphi)) \cap \Gamma$, where Γ is a fixed point set of the mapping $G = T_{\mu_1}^{\Theta_1}(I - \mu_1 B_1) T_{\mu_2}^{\Theta_2}(I - \mu_2 B_2)$, where x^* is the unique solution of the variational inequality

$$\langle (\gamma f - \mu B_3) x^*, z - x^* \rangle \leq 0, \quad \forall z \in \Omega. \tag{1.22}$$

Remark 1.4 Other results on the problem of finding solutions to equilibrium problems and fixed point problems of families of mappings with different approaches can be found in [25, 26].

2 Preliminaries

We indicate weak convergence and strong convergence by using the notation $\rightharpoonup$ and $\to$, respectively. A mapping $f : C \to H$ is called l-Lipschitz continuous if there exists a constant $l \geq 0$ such that

$$\| f(x) - f(y) \| \leq l \| x - y \|, \quad \forall x, y \in C.$$

In particular, if $l = 1$, then f is called a nonexpansive mapping; if $l \in [0, 1)$, then f is a contraction. Recall that a mapping $T : H \to H$ is said to be a firmly nonexpansive mapping if

$$\| Tx - Ty \|^2 \leq \langle Tx - Ty, x - y \rangle, \quad \forall x, y \in H.$$

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.1 *Let X be a real inner product space. Then there holds the following inequality*:

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X.$$

Lemma 2.2 *Let H be a Hilbert space. Then the following equalities hold*:

(a) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ *for all* $x, y \in H$;

(b) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x - y\|^2$ *for all* $x, y \in H$ *and* $\lambda, \mu \in [0, 1]$ *with* $\lambda + \mu = 1$;

(c) *If* $\{x_n\}$ *is a sequence in* H *such that* $x_n \rightharpoonup x$, *it follows that*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|x - y\|^2, \quad \forall y \in H.$$

We have the following crucial lemmas concerning the W-mappings defined by (1.17).

Lemma 2.3 (see [27, Lemma 3.2]) *Let $\{T_n\}_{n=1}^\infty$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$, and let $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Then, for every $x \in C$ and $k \ge 1$, the limit $\lim_{n\to\infty} U_{n,k}$ exists, where $U_{n,k}$ is defined by* (1.17).

Remark 2.1 (see [6, Remark 3.1]) It can be known from Lemma 2.3 that if D is a nonempty bounded subset of C, then for $\epsilon > 0$ there exists $n_0 \ge k$ such that for all $n \ge n_0$,

$$\sup_{x \in D} \|U_{n,k}x - U_k x\| \le \epsilon.$$

Remark 2.2 (see [6, Remark 3.2]) Utilizing Lemma 2.3, we define a mapping $W : C \times C$ as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in C.$$

Such W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. Since W_n is nonexpansive, $W : C \to C$ is also nonexpansive. Indeed, observe that for each $x, y \in C$,

$$\|Wx - Wy\| = \lim_{n\to\infty} \|W_n x - W_n y\| \le \|x - y\|.$$

If $\{x_n\}$ is a bounded sequence in C, then we put $D = \{x_n : n \ge 1\}$. Hence, it is clear from Remark 2.1 that for arbitrary $\epsilon > 0$ there exists $N_0 \ge 1$ such that for all $n > N_0$,

$$\|W_n x_n - Wx_n\| = \|U_{n,1}x_n - U_1 x_n\| \le \sup_{x \in D} \|U_{n,1}x - U_1 x\| \le \epsilon.$$

This implies that

$$\lim_{n\to\infty} \|W_n x_n - W x_n\| = 0.$$

Lemma 2.4 (see [27, Lemma 3.3]) *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, and let $\{\lambda_n\}$ be a sequence in $(0,b]$ for some $b \in (0,1)$. Then $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

Lemma 2.5 (see [28, Demiclosedness principle]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive self-mapping on C with $F(T) \neq \emptyset$. Then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$. Here I is the identity operator of H.*

Lemma 2.6 *Let $A : C \to H$ be a monotone mapping. In the context of the variational inequality problem, the characterization of the projection* (*see Proposition* 1.1(i)) *implies*

$$u \in \mathrm{VI}(C,A) \quad \Leftrightarrow \quad u = P_C(u - \lambda A u), \quad \forall \lambda > 0.$$

Lemma 2.7 (see [29]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \sigma_n \gamma_n, \quad \forall n \ge 1,$$

where γ_n is a sequence in $[0,1]$ *and σ_n is a real sequence such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \sigma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\sigma_n \gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.8 *Each Hilbert space H satisfies Opial's condition, i.e., for the sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$. Then the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf \|x_n - y\|$$

holds for any $y \in H$ such that $y \neq x$.

3 Main result

We will introduce and analyze a general iterative algorithm for finding a common solution of a finite family of MEP (1.2), GSEP (1.5) and the fixed point problems of an infinite family of nonexpansive self-mappings $\{T_n\}_{n=1}^{\infty}$ on C. Under some appropriate conditions imposed on the parameter sequences, we will prove strong convergence of the proposed algorithm.

Theorem 3.1 *Let C be a nonempty closed convex subset of a Hilbert space H. Let F_m be a sequence of bifunctions from $C \times C$ to R satisfying* (A1)-(A4), *and let $\varphi : C \to R$ be a lower semicontinuous and convex function with restriction* (B1) *or* (B2) *for every $1 \le m \le N$, where N denotes some positive integer. Let $\Theta_1, \Theta_2 : C \times C \to R$ be two bifunctions satisfying* (A1)-(A4), *the mapping $B_i : C \to H$ be ζ_i-inverse strongly monotone for $i = 1,2$, B_3 be a κ-Lipschitz and η-strongly monotone operator with constants $\kappa, \eta > 0$, and let*

$f : H \to H$ be an l-Lipschitz mapping with constant $l \geq 0$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings on C and $\{\lambda_n\}$ be a sequence in $(0,b]$ for some $b \in (0,1)$. Suppose that $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma l < \tau$, where $\tau = 1 - \sqrt{1-\mu(2\eta - \mu\kappa^2)}$. Assume that $\Omega := (\bigcap_{n=1}^{\infty} F(T_n)) \cap (\bigcap_{m=1}^{N} \mathrm{MEP}(F_m,\varphi)) \cap \Gamma \neq \emptyset$, where Γ is a fixed point set of the mapping $G = T_{\mu_1}^{\Theta_1}(I-\mu_1 B_1)T_{\mu_2}^{\Theta_2}(I-\mu_2 B_2)$ with $\mu_i \in (0, 2\zeta_i)$ for $i = 1,2$. Let $\{\alpha_n\}, \{\delta_n\}, \{\beta_{n,1}\}, \ldots$ and $\{\beta_{n,N}\}$ be sequences in $[0,1]$ and $\{r_{n,m}\}$ be a sequence in $(0,\infty)$ for every $1 \leq m \leq N$ such that:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(b) $\sum_{m=1}^{N} \beta_{n,m} = 1$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1,m} - \beta_{n,m}| < \infty$ *for each* $1 \leq m \leq N$;
(c) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$;
(d) $\liminf_{n\to\infty} r_{n,m} > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1,m} - r_{n,m}| < \infty$ *for each* $1 \leq m \leq N$.

Given $x_1 \in H$ arbitrarily, the sequence $\{x_n\}$ is generated iteratively by

$$\begin{cases} y_n = \delta_n W_n G(\sum_{m=1}^{N} \beta_{n,m} u_{n,m}) + (1-\delta_n)(\sum_{m=1}^{N} \beta_{n,m} u_{n,m}), \\ x_{n+1} = P_C[\alpha_n \gamma f(x_n) + (1-\alpha_n \mu B_3) W_n G y_n], \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $u_{n,m}$ is such that

$$F_m(u_{n,m}, y) + \varphi(y) - \varphi(u_{n,m}) + \frac{1}{r_{n,m}} \langle y - u_{n,m}, u_{n,m} - x_n \rangle \geq 0, \quad \forall y \in C,$$

for each $1 \leq m \leq N$, W_n is defined by (1.17). *Then the sequence $\{x_n\}$ defined by* (3.1) *converges strongly to $x^* \in \Omega$ as $n \to \infty$, where x^* is the unique solution of the variational inequality*

$$\langle (\gamma f - \mu B_3)x^*, z - x^* \rangle \leq 0, \quad \forall z \in \Omega. \tag{3.2}$$

Proof Let $z_n = \sum_{m=1}^{N} \beta_{n,m} u_{n,m}$ in (3.1), then (3.1) reduces to

$$\begin{cases} z_n = \sum_{m=1}^{N} \beta_{n,m} u_{n,m}, \\ y_n = \delta_n W_n G z_n + (1-\delta_n) z_n, \\ x_{n+1} = P_C[\alpha_n \gamma f(x_n) + (1-\alpha_n \mu B_3) W_n G y_n], \quad \forall n \geq 1. \end{cases} \tag{3.3}$$

