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Points of nonsquareness of Lorentz spaces $\Gamma_{p,w}$

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Abstract

Criteria for nonsquare points of the Lorentz spaces of maximal functions $\Gamma_{p,w}$ are presented under an arbitrary (also degenerated) nonnegative weight function w. The criteria for nonsquareness of Lorentz spaces $\Gamma_{p,w}$ and of their subspaces $(\Gamma_{p,w})_a$ of all order continuous elements, proved directly in (Kolwicz and Panfil in Indag. Math. 24:254-263, 2013), are deduced.

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1 Introduction

The geometry of Banach spaces has been intensively developed during the last decades. Nonsquareness and uniform nonsquareness are important properties in this area. Uniform nonsquareness implies both superreflexivity and the fixed point property (see [1, 2] and [3]). Therefore it is natural to investigate nonsquareness properties in various classes of Banach spaces (see [4–10]). They are also connected with the notion of James constant (see [11–13]) which describes the measure of nonsquareness. The class of uniformly nonsquare Banach spaces is strictly smaller than the class of B-convex Banach spaces. Recall that B-convexity plays an important role in the probability (see [14]).

On the other hand, it is natural to ask whether a separated point x in a Banach function space E has some local property P whenever the whole space E does not possess this property. This leads to the local geometry which has been deeply studied recently (see [15–19]). The monotonicity properties of separated points have applications in best dominated approximation problems in Banach lattices (see [15]). The extreme points, and SU points play a similar role in the theory of Banach spaces.

The purpose of this paper is to characterize nonsquare points of the Lorentz space $\Gamma_{p,w}$. We also give a criterion for a point to be nonsquare in the subspace of order continuous elements $(\Gamma_{p,w})_a$ of $\Gamma_{p,w}$. Since degenerated weight functions w are admitted, such investigations concern the most possible wide class of these spaces. Moreover, the local approach presented in this paper required new techniques and methods (in comparison with the global approach in [20]), which may be of independent interest.

2 Preliminaries

Let $\mathbb{R}$ be the set of real numbers and S_X or $S(X)$ be the unit sphere of a real Banach space X. Denote by $L^0 = L^0[0,\alpha)$ the set of all m-equivalence classes of real-valued mea-

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surable functions defined on $[0,\alpha)$ with m being the Lebesgue measure on $\mathbb{R}$ and $\alpha = 1$ or $\alpha = \infty$.

A Banach lattice $(E, \|\cdot\|_E)$ is called a *Banach function space* (or a *Köthe space*) if it is a sublattice of L^0 satisfying the following conditions:

(1) If $f \in L^0$, $g \in E$ and $|f| \le |g|$ a.e., then $f \in E$ and $\|f\|_E \le \|g\|_E$.

(2) There exists a strictly positive on $[0,\alpha)$, $f \in E$.

The symbol E_+ stands for the positive cone of E, that is, $E_+ = \{x \in E : x \ge 0\}$. We say that E has the *Fatou property* if for any sequence (f_n) such that $0 \le f_n \in E$ for all $n \in \mathbb{N}$, $f \in L^0$, $f_n \uparrow f$ a.e. with $\sup_{n\in\mathbb{N}} \|f_n\|_E < \infty$, we have $f \in E$ and $\|f_n\|_E \uparrow \|f\|_E$.

We say that a Banach function space $(E, \|\cdot\|_E)$ is *rearrangement invariant* (r.i. for short) if whenever $f \in L^0$ and $g \in E$ with $d_f = d_g$, then $f \in E$ and $\|f\|_E = \|g\|_E$ (see [21]). Recall that d_f stands for the *distribution function* of $f \in L^0$, that is, $d_f(\lambda) = \mathrm{m}\{t \in [0,\alpha) \colon |f|(t) > \lambda\}$ for every $\lambda \ge 0$. Then the *nonincreasing rearrangement* f^* of f is defined by

$$f^*(t) = \inf\{\lambda > 0 \colon d_f(\lambda) \le t\}$$

for $t \ge 0$. Given $f \in L^0$, we denote the *maximal function* f^{**} of f^* by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds.$$

It is well known that $f^* \le f^{**}$ and $(f+g)^{**} \le f^{**} + g^{**}$ for any $f, g \in L^0$ (see [22, 23] for other properties of f^* and f^{**}).

A Banach function space E is said to be *strictly monotone* ($E \in (SM)$) if for each $0 \le g \le f$ with $g \ne f$ we have $\|g\|_E < \|f\|_E$. A point $f \in E_+ \setminus \{0\}$ is a *point of lower monotonicity* (*upper monotonicity*) if for any $g \in E_+$ such that $g \le f$ and $g \ne f$ (respectively, $f \le g$ and $g \ne f$), we have $\|g\|_E < \|f\|_E$ (respectively, $\|f\|_E < \|g\|_E$). We will write shortly that f is an *LM point and UM point*, respectively.

Clearly, the following assertions are equivalent:

(i) E is strictly monotone (shortly, $E \in (SM)$);

(ii) each point of $E_+ \setminus \{0\}$ is a point of upper monotonicity;

(iii) each point of $E_+ \setminus \{0\}$ is a point of lower monotonicity.

Definition 2.1 Let $1 \le p < \infty$ and $\alpha = 1$ or $\alpha = \infty$. Let $w \in L^0$ be a nonnegative weight function. The Lorentz space $\Gamma_{p,w} = \Gamma_{p,w}[0,\alpha)$ is a subspace of functions $f \in L^0$ satisfying the following formula:

$$\|f\| := \|f\|_{p,w} = \left(\int_0^\alpha (f^{**})^p w\right)^{1/p} = \left(\int_0^\alpha (f^{**})^p(t)w(t)\,dt\right)^{1/p} < \infty.$$

Throughout the paper, we assume that w satisfies the following conditions:

$$\int_0^t w(s)\,ds < \infty \quad \text{and} \quad \int_t^\alpha s^{-p}w(s)\,ds < \infty$$

for all $0 < t \le \alpha$ if $\alpha = 1$ and for all $0 < t < \alpha$ in the opposite case. These two conditions assure that $\Gamma_{p,w} \ne \{0\}$ and $(\Gamma_{p,w}, \|\cdot\|)$ is a rearrangement invariant Banach function space with the Fatou property (see [24, 25]).

These spaces were introduced by Calderón in [26] and are naturally related to classical Lorentz spaces $\Lambda_{p,w} = \{f \in L^0 : \int_0^\gamma (f^*)^p w < \infty\}$ defined by Lorentz in [27]. Obviously, $\Gamma_{p,w} \subset \Lambda_{p,w}$ for any $0 < p < \infty$ and these spaces coincide if and only if the Hardy operator $H^1 f = f^{**}$ is bounded on $\Lambda_{p,w}$. This condition is equivalent to the so-called B_p condition related to the weight w (see [25, 28–30]). It is also worth mentioning that spaces $\Gamma_{p,w}$ appear naturally in the interpolation theory as a result of the Lions-Peetre K-method. These spaces have been recently intensively investigated from both the isomorphic as well as the isometric point of view (see [24, 25, 31]).

Definition 2.2 A point $f \in S_X$ is a point of nonsquareness (we write shortly f is an NSQ point) provided that

$$\min\{\|f+g\|, \|f-g\|\} < 2$$

for all $g \in S_X$. A Banach space $(X, \|\cdot\|)$ is *nonsquare* ($X \in (NSQ)$ for short) if each point of S_X is an NSQ point.

Notation 2.1 For simplicity, we will sometimes use the following notations:

(a) $(\int_A + \int_B) f = \int_A f + \int_B f$.
(b) By $\mathcal{S}(f)$ we denote the support of $f \in L^0$.
(c) $f^*(\infty) = \lim_{t\to\infty} f^*(t)$ if $\mathrm{m}(\mathcal{S}(f)) = \infty$.
(d) $f^*(\infty) := 0$ if $\mathrm{m}(\mathcal{S}(f)) < \infty$.
(e) For measurable subsets A, B of $\mathbb{R}$, by $A = B$ we mean $\mathrm{m}(A \div B) = 0$.

Let us recall some useful properties of a nonincreasing rearrangement operator.

Lemma 2.1 ([23], Property 7°, p.64) *Let $f \in L^0[0,\infty)$. If $f^*(t) > f^*(\infty)$, then there is a set $e_t(f)$ with $\mathrm{m}(e_t(f)) = t$ and*

$$\int_0^t f^* = \int_{e_t(f)} |f|.$$

Remark 2.1 Let $f \in L^0[0,\alpha)$ with $\alpha = 1$ or $\alpha = \infty$. The above lemma holds (without the assumption $f^*(t) > f^*(\infty)$)

(a) for every $t \in (0, \mathrm{m}(Z))$, where $Z = \{t : |f(t)| \geq f^*(\infty)\}$;
(b) for every $t \in (0,\alpha)$ in the case of $\mathrm{m}(\mathcal{S}(f)) < \infty$.

Lemma 2.2 ([23], Property 8°, p.64) *The equality*

$$\int_0^t f^* = \sup_{\mathrm{m}(e)=t} \int_e |f|$$

holds for $f \in L^0[0,\infty)$.

Remark 2.2 Lemma 2.2 implies

$$\int_0^t (f+g)^* \leq \int_0^t f^* + \int_0^t g^*,$$

i.e., the subadditivity property of the maximal function.

Remark 2.3 Let $f,g \in S(\Gamma_{p,w})$. The inequality

$$(f+g)^{**}(t) < f^{**}(t) + g^{**}(t) \tag{1}$$

for $t \in Z$ with $\mathrm{m}(Z \cap \mathcal{S}(w)) > 0$ implies $\|f+g\| < 2$. Indeed,

$$\left\|\frac{f+g}{2}\right\|^p = \int_0^\alpha \left[\left(\frac{f+g}{2}\right)^{**}\right]^p w < \int_0^\alpha \left[\frac{f^{**}+g^{**}}{2}\right]^p w \le \int_0^\alpha \frac{(f^{**})^p + (g^{**})^p}{2} w = 1.$$

The following result is a generalization of Lemma 1 from [20].

Lemma 2.3 *Let* $x,y \in L^0 \setminus \{0\}$. *If* $\mathrm{m}(\mathcal{S}(x) \cap \mathcal{S}(y)) = 0$, *then*

$$(x+y)^{**}(t) < x^{**}(t) + y^{**}(t)$$

for every $0 < t < \mathrm{m}(\mathcal{S}(x) \cup \mathcal{S}(y))$.

Proof Set

$$t_0 = \sup\bigl\{t \colon (x+y)^*(t) > (x+y)^*(\infty)\bigr\},$$

with the convention $\sup \emptyset = 0$.

Since $\mathrm{m}(\mathcal{S}(x) \cap \mathcal{S}(y)) = 0$, so $(x+y)^*(\infty) = \max\{x^*(\infty), y^*(\infty)\}$. Assume, without loss of generality, that $(x+y)^*(\infty) = x^*(\infty)$. Clearly,

$$\int_0^t y^* > 0 \quad \text{and} \quad \int_0^t x^* > 0 \quad \text{for all } t > 0. \tag{2}$$

Notice that for every $0 < t \le t_0$ if $t_0 < \infty$ and for every $0 < t < t_0$ if $t_0 = \infty$, by Lemma 1 in [20], we have

$$\int_0^t (x+y)^* < \int_0^t x^* + \int_0^t y^* \quad \text{if } t_0 > 0. \tag{3}$$

Moreover, if $t_0 < \infty$, then for $t_0 < t < \mathrm{m}(\mathcal{S}(x) \cup \mathcal{S}(y))$,

$$\begin{aligned}
\int_{t_0}^t x^*(s)\,ds + \int_{t_0}^t y^*(s)\,ds &\ge \int_{t_0}^t x^*(\infty)\,ds + \int_{t_0}^t y^*(s)\,ds \\
&= \int_{t_0}^t (x+y)^*(\infty)\,ds + \int_{t_0}^t y^*(s)\,ds \\
&= \int_{t_0}^t (x+y)^*(s)\,ds + \int_{t_0}^t y^*(s)\,ds \\
&\ge \int_{t_0}^t (x+y)^*(s)\,ds
\end{aligned} \tag{4}$$

since $(x+y)^*(s) = (x+y)^*(\infty)$ for every $s \ge t_0$.

If $t_0 > 0$ then, by (3) and (4), we get

$$\int_0^t (x+y)^* < \int_0^t x^* + \int_0^t y^*$$

for $0 < t < \mathrm{m}(\mathcal{S}(x) \cup \mathcal{S}(y))$.

If $t_0 = 0$ then, by (2) and (4), we have

$$\int_0^t x^*(s)\,ds + \int_0^t y^*(s)\,ds \geq \int_0^t (x+y)^*(s)\,ds + \int_0^t y^*(s)\,ds > \int_0^t (x+y)^*(s)\,ds$$

for $0 < t < \mathrm{m}(\mathcal{S}(x) \cup \mathcal{S}(y))$. □

Remark 2.4 Let $x \in L^0 \setminus \{0\}$ and $C = \{t \colon |x(t)| > x^*(\infty)\}$. Then x^* is constant on $(0,\infty)$ if and only if $\mathrm{m}(C) = 0$.

Proof Clearly, $\mathrm{m}(C) = \mathrm{m}\{t \colon x^*(t) > x^*(\infty)\}$ and $x^*(t) \geq x^*(\infty)$ for all $t \geq 0$. Therefore $\mathrm{m}(C) = 0$ is equivalent to $x^* = a\chi_{(0,\infty)}$ for some $a > 0$. □

Theorem 2.1 *Let E be a symmetric Banach function space, $x \in S(E)$ and $C = \{t \colon |x(t)| > x^*(\infty)\}$. If x is an NSQ point, then* $\mathrm{m}(C) > 0$.

Proof Assume $\mathrm{m}(C) = 0$. By Remark 2.4, $x^*(t) = x^*(\infty) = a > 0$ for every $t \geq 0$. Thus $d_x(\theta) = \infty$ for every $0 \leq \theta < x^*(\infty)$ and $d_x(\theta) = 0$ for every $\theta \geq x^*(\infty)$. Moreover, every function of the type $y = \pm x^*(\infty)\chi_Z$ with $\mathrm{m}(Z) = \infty$ is equimeasurable with x since $d_y(\theta) = \mathrm{m}(Z) = \infty$ for every $0 \leq \theta < x^*(\infty)$ and $d_y(\theta) = 0$ for every $\theta \geq x^*(\infty)$. Therefore $\|y\| = \|x\|$.

Denote

$$A = \{t \colon |x(t)| = x^*(\infty)\}, \qquad B = \{t \colon 0 < |x(t)| < x^*(\infty)\}.$$

Case I. Suppose $\mathrm{m}(A) = \infty$ and denote

$$A_+ = \{t \in A \colon x(t) > 0\}, \qquad A_- = \{t \in A \colon x(t) < 0\}.$$

If $\mathrm{m}(A_+) = \infty$, then take A_+^1 and A_+^2 such that $A_+ = A_+^1 \cup A_+^2$, $A_+^1 \cap A_+^2 = \emptyset$ and $\mathrm{m}(A_+^1) = \mathrm{m}(A_+^2) = \infty$. Define

$$y = x^*(\infty)\chi_{A_+^1} - x^*(\infty)\chi_{A_+^2}.$$

Thus

$$(x+y)^* = (x-y)^* = 2x^*,$$

whence $\|x+y\| = \|x-y\| = 2\|x\|$, *i.e.*, x is not an NSQ point. The case $\mathrm{m}(A_-) = \infty$ goes analogously.