We divide the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. Indeed, take $p \in \Omega$ arbitrarily. Since $p = W_n p = T_{r_{n,m}}^{(F_m,\varphi)} p = Gp$, B_i is ζ_i-inverse-strongly monotone for $i = 1,2$, by Remark 1.1 we deduce from $0 \leq \mu_i \leq 2\zeta_i$, $i = 1,2$ that for any $n \geq 1$,

$$\begin{aligned} \|Gy_n - p\|^2 &= \big\| T_{\mu_1}^{\Theta_1}(I-\mu_1 B_1)T_{\mu_2}^{\Theta_2}(I-\mu_2 B_2)y_n - T_{\mu_1}^{\Theta_1}(I-\mu_1 B_1)T_{\mu_2}^{\Theta_2}(I-\mu_2 B_2)p \big\|^2 \\ &\leq \big\| (I-\mu_1 B_1)T_{\mu_2}^{\Theta_2}(I-\mu_2 B_2)y_n - (I-\mu_1 B_1)T_{\mu_2}^{\Theta_2}(I-\mu_2 B_2)p \big\|^2 \\ &= \big\| \big[T_{\mu_2}^{\Theta_2}(I-\mu_2 B_2)y_n - T_{\mu_2}^{\Theta_2}(I-\mu_2 B_2)p\big] \\ &\quad - \mu_1 \big[B_1 T_{\mu_2}^{\Theta_2}(I-\mu_2 B_2)y_n - B_1 T_{\mu_2}^{\Theta_2}(I-\mu_2 B_2)p\big] \big\|^2 \\ &\leq \big\| T_{\mu_2}^{\Theta_2}(I-\mu_2 B_2)y_n - T_{\mu_2}^{\Theta_2}(I-\mu_2 B_2)p \big\|^2 \\ &\quad + \mu_1(\mu_1 - 2\zeta_1) \big\| B_1 T_{\mu_2}^{\Theta_2}(I-\mu_2 B_2)y_n - B_1 T_{\mu_2}^{\Theta_2}(I-\mu_2 B_2)p \big\|^2 \\ &\leq \big\| T_{\mu_2}^{\Theta_2}(I-\mu_2 B_2)y_n - T_{\mu_2}^{\Theta_2}(I-\mu_2 B_2)p \big\|^2 \end{aligned}$$

$$\begin{aligned}
&\leq \left\|(I-\mu_2 B_2)y_n-(I-\mu_2 B_2)p\right\|^2 \\
&= \left\|(y_n-p)-\mu_2(B_2y_n-B_2p)\right\|^2 \\
&= \|y_n-p\|^2+\mu_2(\mu_2-2\zeta_2)\|B_2y_n-B_2p\|^2 \\
&\leq \|y_n-p\|^2=\left\|\delta_n W_n Gz_n+(1-\delta_n)z_n-p\right\|^2 \\
&\leq \delta_n\|W_nGz_n-p\|^2+(1-\delta_n)\|z_n-p\|^2 \\
&\leq \delta_n\|Gz_n-p\|^2+(1-\delta_n)\|z_n-p\|^2\leq \delta_n\|z_n-p\|^2+(1-\delta_n)\|z_n-p\|^2 \\
&\leq \|z_n-p\|^2=\left\|\sum_{m=1}^N \beta_{n,m}u_{n,m}-p\right\|^2\leq \sum_{m=1}^N\beta_{n,m}\|u_{n,m}-p\|^2 \\
&= \sum_{m=1}^N \beta_{n,m}\left\|T_{r_{n,m}}^{(F_m,\varphi)}x_n-T_{r_{n,m}}^{(F_m,\varphi)}p\right\|^2\leq \sum_{m=1}^N\beta_{n,m}\|x_n-p\|^2=\|x_n-p\|^2. \qquad (3.4)
\end{aligned}$$

(This shows that G is nonexpansive.) It follows that

$$\begin{aligned}
\|x_{n+1}-p\| &= \left\|P_C\left[\alpha_n\gamma f(x_n)+(1-\alpha_n\mu B_3)W_nGy_n\right]-p\right\| \\
&\leq \left\|\alpha_n\gamma f(x_n)+(1-\alpha_n\mu B_3)W_nGy_n-p\right\| \\
&= \left\|\alpha_n\big(\gamma f(x_n)-\mu B_3p\big)+(1-\alpha_n\mu B_3)W_nGy_n-(1-\alpha_n\mu B_3)p\right\| \\
&\leq \alpha_n\gamma l\|x_n-p\|+\alpha_n\left\|(\gamma f-\mu B_3)p\right\|+(1-\alpha_n\tau)\|W_nGy_n-p\| \\
&\leq \alpha_n\gamma l\|x_n-p\|+\alpha_n\left\|(\gamma f-\mu B_3)p\right\|+(1-\alpha_n\tau)\|Gy_n-p\| \\
&\leq \alpha_n\gamma l\|x_n-p\|+\alpha_n\left\|(\gamma f-\mu B_3)p\right\|+(1-\alpha_n\tau)\|x_n-p\| \\
&= \big(1-\alpha_n(\tau-\gamma l)\big)\|x_n-p\|+\alpha_n\left\|(\gamma f-\mu B_3)p\right\| \\
&\leq \max\left\{\|x_n-p\|,\frac{\|(\gamma f-\mu B_3)p\|}{\tau-\gamma l}\right\}.
\end{aligned}$$

By induction, we get

$$\|x_n-p\|\leq \max\left\{\|x_0-p\|,\frac{\|(\gamma f-\mu B_3)p\|}{\tau-\gamma l}\right\},\quad \forall n\geq 0.$$

Therefore, $\{x_n\}$ is bounded and so are the sequences $\{u_{n,m}\}$, $\{z_n\}$, $\{y_n\}$, $\{f(x_n)\}$ and $\{W_nGy_n\}$. Without loss of generality, suppose that there exists a bounded subset $K\subset C$ such that

$$x_n,u_{n,m},z_n,y_n,W_nGx_n,W_nGz_n,W_nGy_n\in K,\quad \forall n\geq 1. \qquad (3.5)$$

Step 2. Show that $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$.

First, we estimate $\|u_{n+1,m}-u_{n,m}\|$. Taking into account that $\liminf_{n\to\infty} r_{n,m}>0$, we may assume, without loss of generality, that $r_{n,m}\subset[\epsilon,\infty)$ for some $\epsilon>0$, for every $1\leq m\leq N$. Utilizing Remark 1.1, we get

$$\begin{aligned}
\|u_{n+1,m}-u_{n,m}\| &= \left\|T_{r_{n+1,m}}^{(F_m,\varphi)}x_{n+1}-T_{r_{n,m}}^{(F_m,\varphi)}x_n\right\| \\
&\leq \left\|T_{r_{n+1,m}}^{(F_m,\varphi)}x_{n+1}-T_{r_{n+1,m}}^{(F_m,\varphi)}x_n\right\|+\left\|T_{r_{n+1,m}}^{(F_m,\varphi)}x_n-T_{r_{n,m}}^{(F_m,\varphi)}x_n\right\|
\end{aligned}$$

$$\begin{aligned}
&\leq \|x_{n+1}-x_n\| + \frac{|r_{n+1,m}-r_{n,m}|}{r_{n+1,m}} \bigl\| T^{(F_m,\varphi)}_{r_{n+1,m}} x_n - x_n \bigr\| \\
&\leq \|x_{n+1}-x_n\| + \frac{|r_{n+1,m}-r_{n,m}|}{\epsilon} \bigl\| T^{(F_m,\varphi)}_{r_{n+1,m}} x_n - x_n \bigr\| \\
&\leq \|x_{n+1}-x_n\| + M|r_{n+1,m}-r_{n,m}|,
\end{aligned} \tag{3.6}$$

where $\sup_{n\geq 1}\{\frac{1}{\epsilon}\| T^{(F_m,\varphi)}_{r_{n+1,m}} x_n - x_n\|\} \leq M$ for some $M > 0$. Next, we estimate $\|z_{n+1}-z_n\|$.

$$\begin{aligned}
\|z_{n+1}-z_n\| &= \left\| \sum_{m=1}^{N} \beta_{n+1,m} u_{n+1,m} - \sum_{m=1}^{N} \beta_{n,m} u_{n,m} \right\| \\
&= \left\| \sum_{m=1}^{N} (\beta_{n+1,m} u_{n+1,m} - \beta_{n,m} u_{n,m}) \right\| \\
&\leq \sum_{m=1}^{N} \bigl(\|\beta_{n+1,m} u_{n+1,m} - \beta_{n,m} u_{n+1,m}\| \bigr) + \sum_{m=1}^{N} \bigl(\|\beta_{n,m} u_{n+1,m} - \beta_{n,m} u_{n,m}\| \bigr) \\
&\leq \sum_{m=1}^{N} \bigl(|\beta_{n+1,m} - \beta_{n,m}| \|u_{n+1,m}\| \bigr) + \sum_{m=1}^{N} \beta_{n,m} \|u_{n+1,m} - u_{n,m}\| \\
&\leq \sum_{m=1}^{N} \bigl(|\beta_{n+1,m} - \beta_{n,m}| \|u_{n+1,m}\| \bigr) + \sum_{m=1}^{N} \beta_{n,m} \bigl(\|x_{n+1}-x_n\| + M|r_{n+1,m}-r_{n,m}| \bigr) \\
&\leq \|x_{n+1}-x_n\| + \sum_{m=1}^{N} \beta_{n,m} M |r_{n+1,m}-r_{n,m}| + \|u_{n+1,m}\| \\
&\leq \|x_{n+1}-x_n\| + M \sum_{m=1}^{N} |r_{n+1,m}-r_{n,m}| + M_0 \sum_{m=1}^{N} |\beta_{n+1,m} - \beta_{n,m}|,
\end{aligned} \tag{3.7}$$

where $M_0 = \sum_{m=1}^{N} \|u_{n+1,m}\|$.