Case II. Let $\mathrm{m}(A) < \infty$ and $\mathrm{m}(B) = \infty$. Define

$$B_+ = \{t \in B \colon x(t) > 0\}, \qquad B_- = \{t \in B \colon x(t) < 0\}.$$

Note that either $d_{x\chi_{B_+}}(\theta) = \infty$ for all $0 \leq \theta < x^*(\infty)$ or $d_{x\chi_{B_-}}(\theta) = \infty$ for all $0 \leq \theta < x^*(\infty)$, since in the opposite case there are $\theta_1, \theta_2 < x^*(\infty)$ that

$$d_{x\chi_{B_+}}(\theta_1) < \infty \quad \text{and} \quad d_{x\chi_{B_-}}(\theta_2) < \infty.$$

Thus, taking $\theta_0 = \max\{\theta_1, \theta_2\}$, we get

$$d_{x\chi_B}(\theta_0) = d_{x\chi_{B_+}}(\theta_0) + d_{x\chi_{B_-}}(\theta_0) < \infty,$$

whence

$$d_x(\theta_0) = d_{x\chi_A}(\theta_0) + d_{x\chi_B}(\theta_0) < \infty,$$

a contradiction.

Without loss of generality, we may assume $d_{x\chi_{B_+}}(\theta) = \infty$ for all $0 \le \theta < x^*(\infty)$. For $n \in \mathbb{N}$, let

$$B_n = \left\{t \in B_+ : \left(1 - \frac{1}{n}\right)x^*(\infty) < x(t) \le \left(1 - \frac{1}{n+1}\right)x^*(\infty)\right\},$$

and note $\bigcup_n B_n = B_+$, $B_n \cap B_m = \emptyset$ for all $n, m \in \mathbb{N}$ and $n \neq m$. Let B_n^1, B_n^2 satisfy $B_n = B_n^1 \cup B_n^2$, $\mathrm{m}(B_n^1) = \mathrm{m}(B_n^2)$, and $B_n^1 \cap B_n^2 = \emptyset$. Denote

$$D_1 = \bigcup_n B_n^1 \quad \text{and} \quad D_2 = \bigcup_n B_n^2.$$

Notice $D_1 \cap D_2 = \emptyset$, $D_1 \cup D_2 = B_+$ and $\mathrm{m}(D_1) = \mathrm{m}(D_2) = \infty$. We claim that

$$d_{x\chi_{D_1}}(\theta) = \infty \quad \text{and} \quad d_{x\chi_{D_2}}(\theta) = \infty \tag{5}$$

for all $0 < \theta < x^*(\infty)$. If there exists $0 < \theta_0 < x^*(\infty)$ such that $d_{x\chi_{D_1}}(\theta_0) < \infty$, then take n_0 satisfying $(1 - 1/n_0)x^*(\infty) \le \theta_0 < (1 - 1/(n_0 + 1))x^*(\infty)$. Thus

$$\begin{aligned}\infty > d_{x\chi_{D_1}}\left(\left(1 - \frac{1}{n_0+1}\right)x^*(\infty)\right) &= \mathrm{m}\left\{t \in D_1 : x(t) > \left(\left(1 - \frac{1}{n_0+1}\right)x^*(\infty)\right)\right\} \\ &= \mathrm{m}\left(\bigcup_{n=n_0+1} B_n^1\right) = \mathrm{m}\left(\bigcup_{n=n_0+1} B_n^2\right) = d_{x\chi_{D_2}}\left(\left(1 - \frac{1}{n_0+1}\right)x^*(\infty)\right).\end{aligned}$$

Therefore $d_{x\chi_{B_+}}((1 - 1/(n_0 + 1))x^*(\infty)) < \infty$, a contradiction. The case $d_{x\chi_{D_2}}(\theta_0) < \infty$ is analogous, which proves claim (5).

Let $y = x^*(\infty)\chi_{D_1} - x^*(\infty)\chi_{D_2}$. Then, for all $0 < \theta < x^*(\infty)$, we have

$$\begin{aligned}d_{(x+y)/2}(\theta) &= \mathrm{m}\left\{t : \left|\left(\frac{x+y}{2}\right)(t)\right| > \theta\right\} \ge \mathrm{m}\left\{t \in D_1 : \left|\left(\frac{x+y}{2}\right)(t)\right| > \theta\right\} \\ &= \mathrm{m}\left\{t \in D_1 : \frac{x(t) + x^*(\infty)}{2} > \theta\right\} = \mathrm{m}\{t \in D_1 : x(t) > 2\theta - x^*(\infty)\} \\ &= d_{x\chi_{D_1}}(\theta_0) = \infty,\end{aligned}$$

where $\theta_0 = \max\{0, 2\theta - x^*(\infty)\} < x^*(\infty)$, since $x\chi_{D_1} \ge 0$ and $\mathrm{m}(D_1) = \infty$. Analogously, for $0 < \theta < x^*(\infty)$, $d_{(x-y)/2}(\theta) \ge d_{x\chi_{D_2}}(\theta_0) = \infty$.

Obviously, by assumption that $\mathrm{m}(C) = 0$, $|x(t)| \le x^*(\infty)$ and $|y(t)| \le x^*(\infty)$. Thus

$$\left|\frac{x \pm y}{2}\right| \le x^*(\infty),$$

whence $d_{(x\pm y)/2}(\theta) = d_x(\theta) = 0$ for every $\theta \geq x^*(\infty)$. Therefore $(\frac{x\pm y}{2})^* = x^*$, whence x is not an NSQ point. □

In the sequel we will use the following notations:

$$\begin{aligned} &\gamma = \inf\{t\colon \mathrm{m}(\mathcal{S}(w) \cap (t,\alpha)) = 0\} \quad \text{with } \alpha = 1 \text{ or } \alpha = \infty, \\ &\beta = \sup\{t\colon \mathrm{m}(\mathcal{S}(w) \cap [0,t)) = 0\}, \end{aligned} \tag{6}$$

with the convention $\inf \emptyset = \alpha$, $\sup \emptyset = 0$.

Theorem 2.2 *Let $x \in S(\Gamma_{p,w})$. If x is an NSQ point, then*

(i) *x^* is not constant on $(0,2\gamma)$ if $\alpha = \infty$,*

(ii) *x^* is not constant on $(0,2\gamma)$ if $\alpha = 1$ and $\gamma \leq 1/2$,*

where γ is defined in (6).

Proof The case $\gamma = \infty$ follows from Remark 2.4 and Theorem 2.1.

Consider $\gamma < \infty$ if $\alpha = \infty$ or $\gamma \leq 1/2$ if $\alpha = 1$. For the contrary, assume that $x^*\chi_{(0,2\gamma)} = a\chi_{(0,2\gamma)}$ for some $a > 0$. Let $C = \{t\colon |x(t)| > x^*(\infty)\}$. Then, in the case of $\alpha = \infty$, Theorem 2.1 implies $\mathrm{m}(C) \geq 2\gamma$. Thus $a > x^*(\infty)$. By Lemma 2.1, there are disjoint sets e_1 and e_2, both of measure γ, such that $\int_0^\gamma x^* = \int_{e_1} |x|$ and $\int_\gamma^{2\gamma} x^* = \int_{e_2} |x|$. Moreover, $e_1 \cup e_2 \subset C$ and $|x(t)| = a$ for $t \in e_1 \cup e_2$. Taking $y = x\chi_{e_1} - x\chi_{e_2}$, we get $y^*\chi_{(0,2\gamma)} = x^*\chi_{(0,2\gamma)}$ and $(x+y)^*\chi_{(0,\gamma)} = (x-y)^*\chi_{(0,\gamma)} = 2x^*\chi_{(0,\gamma)}$. Since $\mathrm{m}(\mathcal{S}(w) \cap (\gamma,\alpha)) = 0$, then $\|x \pm y\| = 2\|x\|$, *i.e.*, x is not an NSQ point. □

Theorem 2.3 *If $x \in S(\Gamma_{p,w})$ is an NSQ point, then* $\mathrm{m}(\mathcal{S}(x)) \geq \beta$, *where β is defined in* (6).