On the other hand, from (1.17), since W_n, T_n and $U_{n,i}$ are all nonexpansive, we have

$$\begin{aligned}
\|W_{n+1}Gz_n - W_nGz_n\| &= \|\lambda_1 T_1 U_{n+1,2} Gz_n - \lambda_1 T_1 U_{n,2} Gz_n\| \\
&\leq \lambda_1 \|U_{n+1,2} Gz_n - U_{n,2} Gz_n\| \\
&= \lambda_1 \|\lambda_2 T_2 U_{n+1,3} Gz_n - \lambda_2 T_2 U_{n,3} Gz_n\| \\
&\leq \lambda_1 \lambda_2 \|U_{n+1,3} Gz_n - U_{n,3} Gz_n\| \\
&\quad \cdots \\
&\leq \lambda_1 \lambda_2 \cdots \lambda_n \|U_{n+1,n+1} Gz_n - U_{n,n+1} Gz_n\| \\
&\leq M_1 \prod_{i=1}^{n} \lambda_i,
\end{aligned} \tag{3.8}$$

where $\sup_{n\geq 1}\{\|U_{n+1,n+1}Gz_n\| + \|U_{n,n+1}Gz_n\|\} \leq M_1$ for some $M_1 > 0$. Hence, we have

$$\begin{aligned}
&\|W_{n+1}Gz_{n+1} - W_nGz_n\| \\
&\quad \leq \|W_{n+1}Gz_{n+1} - W_{n+1}Gz_n\| + \|W_{n+1}Gz_n - W_nGz_n\|
\end{aligned}$$

$$\begin{aligned}
&\leq \|z_{n+1} - z_n\| + M_1 \prod_{i=1}^{n} \lambda_i \\
&\leq \|x_{n+1} - x_n\| + M \sum_{m=1}^{N} |r_{n+1,m} - r_{n,m}| + M_0 \sum_{m=1}^{N} |\beta_{n+1,m} - \beta_{n,m}| + M_1 \prod_{i=1}^{n} \lambda_i. \qquad (3.9)
\end{aligned}$$

Putting (3.9) and (3.7) into (3.3), we have

$$\begin{aligned}
&\|y_{n+1} - y_n\| \\
&\quad \leq \delta_{n+1} \|W_{n+1}Gz_{n+1} - W_nGz_n\| + (1 - \delta_{n+1})\|z_{n+1} - z_n\| + |\delta_{n+1} - \delta_n| \|W_nGz_n - z_n\| \\
&\quad \leq \delta_{n+1} \left(\|z_{n+1} - z_n\| + M_1 \prod_{i=1}^{n} \lambda_i \right) + (1 - \delta_{n+1})\|z_{n+1} - z_n\| + |\delta_{n+1} - \delta_n| \|W_nGz_n - z_n\| \\
&\quad \leq \|z_{n+1} - z_n\| + \delta_{n+1} M_1 \prod_{i=1}^{n} \lambda_i + |\delta_{n+1} - \delta_n| \|W_nGz_n - z_n\| \\
&\quad \leq \|x_{n+1} - x_n\| + M \sum_{m=1}^{N} |r_{n+1,m} - r_{n,m}| + M_0 \sum_{m=1}^{N} |\beta_{n+1,m} - \beta_{n,m}| \\
&\qquad + \delta_{n+1} M_1 \prod_{i=1}^{n} \lambda_i + |\delta_{n+1} - \delta_n| \|W_nGz_n - z_n\|. \qquad (3.10)
\end{aligned}$$

Similarly to (3.8), we have

$$\begin{aligned}
\|W_{n+1}Gy_n - W_nGy_n\| &= \|\lambda_1 T_1 U_{n+1,2}Gy_n - \lambda_1 T_1 U_{n,2}Gy_n\| \\
&\leq \lambda_1 \|U_{n+1,2}Gy_n - U_{n,2}Gy_n\| \\
&= \lambda_1 \|\lambda_2 T_2 U_{n+1,3}Gy_n - \lambda_2 T_2 U_{n,3}Gy_n\| \\
&\leq \lambda_1 \lambda_2 \|U_{n+1,3}Gy_n - U_{n,3}Gy_n\| \\
&\quad \cdots \\
&\leq \lambda_1 \lambda_2 \cdots \lambda_n \|U_{n+1,n+1}Gy_n - U_{n,n+1}Gy_n\| \\
&\leq M_2 \prod_{i=1}^{n} \lambda_i, \qquad (3.11)
\end{aligned}$$

where $\sup_{n \geq 1}\{\|U_{n+1,n+1}Gy_n\| + \|U_{n,n+1}Gy_n\|\} \leq M_2$ for some $M_2 > 0$. Then we have

$$\begin{aligned}
&\|W_{n+1}Gy_{n+1} - W_nGy_n\| \\
&\quad \leq \|W_{n+1}Gy_{n+1} - W_{n+1}Gy_n\| + \|W_{n+1}Gy_n - W_nGy_n\| \\
&\quad \leq \|y_{n+1} - y_n\| + M_2 \prod_{i=1}^{n} \lambda_i \\
&\quad \leq \|x_{n+1} - x_n\| + M \sum_{m=1}^{N} |r_{n+1,m} - r_{n,m}| + M_0 \sum_{m=1}^{N} |\beta_{n+1,m} - \beta_{n,m}| \\
&\qquad + \delta_{n+1} M_1 \prod_{i=1}^{n} \lambda_i + |\delta_{n+1} - \delta_n| \|W_nGz_n - z_n\| + M_2 \prod_{i=1}^{n} \lambda_i. \qquad (3.12)
\end{aligned}$$

Hence, it follows from (3.3)-(3.12) that

$$\begin{aligned}
&\|x_{n+2}-x_{n+1}\| \\
&\quad= \big\|P_C\big[\alpha_{n+1}\gamma f(x_{n+1})+(1-\alpha_{n+1}\mu B_3)W_{n+1}Gy_{n+1}\big] \\
&\qquad - P_C\big[\alpha_n\gamma f(x_n)+(1-\alpha_n\mu B_3)W_nGy_n\big]\big\| \\
&\quad\le \big\|\big[\alpha_{n+1}\gamma f(x_{n+1})+(1-\alpha_{n+1}\mu B_3)W_{n+1}Gy_{n+1}\big]-\big[\alpha_n\gamma f(x_n)+(1-\alpha_n\mu B_3)W_nGy_n\big]\big\| \\
&\quad\le \big\|\alpha_n\gamma\big(f(x_{n+1})-f(x_n)\big)+\gamma(\alpha_{n+1}-\alpha_n)f(x_{n+1}) \\
&\qquad +(1-\alpha_n\mu B_3)(W_{n+1}Gy_{n+1}-W_nGy_n)+\mu(\alpha_n-\alpha_{n+1})B_3W_{n+1}Gy_{n+1}\big\| \\
&\quad\le \alpha_n\gamma l\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\big(\gamma\big\|f(x_{n+1})\big\|+\mu\|B_3W_{n+1}Gy_{n+1}\|\big) \\
&\qquad +(1-\alpha_n\tau)\|W_{n+1}Gy_{n+1}-W_nGy_n\| \\
&\quad\le \big(1-\alpha_n(\tau-\gamma l)\big)\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\big(\gamma\big\|f(x_{n+1})\big\|+\mu\|B_3W_{n+1}Gy_{n+1}\|\big) \\
&\qquad +(1-\alpha_n\tau)\Bigg(M\sum_{m=1}^N|r_{n+1,m}-r_{n,m}|+M_0\sum_{m=1}^N|\beta_{n+1,m}-\beta_{n,m}|+\delta_{n+1}M_1\prod_{i=1}^n\lambda_i \\
&\qquad +|\delta_{n+1}-\delta_n|\|W_nGz_n-z_n\|+M_2\prod_{i=1}^n\lambda_i\Bigg) \\
&\quad\le \big(1-\alpha_n(\tau-\gamma l)\big)\|x_{n+1}-x_n\|+M_3\Bigg(|\alpha_{n+1}-\alpha_n|+\sum_{m=1}^N|r_{n+1,m}-r_{n,m}| \\
&\qquad +\sum_{m=1}^N|\beta_{n+1,m}-\beta_{n,m}|+|\delta_{n+1}-\delta_n|+b^n\Bigg),
\end{aligned} \tag{3.13}$$

where $\sup_{n\ge1}\{\gamma\|f(x_{n+1})\|+\mu\|B_3W_{n+1}Gy_{n+1}\|+M+M_0+\|W_nGz_n-z_n\|+\delta_{n+1}M_1+M_2\}\le M_3$ for some $M_3>0$. Noticing conditions (a), (b), (c), (d) and Lemma 2.7, we get $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$.

Step 3. We show that

$$\lim_{n\to\infty}\|y_n-Gy_n\|=0, \tag{3.14}$$

$$\lim_{n\to\infty}\|x_n-u_{n,m}\|=0,\quad \forall 1\le m\le N, \tag{3.15}$$

$$\lim_{n\to\infty}\|x_n-WGx_n\|=0. \tag{3.16}$$

First, we show $\lim_{n\to\infty}\|y_n-Gy_n\|=0$. Indeed, for simplicity, we write $\tilde{y}_n=T_{\mu_2}^{\Theta_2}(I-\mu_2B_2)y_n$, $\tilde{p}=T_{\mu_2}^{\Theta_2}(I-\mu_2B_2)p$, $w_n=T_{\mu_1}^{\Theta_1}(I-\mu_1B_1)\tilde{y}_n$. Then $w_n=Gy_n$ and $p=Gp$. Similar to the proof of (3.4), we get

$$\|Gy_n-p\|^2\le\|y_n-p\|^2+\mu_2(\mu_2-2\zeta_2)\|B_2y_n-B_2p\|^2+\mu_1(\mu_1-2\zeta_1)\|B\tilde{y}_n-B_1\tilde{p}\|^2. \tag{3.17}$$

From (3.3), (3.4), (3.17), we obtain that for $p\in\Omega$,

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad\le \big\|\alpha_n\gamma f(x_n)+(1-\alpha_n\mu B_3)W_nGy_n-p\big\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \left\| \alpha_n\left(\gamma f(x_n) - \mu B_3 W_n G y_n\right) + W_n G y_n - p \right\|^2 \\
&= \| W_n G y_n - p\|^2 + 2\alpha_n\left\langle \gamma f(x_n) - \mu B_3 W_n G y_n, W_n G y_n - p\right\rangle \\
&\quad + {\alpha_n}^2 \left\| \gamma f(x_n) - \mu B_3 W_n G y_n \right\|^2 \\
&\le \|G y_n - p\|^2 + \alpha_n \left\| \gamma f(x_n) - \mu B_3 W_n G y_n \right\| \\
&\quad \times \left[2\| W_n G y_n - p\| + \alpha_n \left\| \gamma f(x_n) - \mu B_3 W_n G y_n \right\|\right] \\
&\le \|y_n - p\|^2 + \mu_2(\mu_2 - 2\zeta_2)\|B_2 y_n - B_2 p\|^2 + \mu_1(\mu_1 - 2\zeta_1)\|B\tilde{y}_n - B_1\tilde{p}\|^2 \\
&\quad + \alpha_n \left\| \gamma f(x_n) - \mu B_3 W_n G y_n \right\| \left[2\| W_n G y_n - p\| + \alpha_n \left\| \gamma f(x_n) - \mu B_3 W_n G y_n \right\|\right] \\
&\le \|x_n - p\|^2 + \mu_2(\mu_2 - 2\zeta_2)\|B_2 y_n - B_2 p\|^2 + \mu_1(\mu_1 - 2\zeta_1)\|B\tilde{y}_n - B_1\tilde{p}\|^2 \\
&\quad + \alpha_n \left\| \gamma f(x_n) - \mu B_3 W_n G y_n \right\| \left[2\| W_n G y_n - p\| + \alpha_n \left\| \gamma f(x_n) - \mu B_3 W_n G y_n \right\|\right], \quad (3.18)
\end{aligned}
$$

which immediately implies that

$$
\begin{aligned}
&\mu_2(2\zeta_2 - \mu_2)\|B_2 y_n - B_2 p\|^2 + \mu_1(2\zeta_1 - \mu_1)\|B_1\tilde{y}_n - B_1\tilde{p}\|^2 \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n \left\| \gamma f(x_n) - \mu B_3 W_n G y_n \right\| \\
&\quad\quad \times \left[2\| W_n G y_n - p\| + \alpha_n \left\| \gamma f(x_n) - \mu B_3 W_n G y_n \right\|\right] \\
&\quad \le \|x_n - x_{n-1}\|\left(\|x_n - p\| + \|x_{n+1} - p\|\right) \\
&\quad\quad + \alpha_n \left\| \gamma f(x_n) - \mu B_3 W_n G y_n \right\| \left[2\| W_n G y_n - p\| + \alpha_n \left\| \gamma f(x_n) - \mu B_3 W_n G y_n \right\|\right].
\end{aligned}
$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$ and $\mu_i \in (0, 2\zeta_i)$, $i = 1, 2$, we deduce from the boundedness of $\{x_n\}, f(x_n)$ and $\{W_n G y_n\}$ that

$$
\lim_{n\to\infty} \|B_2 y_n - B_2 p\| = 0, \qquad \lim_{n\to\infty} \|B_1\tilde{y}_n - B_1\tilde{p}\| = 0. \tag{3.19}
$$

Also, in terms of the firm nonexpansivity of $T^{\Theta_1}_{\mu_1}$, $T^{\Theta_2}_{\mu_2}$, we obtain from $\mu_i \in (0, 2\zeta_i)$, $i = 1, 2$, that

$$
\begin{aligned}
\|\tilde{y}_n - \tilde{p}\|^2 &= \left\| T^{\Theta_2}_{\mu_2}(I - \mu_2 B_2)y_n - T^{\Theta_2}_{\mu_2}(I - \mu_2 B_2)p \right\|^2 \\
&\le \left\langle (I - \mu_2 B_2)y_n - (I - \mu_2 B_2)p, \tilde{y}_n - \tilde{p}\right\rangle \\
&= \frac{1}{2}\Big[\left\|(I - \mu_2 B_2)y_n - (I - \mu_2 B_2)p\right\|^2 + \|\tilde{y}_n - \tilde{p}\|^2 \\
&\quad - \left\|(I - \mu_2 B_2)y_n - (I - \mu_2 B_2)p - (\tilde{y}_n - \tilde{p})\right\|^2\Big] \\
&\le \frac{1}{2}\Big[\|y_n - p\|^2 + \|\tilde{y}_n - \tilde{p}\|^2 - \left\|(y_n - \tilde{y}_n) - \mu_2(B_2 y_n - B_2 p) - (p - \tilde{p})\right\|^2\Big] \\
&\le \frac{1}{2}\Big[\|x_n - p\|^2 + \|\tilde{y}_n - \tilde{p}\|^2 - \left\|(y_n - \tilde{y}_n) - (p - \tilde{p})\right\|^2 \\
&\quad + 2\mu_2\left\langle (y_n - \tilde{y}_n) - (p - \tilde{p}), B_2 y_n - B_2 p\right\rangle\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
\|w_n - p\|^2 &= \left\| T^{\Theta_1}_{\mu_1}(I - \mu_1 B_1)\tilde{y}_n - T^{\Theta_1}_{\mu_1}(I - \mu_1 B_1)\tilde{p} \right\|^2 \\
&\le \left\langle (I - \mu_1 B_1)\tilde{y}_n - (I - \mu_1 B_1)\tilde{p}, w_n - p\right\rangle
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1}{2}\Big[\big\|(I-\mu_1B_1)\tilde{y}_n-(I-\mu_1B_1)\tilde{p}\big\|^2+\|w_n-p\|^2 \\
&\quad -\big\|(I-\mu_1B_1)\tilde{y}_n-(I-\mu_1B_1)\tilde{p}-(w_n-p)\big\|^2\Big] \\
&\le \frac{1}{2}\Big[\|\tilde{y}_n-\tilde{p}\|^2+\|w_n-p\|^2-\big\|(\tilde{y}_n-w_n)+(p-\tilde{p})\big\|^2 \\
&\quad -2\mu_1\big\langle B_1\tilde{y}_n-B_1\tilde{p},(\tilde{y}_n-w_n)+(p-\tilde{p})\big\rangle-\mu_1{}^2\|B_1\tilde{y}_n-B_1\tilde{p}\|^2\Big] \\
&\le \frac{1}{2}\Big[\|x_n-p\|^2+\|w_n-p\|^2-\big\|(\tilde{y}_n-w_n)+(p-\tilde{p})\big\|^2 \\
&\quad +2\mu_1\big\langle B_1\tilde{y}_n-B_1\tilde{p},(\tilde{y}_n-w_n)+(p-\tilde{p})\big\rangle\Big].
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|\tilde{y}_n-\tilde{p}\|^2 &\le \|x_n-p\|^2-\big\|(y_n-\tilde{y}_n)-(p-\tilde{p})\big\|^2 \\
&\quad +2\mu_2\big\langle(y_n-\tilde{y}_n)-(p-\tilde{p}),B_2y_n-B_2p\big\rangle
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
\|w_n-p\|^2 &\le \|x_n-p\|^2-\big\|(\tilde{y}_n-w_n)+(p-\tilde{p})\big\|^2 \\
&\quad +2\mu_1\big\langle B_1\tilde{y}_n-B_1\tilde{p},(\tilde{y}_n-w_n)+(p-\tilde{p})\big\rangle.
\end{aligned} \tag{3.21}$$