Proof Suppose $\mathrm{m}(\mathcal{S}(x)) < \beta$. This means $\beta > 0$. Take $a = \beta - \mathrm{m}(S(x))$, $y = b\chi_A$, where $\mathrm{m}(A) = a$, $A \cap \mathcal{S}(x) = \emptyset$, and $b = \frac{1}{a}\int_0^\beta x^*$. Then $y^{**}(\beta) = x^{**}(\beta)$ and $\|\frac{x\pm y}{2}\| = \|x\|$ since $(\frac{x\pm y}{2})^{**}(\beta) = \frac{1}{\beta}\int_0^\beta (\frac{x\pm y}{2})^* = \frac{1}{2\beta}(\int_0^\beta x^* + \int_0^\beta y^*) = x^{**}(\beta)$. □

Theorem 2.4 *Let $x \in S(\Gamma_{p,w}[0,\infty))$, β and γ are as in* (6), *and let the weight function be such that $\gamma = \infty$. The function x is an NSQ point if and only if* $\mathrm{m}(\mathcal{S}(x)) \geq \beta$ *and x^* is not constant on* $(0,\infty)$.

Proof Necessity. It follows from Theorems 2.2 and 2.3.

Sufficiency. Let $y \in S(\Gamma_{p,w}[0,\infty))$. If $\mathrm{m}(\mathcal{S}(x) \cap \mathcal{S}(y)) = 0$ then, by Lemma 2.3, $(x+y)^{**}(t) < x^{**}(t) + y^{**}(t)$ for all $t \in (0, \mathrm{m}(\mathcal{S}(x) \cup \mathcal{S}(y)))$. Since $\mathrm{m}(\mathcal{S}(x)) \geq \beta$, so $\mathrm{m}(\mathcal{S}(w) \cap (\beta, \mathrm{m}(\mathcal{S}(x) \cup \mathcal{S}(y)))) > 0$, whence $\|x+y\| < 2$ (see Remark 2.3 and the definition of β), *i.e.*, x is not an NSQ point.

Now assume $\mathrm{m}(\mathcal{S}(x) \cap \mathcal{S}(y)) > 0$. Denote

$$\begin{aligned} &A_1 = \{t \in I\colon x(t)y(t) > 0\}, \qquad A_2 = \{t \in I\colon x(t)y(t) < 0\}, \\ &A_3 = \{t \in I\colon x(t)y(t) = 0 \text{ and } |x(t)| + |y(t)| > 0\}. \end{aligned} \tag{7}$$

We have

$$\mathrm{m}(A_1 \cup A_2) > 0. \tag{8}$$

Obviously,

$$A_i \cap A_j = \emptyset \quad \text{for } i \neq j \text{ and } i,j \in \{1,2,3\}. \tag{9}$$

We will consider the following pairwise independent cases.

Case I. $(|x| + |y|)^*(\infty) = 0$.

Case II. $(|x| + |y|)^*(\infty) > 0$.

Case II.A. There is $t_0 > 0$ such that $(x+y)^*(t_0) < (|x| + |y|)^*(t_0)$, or there is $t_1 > 0$ with $(x-y)^*(t_1) < (|x| + |y|)^*(t_1)$.

Case II.B. For every $t > 0$, $(x+y)^*(t) = (x-y)^*(t) = (|x| + |y|)^*(t)$.

Now let us discuss all the cases.

Proof of Case I. Since $(|x| + |y|)^*(\infty) = 0$ and $\gamma = \infty$, then $|x| + |y|$ satisfies the conditions (i) and (ii) of Theorem 3.1 in [15], whence $|x| + |y|$ is an LM point. Moreover, by (8), at least one of the following inequalities holds:

$$(x+y)\chi_{A_2} < \bigl(|x| + |y|\bigr)\chi_{A_2} \quad \text{or} \quad (x-y)\chi_{A_1} < \bigl(|x| + |y|\bigr)\chi_{A_1}.$$

Since $|x \pm y| \leq |x| + |y|$, one of the following inequalities holds:

$$\|x+y\| < \bigl\||x| + |y|\bigr\| \leq \|x\| + \|y\| = 2 \quad \text{or} \quad \|x-y\| < \bigl\||x| + |y|\bigr\| \leq \|x\| + \|y\| = 2.$$

Proof of Case II.A. Assume that there exists $t_0 > 0$ such that $(x+y)^*(t_0) < (|x| + |y|)^*(t_0)$. Since $|x \pm y| \leq |x| + |y|$, so $(x \pm y)^* \leq (|x| + |y|)^*$. By the right continuity of nonincreasing rearrangement function, there exists $\delta > t_0$ such that $(x+y)^*(t) < (|x| + |y|)^*(t)$ for all $t \in (t_0, \delta)$. Therefore,

$$\int_0^t (x+y)^* < \int_0^t \bigl(|x| + |y|\bigr)^*$$

for $t > t_0$. It is clear that $\int_0^t (|x| + |y|)^* \leq \int_0^t x^* + \int_0^t y^*$ (see Remark 2.2), whence

$$(x+y)^{**}(t) < x^{**}(t) + y^{**}(t)$$

for every $t > t_0$. Since $\mathrm{m}(\mathcal{S}(w) \cap (t_0, \infty)) > 0$, so $\|x+y\| < 2$ (see Remark 2.3).

Notice, if there is $t_1 > 0$ such that $(x-y)^*(t_0) < (|x| + |y|)^*(t_1)$, then analogous reasoning gives $\|x-y\| < 2$.

Proof of Case II.B. Assume

$$(x+y)^*(t) = (x-y)^*(t) = \bigl(|x| + |y|\bigr)^*(t) \tag{10}$$

for every $t > 0$. Consequently,

$$(x \pm y)^*(\infty) = \bigl(|x| + |y|\bigr)^*(\infty). \tag{11}$$

Denote

$$t_0 = \sup\bigl\{t : \bigl(|x| + |y|\bigr)^*(t) > \bigl(|x| + |y|\bigr)^*(\infty)\bigr\}.$$

(a) Suppose $t_0 = \infty$. Then

$$\big(|x| + |y|\big)^* \chi_{(a,\infty)} \text{ and } (x \pm y)^* \chi_{(a,\infty)} \text{ are not constant for every } a > 0. \tag{12}$$

By (12), there is $0 < t_1 < \infty$ satisfying

$$(x + y)^*(t) < \lim_{t \to t_1^-} (x + y)^*(t) \tag{13}$$

for every $t > t_1$. An analogous inequality holds for $(x - y)^*$ and $(|x| + |y|)^*$. Take the sets B_+, B_-, B_0 of measure t_1 such that

$$\int_0^{t_1} (x + y)^* = \int_{B_+} |x + y|, \qquad \int_0^{t_1} (x - y)^* = \int_{B_-} |x - y|,$$
$$\int_0^{t_1} \big(|x| + |y|\big)^* = \int_{B_0} |x| + |y|$$

(see Lemma 2.1). Clearly, by the definition of t_1 and equimeasurability of a function and its nonincreasing rearrangement, we get

$$B_+ = \big\{t\colon \big|(x + y)(t)\big| > (x + y)^*(t_1)\big\},$$
$$B_- = \big\{t\colon \big|(x - y)(t)\big| > (x - y)^*(t_1)\big\},$$
$$B_0 = \big\{t\colon \big(|x| + |y|\big)(t) > \big(|x| + |y|\big)^*(t_1)\big\}.$$

Let

$$B_+^i = B_+ \cap A_i, \qquad B_-^i = B_- \cap A_i, \qquad B_0^i = B_0 \cap A_i \quad \text{for } i \in \{1, 2, 3\} \tag{14}$$

(see notation (7)). By $|x \pm y|\chi_{A_3} = (|x| + |y|)\chi_{A_3}$, (10), (11) and (13),

$$B_+^3 = B_-^3 = B_0^3. \tag{15}$$

Analogously, the equality $|x + y|\chi_{A_1} = (|x| + |y|)\chi_{A_1}$ gives

$$B_+^1 = B_0^1,$$

and $|x - y|\chi_{A_2} = (|x| + |y|)\chi_{A_2}$ yields

$$B_-^2 = B_0^2.$$

We claim that $\mathrm{m}(B_+^2) = \mathrm{m}(B_-^1) = 0$. Indeed, if $\mathrm{m}(B_+^2) > 0$, then by Lemma 2.2 and (10), we get

$$\begin{aligned}\int_0^{t_1} (x + y)^* &= \int_{B_+} |x + y| = \bigg(\int_{B_+^2} + \int_{B_+^1 \cup B_+^3}\bigg) |x + y| \\ &< \bigg(\int_{B_+^2} + \int_{B_+^1 \cup B_+^3}\bigg) |x| + |y| = \int_{B_+} |x| + |y| \le \int_0^{t_1} \big(|x| + |y|\big)^* = \int_0^{t_1} (x + y)^*,\end{aligned}$$

a contradiction. Analogous reasoning goes for $\mathrm{m}(B_-^1) > 0$, which proves the claim.

Moreover, the above arguments and (9) imply $\mathrm{m}(B_0^2) = \mathrm{m}(B_0^1) = 0$, since

$$\mathrm{m}\big(B_+^1 \cup B_+^3\big) = \mathrm{m}(B_+) = \mathrm{m}(B_0) = \mathrm{m}\big(B_0^1 \cup B_0^2 \cup B_0^3\big) = \mathrm{m}\big(B_+^1 \cup B_0^2 \cup B_+^3\big)$$

and

$$\mathrm{m}\big(B_-^2 \cup B_-^3\big) = \mathrm{m}(B_-) = \mathrm{m}(B_0) = \mathrm{m}\big(B_0^1 \cup B_0^2 \cup B_0^3\big) = \mathrm{m}\big(B_0^1 \cup B_-^2 \cup B_-^3\big).$$

Thus $B_0 = B_0^3$, whence (15) implies $\mathrm{m}(B_+^1) = \mathrm{m}(B_-^2) = 0$. Summarizing, we get

$$B_0 = B_+ = B_- \subset A_3. \tag{16}$$

For $0 < t \leq t_1$ there exists a set $B_+(t)$ of measure t such that

$$\int_0^t (x+y)^* = \int_{B_+(t)} |x+y| = \int_{B_+^x(t)} |x| + \int_{B_+^y(t)} |y|,$$

where

$$B_+^x(t) = B_+(t) \cap \mathcal{S}(x) \quad \text{and} \quad B_+^y(t) = B_+(t) \cap \mathcal{S}(y).$$

We have $B_+(t) = B_+^x(t) \cup B_+^y(t)$ and, by (16), $\mathrm{m}(B_+^x \cap B_+^y) = 0$. By the above argumentation, together with Lemma 2.2 and $\mathrm{m}(\mathcal{S}(x) \cup \mathcal{S}(y)) = \infty$, at least one of the following inequalities holds:

$$\int_{B_+^x(t)} |x| \leq \int_0^{\mathrm{m}(B_+^x(t))} x^* < \int_0^t x^* \quad \text{or} \quad \int_{B_+^y(t)} |y| \leq \int_0^{\mathrm{m}(B_+^y(t))} y^* < \int_0^t y^*$$

for $0 < t \leq t_1$. Thus, for every $0 < t \leq t_1$,

$$\int_0^t (x+y)^* < \int_0^t x^* + \int_0^t y^*. \tag{17}$$

By (12), we may find a sequence (t_n), $t_n \to \infty$, such that inequality (13) is satisfied for each t_n. Similarly as above, we conclude inequality (17) with t_n instead of t_1. Consequently, (17) holds for all $t > 0$. This means $\|x+y\| < 2$ (see Remark 2.3).

(b) Assume $0 < t_0 < \infty$ and take the sets B_+, B_-, B_0 of measure t_0 such that

$$\int_0^{t_0} (x+y)^* = \int_{B_+} |x+y|, \qquad \int_0^{t_0} (x-y)^* = \int_{B_-} |x-y|,$$
$$\int_0^{t_0} \big(|x|+|y|\big)^* = \int_{B_0} |x|+|y|$$

(see Lemma 2.1). Clearly, by the definition of t_0 and equimeasurability of a function and its nonincreasing rearrangement, we get

$$B_+ = \big\{t\colon \big|(x+y)(t)\big| > (x+y)^*(\infty)\big\},$$
$$B_- = \big\{t\colon \big|(x-y)(t)\big| > (x-y)^*(\infty)\big\},$$
$$B_0 = \big\{t\colon \big(|x|+|y|\big)(t) > \big(|x|+|y|\big)^*(\infty)\big\}.$$

Let B_+^i, B_-^i and B_0^i for $i \in \{1,2,3\}$ be defined as in (14). By $|x \pm y|\chi_{A_3} = (|x| + |y|)\chi_{A_3}$, (10) and (11) we conclude (15). Moreover, similarly as above, we get (17) for t_0 instead of t_1.

Moreover, by the definition of t_0, for every $t > t_0$,

$$\begin{aligned}\int_{t_0}^t (x \pm y)^* &= \int_{t_0}^t \bigl(|x| + |y|\bigr)^* = \int_{t_0}^t \bigl(|x| + |y|\bigr)^*(\infty) \\ &\leq \int_{t_0}^t x^*(\infty) + \int_{t_0}^t y^*(\infty) \leq \int_{t_0}^t x^* + \int_{t_0}^t y^*.\end{aligned}$$

Finally, by (17) and the above inequality, we get

$$\int_0^t (x + y)^* < \int_0^t x^* + \int_0^t y^*$$

for every $t > 0$. Therefore, $\|x + y\| < 2$ (see Remark 2.3).

(c) Suppose $t_0 = 0$, *i.e.*,

$$(x \pm y)^* \chi_{(0,\infty)} = \bigl(|x| + |y|\bigr)^* \chi_{(0,\infty)} = \bigl(|x| + |y|\bigr)^*(\infty)\chi_{(0,\infty)} = a\chi_{(0,\infty)}$$

for some $a > 0$. Notice $\mathrm{m}(C) > 0$. Then, for every $0 < t < \mathrm{m}(C)$, we have

$$\bigl(|x| + |y|\bigr)^*(t) = \bigl(|x| + |y|\bigr)^*(\infty) \leq x^*(\infty) + y^*(\infty) < x^*(t) + y^*(t).$$

Additionally, for all $t > 0$,

$$\bigl(|x| + |y|\bigr)^*(t) = \bigl(|x| + |y|\bigr)^*(\infty) \leq x^*(\infty) + y^*(\infty) \leq x^*(t) + y^*(t).$$

Since $\gamma = \infty$ and $(|x| + |y|)^*$ satisfies the conditions (i) and (ii) of Theorem 3.2 in [15], so $(|x| + |y|)^*$ is a UM point. Thus $\|(|x| + |y|)^*\| < \|x^* + y^*\|$. Therefore,

$$\|x + y\| = \bigl\|\bigl(|x| + |y|\bigr)^*\bigr\| < \bigl\|x^* + y^*\bigr\| \leq \bigl\|x^*\bigr\| + \bigl\|y^*\bigr\| = \|x\| + \|y\|,$$

which finishes the proof. □

Theorem 2.5 *An element $x \in S(\Gamma_{p,w}[0,1))$ is an NSQ point if and only if $\mathrm{m}(\mathcal{S}(x)) \geq \beta$ and, if $\gamma \leq 1/2$, x^* is not constant on $[0,2\gamma]$, where β and γ are defined in* (6).

Proof Necessity. It follows from Theorems 2.2 and 2.3.

Sufficiency. Let $y \in S(\Gamma_{p,w}[0,\infty))$. If x and y have disjoint supports, then, by Lemma 1 in [20], $(x + y)^{**}(t) < x^{**}(t) + y^{**}(t)$ for all $t \in (0, \mathrm{m}(\mathcal{S}(x) \cup \mathcal{S}(y)))$. Since $\mathrm{m}(\mathcal{S}(x)) \geq \beta$, so $\mathrm{m}(\mathcal{S}(w) \cap (\beta, \mathrm{m}(\mathcal{S}(x) \cup \mathcal{S}(y)))) > 0$, whence $\|x + y\| < 2$ (see Remark 2.3).