Consequently, it follows from (3.4), (3.18) and (3.20) that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad \le \|Gy_n-p\|^2+\alpha_n\big\|\gamma f(x_n)-\mu B_3W_nGy_n\big\| \\
&\qquad \times\big[2\|W_nGy_n-p\|+\alpha_n\big\|\gamma f(x_n)-\mu B_3W_nGy_n\big\|\big] \\
&\quad \le \|\tilde{y}_n-\tilde{p}\|^2+\alpha_n\big\|\gamma f(x_n)-\mu B_3W_nGy_n\big\| \\
&\qquad \times\big[2\|W_nGy_n-p\|+\alpha_n\big\|\gamma f(x_n)-\mu B_3W_nGy_n\big\|\big] \\
&\quad \le \|x_n-p\|^2-\big\|(y_n-\tilde{y}_n)-(p-\tilde{p})\big\|^2+2\mu_2\big\langle(y_n-\tilde{y}_n)-(p-\tilde{p}),B_2y_n-B_2p\big\rangle \\
&\qquad +\alpha_n\big\|\gamma f(x_n)-\mu B_3W_nGy_n\big\|\big[2\|W_nGy_n-p\|+\alpha_n\big\|\gamma f(x_n)-\mu B_3W_nGy_n\big\|\big],
\end{aligned}$$

which yields

$$\begin{aligned}
&\big\|(y_n-\tilde{y}_n)-(p-\tilde{p})\big\|^2 \\
&\quad \le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\mu_2\big\|(y_n-\tilde{y}_n)-(p-\tilde{p})\big\|\|B_2y_n-B_2p\| \\
&\qquad +\alpha_n\big\|\gamma f(x_n)-\mu B_3W_nGy_n\big\|\big[2\|W_nGy_n-p\|+\alpha_n\big\|\gamma f(x_n)-\mu B_3W_nGy_n\big\|\big] \\
&\quad \le \big(\|x_n-x_{n+1}\|\big)\big(\|x_n-p\|+\|x_{n+1}-p\|\big)+2\mu_2\big\|(y_n-\tilde{y}_n)-(p-\tilde{p})\big\|\|B_2y_n-B_2p\| \\
&\qquad +\alpha_n\big\|\gamma f(x_n)-\mu B_3W_nGy_n\big\|\big[2\|W_nGy_n-p\|+\alpha_n\big\|\gamma f(x_n)-\mu B_3W_nGy_n\big\|\big].
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ and $\lim_{n\to\infty}\|B_2y_n-B_2p\|=0$, we deduce that

$$\lim_{n\to\infty}\big\|(y_n-\tilde{y}_n)-(p-\tilde{p})\big\|=0. \tag{3.22}$$

Furthermore, it follows from (3.4), (3.18) and (3.21) that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad \le \|Gy_n-p\|^2+\alpha_n\big\|\gamma f(x_n)-\mu B_3W_nGy_n\big\| \\
&\qquad \times\big[2\|W_nGy_n-p\|+\alpha_n\big\|\gamma f(x_n)-\mu B_3W_nGy_n\big\|\big] \\
&\quad \le \|w_n-p\|^2+\alpha_n\big\|\gamma f(x_n)-\mu B_3W_nGy_n\big\| \\
&\qquad \times\big[2\|W_nGy_n-p\|+\alpha_n\big\|\gamma f(x_n)-\mu B_3W_nGy_n\big\|\big] \\
&\quad \le \|x_n-p\|^2-\big\|(\tilde{y}_n-w_n)+(p-\tilde{p})\big\|^2+2\mu_1\langle B_1\tilde{y}_n-B_1\tilde{p},(\tilde{y}_n-w_n)+(p-\tilde{p})\rangle \\
&\qquad +\alpha_n\big\|\gamma f(x_n)-\mu B_3W_nGy_n\big\|\big[2\|W_nGy_n-p\|+\alpha_n\big\|\gamma f(x_n)-\mu B_3W_nGy_n\big\|\big],
\end{aligned}$$

which leads to

$$\begin{aligned}
&\big\|(\tilde{y}_n-w_n)+(p-\tilde{p})\big\|^2 \\
&\quad \le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\mu_1\|B_1\tilde{y}_n-B_1\tilde{p}\|\big\|(\tilde{y}_n-w_n)+(p-\tilde{p})\big\| \\
&\qquad +\alpha_n\big\|\gamma f(x_n)-\mu B_3W_nGy_n\big\|\big[2\|W_nGy_n-p\|+\alpha_n\big\|\gamma f(x_n)-\mu B_3W_nGy_n\big\|\big] \\
&\quad \le \big(\|x_n-x_{n+1}\|\big)\big(\|x_n-p\|+\|x_{n+1}-p\|\big)+2\mu_1\|B_1\tilde{y}_n-B_1\tilde{p}\|\big\|(\tilde{y}_n-w_n)+(p-\tilde{p})\big\| \\
&\qquad +\alpha_n\big\|\gamma f(x_n)-\mu B_3W_nGy_n\big\|\big[2\|W_nGy_n-p\|+\alpha_n\big\|\gamma f(x_n)-\mu B_3W_nGy_n\big\|\big].
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ and $\lim_{n\to\infty}\|B_1\tilde{y}_n-B_1\tilde{p}\|=0$, we deduce that

$$\lim_{n\to\infty}\big\|(\tilde{y}_n-w_n)+(p-\tilde{p})\big\|=0. \tag{3.23}$$

Note that

$$\|y_n-w_n\|\le\big\|(y_n-\tilde{y}_n)-(p-\tilde{p})\big\|+\big\|(\tilde{y}_n-w_n)+(p-\tilde{p})\big\|.$$

Hence from (3.20) and (3.21), we get

$$\lim_{n\to\infty}\|y_n-w_n\|=\lim_{n\to\infty}\|y_n-Gy_n\|=0.$$

Next, we show that $\lim_{n\to\infty}\|x_n-u_{n,m}\|=0$ for every $1\le m\le N$ and $\lim_{n\to\infty}\|x_n-WGx_n\|=0$. Indeed, by Proposition 1.2(c), we obtain that for any $p\in\Omega$ and for each $1\le m\le N$,

$$\begin{aligned}
\|u_{n,m}-p\|^2 &= \big\|T_{r_{n,m}}^{(F_m,\varphi)}x_n-T_{r_{n,m}}^{(F_m,\varphi)}p\big\|^2 \\
&\le \langle u_{n,m}-p,x_n-p\rangle \\
&= \frac{1}{2}\big[\|u_{n,m}-p\|^2+\|x_n-p\|^2-\|u_{n,m}-x_n\|^2\big].
\end{aligned}$$

That is,

$$\|u_{n,m}-p\|^2\le\|x_n-p\|^2-\|u_{n,m}-x_n\|^2.$$

Then we have

$$\begin{aligned}\|z_n-p\|^2 &= \left\|\sum_{m=1}^{N}\beta_{n,m}u_{n,m}-p\right\|^2 \le \sum_{m=1}^{N}\beta_{n,m}\|u_{n,m}-p\|^2\\ &\le \sum_{m=1}^{N}\bigl(\|x_n-p\|^2-\|u_{n,m}-x_n\|^2\bigr)\\ &= \|x_n-p\|^2-\sum_{m=1}^{N}\|u_{n,m}-x_n\|^2.\end{aligned}$$

It follows that

$$\|y_n-p\|^2 \le \|z_n-p\|^2 \le \|x_n-p\|^2-\sum_{m=1}^{N}\|u_{n,m}-x_n\|^2. \tag{3.24}$$