Assume that x and y have not disjoint supports, *i.e.*,

$$\mathrm{m}(A_1 \cup A_2) > 0, \tag{18}$$

where

$$\begin{aligned}&A_1 = \bigl\{t \in (0,1) \colon x(t)y(t) > 0\bigr\}, \qquad A_2 = \bigl\{t \in (0,1) \colon x(t)y(t) < 0\bigr\}, \\ &A_3 = \bigl\{t \in (0,1) \colon x(t)y(t) = 0 \text{ and } \bigl|x(t)\bigr| + \bigl|y(t)\bigr| > 0\bigr\}.\end{aligned} \tag{19}$$

$\mathcal{A}$. Assume $\mathrm{m}(\mathcal{S}(|x|+|y|)) \leq \gamma$. By Theorem 3.1 in [15], $|x|+|y|$ is an LM point. By (18), at least one of the inequalities holds:

$$|x+y|\chi_{A_2} < \big(|x|+|y|\big)\chi_{A_2} \quad \text{or} \quad |x-y|\chi_{A_1} < \big(|x|+|y|\big)\chi_{A_1}.$$

Obviously, $|x \pm y| \leq |x|+|y|$, whence at least one of the inequalities holds: $\|x+y\| < \||x|+|y|\| \leq \|x\|+\|y\|$ or $\|x-y\| < \||x|+|y|\| \leq \|x\|+\|y\|$.

$\mathcal{B}$. Suppose

$$\mathrm{m}\big(\mathcal{S}\big(|x|+|y|\big)\big) > \gamma. \tag{20}$$

Denote the sets B_+, B_-, B_0, B_x, B_y of measure γ satisfying

$$\begin{aligned} &\int_0^\gamma (x+y)^* = \int_{B_+} |x+y|, \qquad \int_0^\gamma (x-y)^* = \int_{B_-} |x-y|, \\ &\int_0^\gamma \big(|x|+|y|\big)^* = \int_{B_0} |x|+|y|, \qquad \int_0^\gamma x^* = \int_{B_x} |x|, \qquad \int_0^\gamma y^* = \int_{B_y} |y| \end{aligned} \tag{21}$$

(see Lemma 2.1 and Remark 2.1). Notice

$$A_i \cap A_j = \emptyset \quad \text{for } i \neq j \text{ and } i,j \in \{1,2,3\}. \tag{22}$$

Moreover,

$$A_1 \cup A_2 \subset \mathcal{S}(x), \qquad A_1 \cup A_2 \subset \mathcal{S}(y), \qquad \mathcal{S}\big(|x|+|y|\big) = A_1 \cup A_2 \cup A_3.$$

Consider the following cases.

Case I. There is $t_0 \in (0,\gamma)$ such that $(x+y)^*(t_0) < (|x|+|y|)^*(t_0)$, or there is $t_1 \in (0,\gamma)$ such that $(x-y)^*(t_1) < (|x|+|y|)^*(t_1)$.

Case II. $(x+y)^*(t) = (x-y)^*(t) = (|x|+|y|)^*(t)$ for all $t \in (0,\gamma)$.

Case II.1. $\mathrm{m}(B_+ \cap A_3) > 0$ or $\mathrm{m}(B_- \cap A_3) > 0$.

Case II.1.A. $S(y) = B_+ \cap A_1$ or $S(x) = B_+ \cap A_1$.

Case II.1.B. $S(x) \supsetneq B_+ \cap A_1$ and $S(y) \supsetneq B_+ \cap A_1$.

Case II.2. $\mathrm{m}(B_+ \cap A_3) = 0$ and $\mathrm{m}(B_- \cap A_3) = 0$.

Now let us discuss all the cases.

Proof of Case I. Assume that there is $t_0 \in (0,\gamma)$ such that $(x+y)^*(t_0) < (|x|+|y|)^*(t_0)$. Since $(x \pm y) \leq |x|+|y|$ so $(x \pm y)^* \leq (|x|+|y|)^*$. By the right continuity of nonincreasing rearrangement, there is $\delta > t_0$ such that $(x+y)^*(t) < (|x|+|y|)^*(t)$ for all $t \in (t_0,\delta)$. Therefore,

$$\int_0^t (x+y)^* < \int_0^t \big(|x|+|y|\big)^*$$

for $t \in (t_0,\gamma)$. It is clear that $\int_0^t (|x|+|y|)^* \leq \int_0^t x^* + \int_0^t y^*$ (see Remark 2.2), whence

$$(x+y)^{**}(t) < x^{**}(t) + y^{**}(t)$$

for every $t \in (t_0,\gamma)$. By the definition of γ, $\mathrm{m}(\mathcal{S}(w) \cap (t_0,\gamma)) > 0$, so $\|x+y\| < 2$ (see Remark 2.3).

Notice, if there is $t_0 \in (0,\gamma)$ that $(x-y)^*(t_0) < (|x|+|y|)^*(t_0)$, then analogous reasoning gives $\|x-y\| < 2$.

Proof of Case II. Suppose

$$(x+y)^*(t) = (x-y)^*(t) = \bigl(|x|+|y|\bigr)^*(t) \tag{23}$$

for every $t \in (0,\gamma)$. Condition (23) implies

$$\mathrm{m}(B_+ \cap A_2) = 0 \quad \text{and} \quad \mathrm{m}(B_- \cap A_1) = 0, \tag{24}$$

since otherwise if, for example, $\mathrm{m}(B_+ \cap A_2) > 0$, then, by $\mathcal{S}(x+y) \subset A_1 \cup A_2 \cup A_3$,

$$\begin{aligned}\int_0^\gamma (x+y)^* &= \int_{B_+} |x+y| = \int_{B_+\cap A_2} |x+y| + \int_{B_+\cap(A_1\cup A_3)} |x|+|y| \\ &< \int_{B_+\cap A_2} |x|+|y| + \int_{B_+\cap(A_1\cup A_3)} |x|+|y| = \int_{B_+} |x|+|y| \le \int_0^\gamma \bigl(|x|+|y|\bigr)^*,\end{aligned}$$

a contradiction with (23).

Notice

$$\min\bigl\{\mathrm{m}\bigl(\mathcal{S}(x+y)\bigr), \mathrm{m}\bigl(\mathcal{S}(x-y)\bigr)\bigr\} \ge \gamma \tag{25}$$

by (20) and (23).

Case II.1. Assume $\mathrm{m}(B_+ \cap A_3) > 0$. Then $\mathrm{m}(B_+) = \gamma$ implies

$$\mathrm{m}(B_+ \cap A_1) < \gamma. \tag{26}$$

By (24), we have

$$\begin{aligned}\int_0^\gamma (x+y)^* &= \int_{B_+} |x+y| = \int_{B_+\cap A_3} |x+y| + \int_{B_+\cap A_1} |x+y| \\ &= \int_{B_+\cap A_3\cap\mathcal{S}(x)} |x| + \int_{B_+\cap A_3\cap\mathcal{S}(y)} |y| + \int_{B_+\cap A_1} |x|+|y|.\end{aligned} \tag{27}$$

Case II.1.A. Assume $S(y) = B_+ \cap A_1$. Then $\mathrm{m}(\mathcal{S}(y)) < \gamma$ by (26), and $\mathcal{S}(y) = A_1$ since $A_1 \subset \mathcal{S}(y)$. Thus

$$\begin{aligned}&A_3 \subset \mathcal{S}(x), \qquad \mathrm{m}(A_2) = 0, \\ &\int_{B_+\cap A_3\cap\mathcal{S}(y)} |y| = 0 \quad \text{and} \quad \int_{B_+\cap A_1} |y| = \int_{B_y} |y|.\end{aligned} \tag{28}$$

Moreover, by (20) and $\mathcal{S}(|x|+|y|) = A_1 \cup A_3 = \mathcal{S}(x)$, we get $\mathrm{m}(\mathcal{S}(x)) > \gamma$.

We claim that

$$\int_{B_+\cap A_3\cap\mathcal{S}(x)} |x| + \int_{B_+\cap A_1} |x| < \int_{B_x} |x|. \tag{29}$$

Assume for the contrary that $\int_{B_+\cap A_3\cap\mathcal{S}(x)}|x| + \int_{B_+\cap A_1}|x| = \int_{B_x}|x|$. By (27) and (28),

$$\int_0^\gamma (x+y)^* = \int_{B_x}|x| + \int_{B_y}|y| = \int_0^\gamma x^* + \int_0^\gamma y^*. \tag{30}$$

Moreover, by (24) and $\mathrm{m}(A_2) = 0$, we get $B_-\cap(\mathcal{S}(x-y))\subset A_3\subset\mathcal{S}(x)$. Furthermore, by (21) and (23), we get

$$\int_0^\gamma (x+y)^* = \int_0^\gamma (x-y)^* = \int_{B_-}|x-y| = \int_{B_-}|x| \leq \int_0^\gamma x^* < \int_0^\gamma x^* + \int_0^\gamma y^*,$$

a contradiction with (30). This proves claim (29). Therefore, (27), (28), (29) imply $\int_0^\gamma (x+y)^* < \int_0^\gamma x^* + \int_0^\gamma y^*$, which finishes the proof (see Remark 2.3 and the definition of γ).

It is clear that analogous reasoning holds for the case of $S(x) = B_+\cap A_1$.

Case II.1.B. Assume $\mathcal{S}(x) \supsetneq B_+\cap A_1$ and $\mathcal{S}(y)\supsetneq B_+\cap A_1$. Then

$$\mathrm{m}\big(S(x)\big) > \mathrm{m}(B_+\cap A_1) \quad \text{and} \quad \mathrm{m}\big(S(y)\big) > \mathrm{m}(B_+\cap A_1). \tag{31}$$

We claim that at least one of inequalities (32) or (33) holds,

$$\int_{B_+\cap A_3\cap\mathcal{S}(y)}|y| + \int_{B_+\cap A_1}|y| < \int_{B_y}|y|, \tag{32}$$

$$\int_{B_+\cap A_3\cap\mathcal{S}(x)}|x| + \int_{B_+\cap A_1}|x| < \int_{B_x}|x|. \tag{33}$$

If $\mathrm{m}(B_+\cap A_3\cap\mathcal{S}(y)) = 0$ or $\mathrm{m}(B_+\cap A_3\cap\mathcal{S}(x)) = 0$, then, by (26) and (31), we get (32) or (33), respectively.

If $\mathrm{m}(B_+\cap A_3\cap\mathcal{S}(x)) > 0$ and $\mathrm{m}(B_+\cap A_3\cap\mathcal{S}(y)) > 0$, then

$$\mathrm{m}\big(B_+\cap\mathcal{S}(x)\big) < \gamma \quad \text{and} \quad \mathrm{m}\big(B_+\cap\mathcal{S}(y)\big) < \gamma. \tag{34}$$

Assume for the contrary that (32) and (33) do not hold, *i.e.*,

$$\int_{B_+\cap[A_1\cup(A_3\cap\mathcal{S}(y))]}|y| = \int_{B_y}|y| \quad \text{and} \quad \int_{B_+\cap[A_1\cup(A_3\cap\mathcal{S}(x))]}|x| = \int_{B_x}|x|. \tag{35}$$

The equality $\mathcal{S}(y) = A_1\cup A_2\cup(A_3\cap\mathcal{S}(y))$ and (24) imply

$$B_+\cap\mathcal{S}(y) = B_+\cap\big[A_1\cup A_2\cup\big(A_3\cap\mathcal{S}(y)\big)\big] = B_+\cap\big[A_1\cup\big(A_3\cap\mathcal{S}(y)\big)\big],$$

and analogously we get

$$B_+\cap\mathcal{S}(x) = B_+\cap\big[A_1\cup\big(A_3\cap\mathcal{S}(x)\big)\big].$$

Therefore, by (35), we have

$$\int_{B_+\cap\mathcal{S}(y)}|y| = \int_{B_+\cap[A_1\cup(A_3\cap\mathcal{S}(y))]}|y| = \int_{B_y}|y| = \int_0^\gamma y^*$$

and

$$\int_{B_+\cap\mathcal{S}(x)}|x| = \int_{B_+\cap[A_1\cup(A_3\cap\mathcal{S}(x))]}|x| = \int_{B_x}|x| = \int_0^\gamma x^*.$$

Thus by (34), $\mathcal{S}(x)\subset B_+$ and $\mathcal{S}(y)\subset B_+$, whence $\mathcal{S}(|x|+|y|)\subset B_+$. Since $\mathrm{m}(B_+)=\gamma$, we get a contradiction with (20). This proves that (32) or (33) holds.

Finally, by (27) and (32) or (33), we get $\int_0^\gamma (x+y)^* < \int_0^\gamma x^* + \int_0^\gamma y^*$, which finishes the proof (see Remark 2.3 and the definition of γ).

Considering the case of $\mathrm{m}(B_-\cap A_3)>0$, we may follow analogously but with element $(x-y)^*$.

Case II.2. Suppose $\mathrm{m}(B_+\cap A_3)=0$ and $\mathrm{m}(B_-\cap A_3)=0$. Then, by (24),

$$B_+\cap\mathcal{S}(x+y)\subset A_1 \quad\text{and}\quad B_-\cap\mathcal{S}(x-y)\subset A_2. \tag{36}$$

We claim that $\mathrm{m}(A_1)\geq\gamma$ and $\mathrm{m}(A_2)\geq\gamma$. If $\mathrm{m}(A_1)<\gamma$, then $\mathrm{m}(B_+\cap\mathcal{S}(x+y))<\gamma$, whence $\mathrm{m}(\mathcal{S}(x+y))<\gamma$ by the definition of set B_+, a contradiction with (25). The case of $\mathrm{m}(A_2)<\gamma$ goes analogously and proves the claim.

By (22), $\gamma\leq 1/2$ and $\mathrm{m}(\mathcal{S}(x))\geq 2\gamma$. Since x^* is not constant on $(0,2\gamma)$, then

$$\int_0^{2\gamma} x^* < 2\int_0^\gamma x^*.$$

Conditions (36) imply $(B_+\cap\mathcal{S}(x+y))\cap(B_-\cap\mathcal{S}(x-y))=\emptyset$. Consequently,

$$\int_{B_+\cap\mathcal{S}(x+y)}|x| + \int_{B_-\cap\mathcal{S}(x-y)}|x| \leq \int_0^{2\gamma}x^* < 2\int_0^\gamma x^*.$$

Thus

$$\int_{B_+\cap\mathcal{S}(x+y)}|x| < \int_0^\gamma x^* \quad\text{or}\quad \int_{B_-\cap\mathcal{S}(x-y)}|x| < \int_0^\gamma x^*.$$

Finally, by (36), one of the following holds:

$$\int_0^\gamma (x+y)^* = \int_{B_+\cap\mathcal{S}(x+y)}|x+y| = \int_{B_+\cap\mathcal{S}(x+y)}|x|+|y| < \int_0^\gamma x^* + \int_0^\gamma y^*$$

or

$$\int_0^\gamma (x-y)^* = \int_{B_-\cap\mathcal{S}(x-y)}|x-y| = \int_{B_-\cap\mathcal{S}(x-y)}|x|+|y| < \int_0^\gamma x^* + \int_0^\gamma y^*,$$

which finishes the proof (see Remark 2.3 and the definition of γ). □

Below we present some modification of Lemma 2.1 from [15].

Lemma 2.4 *Let $x,y\in L^0$ satisfy $|x|\leq|y|$, $|x(t)|<|y(t)|$ for $t\in A$, $\mathrm{m}(A)>0$ and $|y(t)|>x^*(\infty)$ for every $t\in A$. Then there is a set B of positive measure such that $x^*(t)<y^*(t)$ for $t\in B$.*

Proof Denote $D = \{t\colon 0 < |x(t)| < x^*(\infty)\}$. The case of $\mathrm{m}(D) = 0$ is done in Lemma 2.1 in [15]. Assume $\mathrm{m}(D) > 0$. Thus $x^*(\infty) > 0$. Define

$$\tilde{x} = |x|\chi_{I\setminus D} + x^*(\infty)\chi_D \quad \text{and} \quad \tilde{y} = |y|\chi_{I\setminus D} + |y|\chi_{D_2} + x^*(\infty)\chi_{D_1},$$

where $D_1 = \{t \in D\colon |y(t)| < x^*(\infty)\}$, $D_2 = D \setminus D_1$.