It follows from (3.18) and (3.24) that

$$\begin{aligned}&\|x_{n+1}-p\|^2\\ &\quad\le \|y_n-p\|^2+\alpha_n\bigl\|\gamma f(x_n)-\mu B_3W_nGy_n\bigr\|\\ &\qquad\times\bigl[2\|W_nGy_n-p\|+\alpha_n\bigl\|\gamma f(x_n)-\mu B_3W_nGy_n\bigr\|\bigr]\\ &\quad\le \|x_n-p\|^2-\sum_{m=1}^{N}\|u_{n,m}-x_n\|^2+\alpha_n\bigl\|\gamma f(x_n)-\mu B_3W_nGy_n\bigr\|\\ &\qquad\times\bigl[2\|W_nGy_n-p\|+\alpha_n\bigl\|\gamma f(x_n)-\mu B_3W_nGy_n\bigr\|\bigr],\end{aligned}\tag{3.25}$$

which immediately implies that

$$\begin{aligned}&\sum_{m=1}^{N}\|u_{n,m}-x_n\|^2\\ &\quad\le \|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\qquad+\alpha_n\bigl\|\gamma f(x_n)-\mu B_3W_nGy_n\bigr\|\bigl[2\|W_nGy_n-p\|+\alpha_n\bigl\|\gamma f(x_n)-\mu B_3W_nGy_n\bigr\|\bigr]\\ &\quad\le \bigl(\|x_n-x_{n+1}\|\bigr)\bigl(\|x_n-p\|+\|x_{n+1}-p\|\bigr)\\ &\qquad+\alpha_n\bigl\|\gamma f(x_n)-\mu B_3W_nGy_n\bigr\|\bigl[2\|W_nGy_n-p\|+\alpha_n\bigl\|\gamma f(x_n)-\mu B_3W_nGy_n\bigr\|\bigr].\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}\|x_n-x_{n+1}\|=0$, we deduce that

$$\lim_{n\to\infty}\|u_{n,m}-x_n\|=0,\quad \forall 1\le m\le N. \tag{3.26}$$

Since

$$\|z_n-x_n\|=\left\|\sum_{m=1}^{N}\beta_{n,m}u_{n,m}-x_n\right\|\le\sum_{m=1}^{N}\beta_{n,m}\|u_{n,m}-x_n\|,$$

from $\lim_{n\to\infty}\|u_{n,m}-x_n\|=0$, we get

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{3.27}$$

Notice that

$$\begin{aligned}\|y_n-x_n\| &\le \|y_n-z_n\|+\|z_n-x_n\| \\ &\le \bigl\|\delta_n W_n G z_n+(1-\delta_n)z_n-z_n\bigr\|+\|z_n-x_n\| \\ &= \delta_n\|W_n G z_n-z_n\|+\|z_n-x_n\|.\end{aligned}$$

Since $\delta_n\to 0$ and $\|z_n-x_n\|\to 0$, we get

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.28}$$

Note that

$$\begin{aligned}\|x_n-W_nGx_n\| &\le \|x_n-W_nGy_n\|+\|W_nGy_n-W_nGx_n\| \\ &\le \|x_n-W_nGy_n\|+\|y_n-x_n\|.\end{aligned} \tag{3.29}$$

On the other hand,

$$\begin{aligned}\|x_n-W_nGy_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-W_nGy_n\| \\ &= \|x_n-x_{n+1}\|+\bigl\|P_C\bigl[\alpha_n\gamma f(x_n)+(1-\alpha_n\mu B_3)W_nGy_n\bigr]-P_CW_nGy_n\bigr\| \\ &\le \|x_n-x_{n+1}\|+\bigl\|\alpha_n\gamma f(x_n)+(1-\alpha_n\mu B_3)W_nGy_n-W_nGy_n\bigr\| \\ &= \|x_n-x_{n+1}\|+\alpha_n\bigl\|\gamma f(x_n)-\mu B_3W_nGy_n\bigr\|\to 0.\end{aligned}$$

From $\lim_{n\to\infty}\|x_n-x_{n+1}\|=0$ and $\lim_{n\to\infty}\alpha_n=0$, we get

$$\lim_{n\to\infty}\|x_n-W_nGx_n\|=0. \tag{3.30}$$

Note that

$$\|x_n-WGx_n\|\le\|x_n-W_nGx_n\|+\|W_nGx_n-WGx_n\|.$$

From (3.30) and Remark 2.2, we see

$$\lim_{n\to\infty}\|x_n-WGx_n\|=0.$$

Step 4. Now we shall prove

$$\limsup_{n\to\infty}\bigl\langle x_n-x^*,(\gamma f-\mu B_3)x^*\bigr\rangle\le 0, \tag{3.31}$$

where x^* is the unique solution of variational inequality (3.2). To show this, we take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\bigl\langle x_n-x^*,(\gamma f-\mu B_3)x^*\bigr\rangle=\lim_{i\to\infty}\bigl\langle x_{n_i}-x^*,(\gamma f-\mu B_3)x^*\bigr\rangle. \tag{3.32}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence of $\{x_{n_i}\}$. Without loss of generality, we can still denote it by $\{x_{n_i}\}$ such that $x_{n_i} \rightharpoonup \omega$. Let us show $\omega \in \Omega := (\bigcap_{n=1}^{\infty} F(T_n)) \cap (\bigcap_{m=1}^{N} \operatorname{MEP}(F_m, \varphi)) \cap \Gamma$.

We first show $\omega \in \Gamma$. From $\|y_n - Gy_n\| \to 0$ and $\|x_n - y_n\| \to 0$ and Lemma 2.5 (demiclosedness principle), we have $\omega \in F(G) = \Gamma$.

Next we show $\omega \in \bigcap_{m=1}^{N} \operatorname{MEP}(F_m, \varphi)$. Since $u_{n,m} = T_{r_{n,m}}^{(F_m,\varphi)} x_n$, we have

$$F_m(u_{n,m}, y) + \varphi(y) - \varphi(u_{n,m}) + \frac{1}{r_{n,m}} \langle y - u_{n,m}, u_{n,m} - x_n \rangle \geq 0, \quad \forall y \in C.$$

It follows from (A2) that

$$\varphi(y) - \varphi(u_{n,m}) + \frac{1}{r_{n,m}} \langle y - u_{n,m}, u_{n,m} - x_n \rangle \geq F_m(y, u_{n,m}), \quad \forall y \in C.$$

Replacing n by n_i, we arrive at

$$\varphi(y) - \varphi(u_{n_i,m}) + \left\langle \frac{u_{n_i,m} - x_{n_i}}{r_{n_i,m}}, y - u_{n_i,m} \right\rangle \geq F_m(y, u_{n_i,m}), \quad \forall y \in C. \tag{3.33}$$

Put $y_{t_m} = t_m y + (1 - t_m)\omega$ for all $t_m \in (0,1]$ and $y \in C$. Then from (3.33) we have

$$0 \geq -\varphi(y_{t_m}) + \varphi(u_{n_i,m}) - \left\langle \frac{u_{n_i,m} - x_{n_i}}{r_{n_i,m}}, y_{t_m} - u_{n_i,m} \right\rangle + F_m(y_{t_m}, u_{n_i,m}).$$

So, from (A4), the weak lower semicontinuity of φ, $\frac{u_{n_i,m} - x_{n_i}}{r_{n_i,m}} \to 0$ and $u_{n_i} \rightharpoonup \omega$, we have

$$0 \geq -\varphi(y_{t_m}) + \varphi(\omega) + F_m(y_{t_m}, \omega) \quad \text{as } i \to \infty. \tag{3.34}$$

From (A1), (A4) and (3.34), we also have

$$\begin{aligned} 0 &= F_m(y_{t_m}, y_{t_m}) + \varphi(y_{t_m}) - \varphi(y_{t_m}) \\ &\leq t_m F_m(y_{t_m}, y) + (1 - t_m) F_m(y_{t_m}, \omega) + t_m \varphi(y) - (1 - t_m)\varphi(\omega) - \varphi(y_{t_m}) \\ &= t_m \bigl[F_m(y_{t_m}, y) + \varphi(y) - \varphi(y_{t_m})\bigr] + (1 - t_m)\bigl[F_m(y_{t_m}, \omega) + \varphi(\omega) - \varphi(y_{t_m})\bigr] \\ &\leq t_m \bigl[F_m(y_{t_m}, y) + \varphi(y) - \varphi(y_{t_m})\bigr], \end{aligned}$$

and hence

$$0 \leq F_m(y_{t_m}, y) + \varphi(y) - \varphi(y_{t_m}).$$

Letting $t_m \to 0$, we have, for each $y \in C$,

$$0 \leq F_m(\omega, y) + \varphi(y) - \varphi(\omega).$$

This implies $\omega \in \operatorname{MEP}(F_m, \varphi)$ for each $1 \leq m \leq N$. Therefore $\omega \in \bigcap_{m=1}^{N} \operatorname{MEP}(F_m, \varphi)$.

Last we show $\omega \in F(W) = \bigcap_{n=1}^{\infty} F(T_n)$. If $\omega \notin F(W)$. From Opial's lemma (Lemma 2.8), we have

$$\begin{aligned}\liminf_{i\to\infty} \|x_{n_i} - \omega\| &< \liminf_{i\to\infty} \|x_{n_i} - WG\omega\| \\ &\le \liminf_{i\to\infty} \bigl(\|x_{n_i} - WGx_{n_i}\| + \|WGx_{n_i} - WG\omega\|\bigr) \\ &\le \liminf_{i\to\infty} \bigl(\|x_{n_i} - WGx_{n_i}\| + \|x_{n_i} - \omega\|\bigr).\end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - WGx_n\| = 0$, we have

$$\liminf_{i\to\infty} \|x_{n_i} - \omega\| < \liminf_{i\to\infty} \|x_{n_i} - \omega\|.$$

This is a contradiction. Therefore, we have $\omega \in F(W) = \bigcap_{n=1}^{\infty} F(T_n)$, that is,

$$\omega \in F(W) \cap \left(\bigcap_{m=1}^{N} \mathrm{MEP}(F_m, \varphi)\right) \cap \Gamma = \left(\bigcap_{n=1}^{\infty} F(T_n)\right) \cap \left(\bigcap_{m=1}^{N} \mathrm{MEP}(F_m, \varphi)\right) \cap \Gamma = \Omega.$$

Since $\omega \in \Omega$, due to (3.32) and the property of metric projection, we have

$$\begin{aligned}\limsup_{n\to\infty} \bigl\langle x_n - x^*, (\gamma f - \mu B_3)x^*\bigr\rangle &= \lim_{i\to\infty} \bigl\langle x_{n_i} - x^*, (\gamma f - \mu B_3)x^*\bigr\rangle \\ &= \bigl\langle \omega - x^*, (\gamma f - \mu B_3)x^*\bigr\rangle \le 0. \qquad (3.35)\end{aligned}$$

Step 5. Finally, we prove that $x_n \to x^*$ as $n \to \infty$. Setting $v_n = \alpha_n \gamma f(x_n) + (1 - \alpha_n \mu B_3) W_n G y_n$, $\forall n \ge 1$. Then we can rewrite (3.1) as

$$\begin{cases} z_n = \sum_{m=1}^{N} \beta_{n,m} u_{n,m}, \\ y_n = \delta_n W_n G z_n + (1 - \delta_n) z_n, \\ x_{n+1} = P_C v_n. \end{cases}$$

It follows from (3.3) and Proposition 1.1(i) that

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\quad = \bigl\langle P_C v_n - v_n, P_C v_n - x^*\bigr\rangle + \bigl\langle v_n - x^*, x_{n+1} - x^*\bigr\rangle \\ &\quad \le \bigl\langle v_n - x^*, x_{n+1} - x^*\bigr\rangle \\ &\quad = \bigl\langle \alpha_n \gamma f(x_n) + (1 - \alpha_n \mu B_3) W_n G y_n - x^*, x_{n+1} - x^*\bigr\rangle \\ &\quad = \bigl\langle \alpha_n \bigl(\gamma f(x_n) - \mu B_3 x^*\bigr) + (1 - \alpha_n \mu B_3)\bigl(W_n G y_n - x^*\bigr), x_{n+1} - x^*\bigr\rangle \\ &\quad \le \bigl\langle \alpha_n \gamma \bigl(f(x_n) - f(x^*)\bigr) + (1 - \alpha_n \mu B_3)\bigl(W_n G y_n - x^*\bigr), x_{n+1} - x^*\bigr\rangle \\ &\qquad + \alpha_n \bigl\langle (\gamma f - \mu B_3)x^*, x_{n+1} - x^*\bigr\rangle \\ &\quad \le \bigl\|\alpha_n \gamma \bigl(f(x_n) - f(x^*)\bigr) + (1 - \alpha_n \mu B_3)\bigl(W_n G y_n - x^*\bigr)\bigr\| \|x_{n+1} - x^*\| \\ &\qquad + \alpha_n \bigl\langle (\gamma f - \mu B_3)x^*, x_{n+1} - x^*\bigr\rangle \\ &\quad \le \bigl[\alpha_n \gamma l \|x_n - x^*\| + (1 - \alpha_n \tau)\|W_n G y_n - x^*\|\bigr] \|x_{n+1} - x^*\| \\ &\qquad + \alpha_n \bigl\langle (\gamma f - \mu B_3)x^*, x_{n+1} - x^*\bigr\rangle\end{aligned}$$

$$
\begin{aligned}
&\le \left[\alpha_n \gamma l \|x_n - x^*\| + (1-\alpha_n\tau)\|x_n - x^*\|\right]\|x_{n+1} - x^*\| + \alpha_n\langle(\gamma f - \mu B_3)x^*, x_{n+1} - x^*\rangle \\
&\le \left(1 - \alpha_n(\tau - \gamma l)\right)\|x_n - x^*\|\|x_{n+1} - x^*\| + \alpha_n\langle(\gamma f - \mu B_3)x^*, x_{n+1} - x^*\rangle \\
&\le \frac{1}{2}\left(1 - \alpha_n(\tau - \gamma l)\right)\left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\right) + \alpha_n\langle(\gamma f - \mu B_3)x^*, x_{n+1} - x^*\rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad\le \frac{1 - \alpha_n(\tau - \gamma l)}{1 + \alpha_n(\tau - \gamma l)}\|x_n - x^*\|^2 + \frac{\alpha_n}{1 + \alpha_n(\tau - \gamma l)}\langle(\gamma f - \mu B_3)x^*, x_{n+1} - x^*\rangle \\
&\quad= \left(1 - \frac{2\alpha_n(\tau - \gamma l)}{1 + \alpha_n(\tau - \gamma l)}\right)\|x_n - x^*\|^2 \\
&\qquad+ \frac{2\alpha_n(\tau - \gamma l)}{1 + \alpha_n(\tau - \gamma l)} \cdot \frac{1}{2(\tau - \gamma l)}\langle(\gamma f - \mu B_3)x^*, x_{n+1} - x^*\rangle \\
&\quad= (1 - \gamma_n)\|x_n - x^*\|^2 + \sigma_n\gamma_n,
\end{aligned}
\tag{3.36}
$$

where $\gamma_n = \frac{2\alpha_n(\tau-\gamma l)}{1+\alpha_n(\tau-\gamma l)}$ and

$$
\sigma_n = \frac{1}{2(\tau - \gamma l)}\langle(\gamma f - \mu B_3)x^*, x_{n+1} - x^*\rangle.
$$

It is easily seen that $\sum_{n=1}^{\infty}\gamma_n = \infty$ and $\limsup_{n\to\infty}\sigma_n \le 0$ (due to condition (a) and (3.35)). According to Lemma 2.7 we conclude that $\{x_n\}$ converges strongly to x^*. This completes the proof. □

Putting $T_n \equiv I$ the identity mapping, we obtain from Theorem 3.1 the following.

Corollary 3.1 *Let C be a nonempty closed convex subset of a Hilbert space H. Let F_m be a sequence of bifunctions from $C \times C$ to R satisfying* (A1)-(A4), *and let $\varphi: C \to R$ be a lower semicontinuous and convex function with restriction* (B1) *or* (B2) *for every $1 \le m \le N$, where N denotes some positive integer. Let $\Theta_1, \Theta_2: C \times C \to R$ be two bifunctions satisfying* (A1)-(A4), *the mapping $B_i: C \to H$ be ζ_i-inverse strongly monotone for $i = 1, 2$, B_3 be a κ-Lipschitz and η-strongly monotone operator with constants $\kappa, \eta > 0$, and let $f: H \to H$ be an l-Lipschitz mapping with constant $l > 0$. Suppose that $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that $\Omega := (\bigcap_{m=1}^{N} \mathrm{MEP}(F_m, \varphi)) \cap \Gamma \ne \emptyset$, where Γ is a fixed point set of the mapping $G = T_{\mu_1}^{\Theta_1}(I - \mu_1 B_1)T_{\mu_2}^{\Theta_2}(I - \mu_2 B_2)$ with $\mu_i \in (0, 2\zeta_i)$ for $i = 1, 2$. Let $\{\alpha_n\}, \{\delta_n\}, \{\beta_{n,1}\}, \ldots$ and $\{\beta_{n,N}\}$ be sequences in $[0,1]$ and $\{r_{n,m}\}$ be a sequence in $(0,\infty)$ for every $1 \le m \le N$ such that*:

(a) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ *and* $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$;
(b) $\sum_{m=1}^{N}\beta_{n,m} = 1$ *and* $\sum_{n=1}^{\infty}|\beta_{n+1,m} - \beta_{n,m}| < \infty$ *for each* $1 \le m \le N$;
(c) $\lim_{n\to\infty}\delta_n = 0$ *and* $\sum_{n=1}^{\infty}|\delta_{n+1} - \delta_n| < \infty$;
(d) $\liminf_{n\to\infty} r_{n,m} > 0$ *and* $\sum_{n=1}^{\infty}|r_{n+1,m} - r_{n,m}| < \infty$ *for each* $1 \le m \le N$.

Given $x_1 \in H$ arbitrarily, the sequence $\{x_n\}$ is generated iteratively by

$$
\begin{cases}
y_n = \delta_n G(\sum_{m=1}^{N}\beta_{n,m}u_{n,m}) + (1 - \delta_n)(\sum_{m=1}^{N}\beta_{n,m}u_{n,m}), \\
x_{n+1} = P_C[\alpha_n\gamma f(x_n) + (1 - \alpha_n\mu B_3)Gy_n], \quad \forall n \ge 1,
\end{cases}
\tag{3.37}
$$

where $u_{n,m}$ is such that

$$F_m(u_{n,m},y) + \varphi(y) - \varphi(u_{n,m}) + \frac{1}{r_{n,m}}\langle y - u_{n,m}, u_{n,m} - x_n\rangle \geq 0, \quad \forall y \in C,$$

for each $1 \leq m \leq N$. Then the sequence $\{x_n\}$ defined by (3.37) *converges strongly to $x^* \in \Omega$, as $n \to \infty$, where x^* is the unique solution of the variational inequality*

$$\bigl\langle (\gamma f - \mu B_3)x^*, z - x^*\bigr\rangle \leq 0, \quad \forall z \in \Omega.$$

If $N = 1$, we obtain from Theorem 3.1 the following.

Corollary 3.2 *Let C be a nonempty closed convex subset of a Hilbert space H. Let F be a sequence of bifunctions from $C \times C$ to R satisfying* (A1)-(A4), *and let $\varphi : C \to R$ be a lower semicontinuous and convex function with restriction* (B1) *or* (B2). *Let $\Theta_1, \Theta_2 : C \times C \to R$ be two bifunctions satisfying* (A1)-(A4), *the mapping $B_i : C \to H$ be ζ_i-inverse strongly monotone for $i = 1, 2$, B_3 be a κ-Lipschitz and η-strongly monotone operator with constants $\kappa, \eta > 0$, and let $f : H \to H$ be an l-Lipschitz mapping with constant $l \geq 0$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings on C and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Suppose that $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that $\Omega := (\bigcap_{n=1}^{\infty} F(T_n)) \cap (\mathrm{MEP}(F, \varphi)) \cap \Gamma \neq \emptyset$, where Γ is a fixed point set of the mapping $G = T_{\mu_1}^{\Theta_1}(I - \mu_1 B_1)T_{\mu_2}^{\Theta_2}(I - \mu_2 B_2)$ with $\mu_i \in (0, 2\zeta_i)$ for $i = 1, 2$. Let $\{\alpha_n\}$, $\{\delta_n\}$ be sequences in $[0, 1]$ and $\{r_n\}$ be a sequence in $(0, \infty)$ such that*:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(b) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$;
(c) $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Given $x_1 \in H$ arbitrarily, the sequence $\{x_n\}$ is generated iteratively by

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \delta_n W_n G u_n + (1 - \delta_n)u_n, \\ x_{n+1} = P_C[\alpha_n \gamma f(x_n) + (1 - \alpha_n \mu B_3)W_n G y_n], \quad \forall n \geq 1, \end{cases} \tag{3.38}$$

where W_n is defined by (1.17). *Then the sequence $\{x_n\}$ defined by* (3.38) *converges strongly to $x^* \in \Omega$, as $n \to \infty$, where x^* is the unique solution of the variational inequality*

$$\bigl\langle (\gamma f - \mu B_3)x^*, z - x^*\bigr\rangle \leq 0, \quad \forall z \in \Omega.$$

If $\varphi \equiv 0$, we obtain from Theorem 3.1 the following.

Corollary 3.3 *Let C be a nonempty closed convex subset of a Hilbert space H. Let F_m be a sequence of bifunctions from $C \times C$ to R satisfying* (A1)-(A4) *for every $1 \leq m \leq N$, where N denotes some positive integer. Let $\Theta_1, \Theta_2 : C \times C \to R$ be two bifunctions satisfying* (A1)-(A4), *the mapping $B_i : C \to H$ be ζ_i-inverse strongly monotone for $i = 1, 2$, B_3 be a κ-Lipschitz and η-strongly monotone operator with constants $\kappa, \eta > 0$, and let $f : H \to H$ be an l-Lipschitz mapping with constant $l \geq 0$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings on C and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Suppose that $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that*

$\Omega := (\bigcap_{n=1}^{\infty} F(T_n)) \cap (\bigcap_{m=1}^{N} \mathrm{EP}(F_m)) \cap \Gamma \neq \emptyset$, where Γ is a fixed point set of the mapping $G = T_{\mu_1}^{\Theta_1}(I - \mu_1 B_1) T_{\mu_2}^{\Theta_2}(I - \mu_2 B_2)$ with $\mu_i \in (0, 2\zeta_i)$ for $i = 1, 2$. Let $\{\alpha_n\}, \{\delta_n\}, \{\beta_{n,1}\}, \ldots$ and $\{\beta_{n,N}\}$ be sequences in $[0,1]$ and $\{r_{n,m}\}$ be a sequence in $(0, \infty)$ for every $1 \le m \le N$ such that:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(b) $\sum_{m=1}^{N} \beta_{n,m} = 1$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1,m} - \beta_{n,m}| < \infty$ *for each* $1 \le m \le N$;
(c) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$;
(d) $\liminf_{n\to\infty} r_{n,m} > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1,m} - r_{n,m}| < \infty$ *for each* $1 \le m \le N$.

Given $x_1 \in H$ arbitrarily, the sequence $\{x_n\}$ is generated iteratively by

$$\begin{cases} y_n = \delta_n W_n G(\sum_{m=1}^{N} \beta_{n,m} u_{n,m}) + (1 - \delta_n)(\sum_{m=1}^{N} \beta_{n,m} u_{n,m}), \\ x_{n+1} = P_C[\alpha_n \gamma f(x_n) + (1 - \alpha_n \mu B_3) W_n G y_n], \quad \forall n \ge 1, \end{cases} \tag{3.39}$$

where $u_{n,m}$ is such that

$$F_m(u_{n,m}, y) + \frac{1}{r_{n,m}} \langle y - u_{n,m}, u_{n,m} - x_n \rangle \ge 0, \quad \forall y \in C,$$

for each $1 \le m \le N$, W_n is defined by (1.17). *Then the sequence $\{x_n\}$ defined by* (3.39) *converges strongly to $x^* \in \Omega$, as $n \to \infty$, where x^* is the unique solution of the variational inequality*

$$\langle (\gamma f - \mu B_3) x^*, z - x^* \rangle \le 0, \quad \forall z \in \Omega.$$

If $\Theta_1 \equiv \Theta_2 \equiv 0$, we obtain from Theorem 3.1 the following.

Corollary 3.4 *Let C be a nonempty closed convex subset of a Hilbert space H. Let F_m be a sequence of bifunctions from $C \times C$ to R satisfying* (A1)-(A4), *and let $\varphi : C \to R$ be a lower semicontinuous and convex function with restriction* (B1) *or* (B2) *for every $1 \le m \le N$, where N denotes some positive integer. Let the mapping $B_i : C \to H$ be ζ_i-inverse strongly monotone for $i = 1, 2$, B_3 be a κ-Lipschitz and η-strongly monotone operator with constants $\kappa, \eta > 0$, and let $f : H \to H$ be an l-Lipschitz mapping with constant $l \ge 0$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings on C and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0,1)$. Suppose that $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that $\Omega := (\bigcap_{n=1}^{\infty} F(T_n)) \cap (\bigcap_{m=1}^{N} \mathrm{MEP}(F_m, \varphi)) \cap \Gamma \neq \emptyset$, where Γ is a fixed point set of the mapping $G = P_C(I - \mu_1 B_1) P_C(I - \mu_2 B_2)$ with $\mu_i \in (0, 2\zeta_i)$ for $i = 1, 2$. Let $\{\alpha_n\}, \{\delta_n\}, \{\beta_{n,1}\}, \ldots$ and $\{\beta_{n,N}\}$ be sequences in $[0,1]$ and $\{r_{n,m}\}$ be a sequence in $(0, \infty)$ for every $1 \le m \le N$ such that*:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(b) $\sum_{m=1}^{N} \beta_{n,m} = 1$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1,m} - \beta_{n,m}| < \infty$ *for each* $1 \le m \le N$;
(c) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$;
(d) $\liminf_{n\to\infty} r_{n,m} > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1,m} - r_{n,m}| < \infty$ *for each* $1 \le m \le N$.

Given $x_1 \in H$ arbitrarily, the sequence $\{x_n\}$ is generated iteratively by

$$\begin{cases} y_n = \delta_n W_n G(\sum_{m=1}^{N} \beta_{n,m} u_{n,m}) + (1 - \delta_n)(\sum_{m=1}^{N} \beta_{n,m} u_{n,m}), \\ x_{n+1} = P_C[\alpha_n \gamma f(x_n) + (1 - \alpha_n \mu B_3) W_n G y_n], \quad \forall n \ge 1, \end{cases} \tag{3.40}$$

where $u_{n,m}$ is such that

$$F_m(u_{n,m}, y) + \varphi(y) - \varphi(u_{n,m}) + \frac{1}{r_{n,m}} \langle y - u_{n,m}, u_{n,m} - x_n \rangle \ge 0, \quad \forall y \in C,$$

for each $1 \leq m \leq N$, W_n *is defined by* (1.17). *Then the sequence* $\{x_n\}$ *defined by* (3.40) *converges strongly to* $x^* \in \Omega$, *as* $n \to \infty$, *where* x^* *is the unique solution of the variational inequality*

$$\langle (\gamma f - \mu B_3)x^*, z - x^* \rangle \leq 0, \quad \forall z \in \Omega.$$

Competing interests
The authors declare that they have no competing interest.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the finial manuscript.

Author details
[1]Department of Mathematics, Shanghai Normal University, Shanghai, 200234, China. [2]College of Science, Zhongyuan University of Technology, Zhengzhou, 450007, China.

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