Then $\tilde{x}$ and x as well as $\tilde{y}$ and y are equimeasurable. Since $\tilde{x}$ and $\tilde{y}$ satisfy the assumptions of Lemma 2.1 in [15], there is a set B with $\mathrm{m}(B) > 0$ such that $x^*(t) = \tilde{x}^*(t) < \tilde{y}^*(t) = y^*(t)$ for $t \in B$. □

Theorem 2.6 *Let $x \in S(\Gamma_{p,w}[0,\infty))$, β and γ are as in* (6), *and let the weight function be such that $\gamma < \infty$. Then x is an NSQ point if and only if* $\mathrm{m}(\mathcal{S}(x)) \geq \beta$ *and x^* is not constant on* $(0, 2\gamma)$.

Proof *Necessity.* It follows from Theorems 2.2 and 2.3.

Sufficiency. Let $y \in S(\Gamma_{p,w})$. If $\mathrm{m}(\mathcal{S}(x) \cap \mathcal{S}(y)) = 0$, then by Lemma 2.3, $(x+y)^{**}(t) < x^{**}(t) + y^{**}(t)$ for all $t \in (0, \mathrm{m}(\mathcal{S}(x) \cup \mathcal{S}(y)))$. Since $\mathrm{m}(\mathcal{S}(x)) \geq \beta$, so $\mathrm{m}(\mathcal{S}(w) \cap (\beta, \mathrm{m}(\mathcal{S}(x) \cup \mathcal{S}(y)))) > 0$, whence $\|x + y\| < 2$ (see Remark 2.3 and the definition of β).

Denote

$$\begin{aligned} &A_1 = \{t \in (0,1)\colon x(t)y(t) > 0\}, \qquad A_2 = \{t \in (0,1)\colon x(t)y(t) < 0\}, \\ &A_3 = \{t \in (0,1)\colon x(t)y(t) = 0 \text{ and } |x(t)| + |y(t)| > 0\}, \end{aligned} \tag{37}$$

and assume

$$\mathrm{m}(A_1 \cup A_2) > 0. \tag{38}$$

Obviously,

$$A_i \cap A_j = \emptyset \quad \text{for } i \neq j \text{ and } i,j \in \{1,2,3\}. \tag{39}$$

$\mathcal{A}$. Assume $\mathrm{m}(\mathcal{S}(|x| + |y|)) \leq \gamma$. We follow as in the proof of Theorem 2.5, Case $\mathcal{A}$.

$\mathcal{B}$. Suppose

$$\mathrm{m}\bigl(\mathcal{S}\bigl(|x| + |y|\bigr)\bigr) > \gamma. \tag{40}$$

Consider the following cases (for the definitions of s_0, s_+, s_-, B_+ and B_- see (43) and (45) below).

Case I. There is $t_0 \in (0,\gamma)$ such that $(x+y)^*(t_0) < (|x| + |y|)^*(t_0)$, or there is $t_1 \in (0,\gamma)$ such that $(x-y)^*(t_1) < (|x| + |y|)^*(t_1)$.

Case II. $(x+y)^*(t) = (x-y)^*(t) = (|x| + |y|)^*(t)$ for all $t \in (0,\gamma)$.

Case II.A. $s_0 = 0$ or $s_+ = 0$ or $s_- = 0$.

Case II.B. $s_0 > 0$ and $s_+ > 0$ and $s_- > 0$.

Case II.B.a. $s_0 \leq \gamma$.

Case II.B.a.1. $\mathrm{m}(B_+ \cap A_3) > 0$ or $\mathrm{m}(B_- \cap A_3) > 0$.

Case II.B.a.1.A. $S(y) = B_+ \cap A_1$ or $S(x) = B_+ \cap A_1$.

Case II.B.a.1.B. $S(x) \supsetneq B_+ \cap A_1$ and $S(y) \supsetneq B_+ \cap A_1$.

Case II.B.a.2. $\mathrm{m}(B_+ \cap A_3) = 0$ and $\mathrm{m}(B_- \cap A_3) = 0$.

Case II. B.a.2.A. x^* or y^* is not constant on $(0, 2s_0)$.

Case II. B.a.2.B. $x^*\chi_{(0,2s_0)} = a > 0$ and $y^*\chi_{(0,2s_0)} = b > 0$.

Case II.B.b. $s_0 > \gamma$.

Now let us discuss all the cases.

Proof of Case I. The proof is the same as that of Theorem 2.5, Case $\mathcal{B}$.I.

Proof of Case II. Suppose

$$(x+y)^*(t) = (x-y)^*(t) = \big(|x| + |y|\big)^*(t) \tag{41}$$

for every $t \in (0, \gamma)$. Notice

$$\min\big\{\mathrm{m}\big(\mathcal{S}(x+y)\big), \mathrm{m}\big(\mathcal{S}(x-y)\big)\big\} \geq \gamma \tag{42}$$

by (40) and (41).

Denote

$$\begin{aligned} s_0 &= \sup\big\{t\colon \big(|x| + |y|\big)^*(t) > \big(|x| + |y|\big)^*(\infty)\big\}, \\ s_+ &= \sup\big\{t\colon (x+y)^*(t) > (x+y)^*(\infty)\big\}, \\ s_- &= \sup\big\{t\colon (x-y)^*(t) > (x-y)^*(\infty)\big\}. \end{aligned} \tag{43}$$

Case II.A.

(a) Assume $s_0 = 0$, *i.e.*, for every $t > 0$, $(|x| + |y|)^*(t) = (|x| + |y|)^*(\infty) = a > 0$. Since x^* is not constant on $(0, 2\gamma)$, so by Remark 2.4, $\mathrm{m}(C) > 0$, where $C = \{t\colon |x(t)| > x^*(\infty)\}$. Thus, for every $0 < t < \mathrm{m}(C)$,

$$\big(|x| + |y|\big)^*(t) = \big(|x| + |y|\big)^*(\infty) \leq x^*(\infty) + y^*(\infty) < x^*(t) + y^*(t).$$

Additionally, for all $t > 0$,

$$\big(|x| + |y|\big)^*(t) = \big(|x| + |y|\big)^*(\infty) \leq x^*(\infty) + y^*(\infty) \leq x^*(t) + y^*(t).$$

Since $(|x| + |y|)^*$ is constant on $(0, \infty)$, so it satisfies the conditions (i) and (ii) of Theorem 3.2 in [15]. Thus $(|x| + |y|)^*$ is a UM point, whence $\|(|x| + |y|)^*\| < \|x^* + y^*\|$. Finally,

$$\|x+y\| \leq \big\||x| + |y|\big\| = \big\|\big(|x| + |y|\big)^*\big\| < \big\|x^* + y^*\big\| \leq \big\|x^*\big\| + \big\|y^*\big\| = \|x\| + \|y\|,$$

which finishes the proof.

(b) Assume $s_+ = 0$. By (42), for every $t > 0$, $(x+y)^*(t) = (x+y)^*(\infty) = a > 0$. By (41) and $(x+y)^*(\infty) \leq (|x| + |y|)^*(\infty)$,

$$(x+y)^*(t) = \big(|x| + |y|\big)^*(t) = a$$

for all $t > 0$, whence $s_0 = 0$. The rest of the proof goes as in (a).

(c) If $s_- = 0$, then analogous reasoning as in (b) goes for element $(x-y)^*$.

Case II.B. Suppose $s_0 > 0$ and $s_+ > 0$ and $s_- > 0$. Clearly,

$$\min\{s_+, s_-\} \geq \min\{s_0, \gamma\} \tag{44}$$

by (41) and

$$(x \pm y)^*(\infty) \leq \bigl(|x| + |y|\bigr)^*(\infty) < \bigl(|x| + |y|\bigr)^*(t) = (x \pm y)^*(t)$$

for all $t \in (0, \min\{s_0, \gamma\})$.

Case II.B.a. Assume $s_0 \leq \gamma$. Then $(|x| + |y|)^*(\infty) > 0$, since the opposite case (40) implies $s_0 > \gamma$.

By Lemma 2.1 and Remark 2.1, we find the sets B_+, B_-, B_0 of measure s_0 satisfying

$$\begin{aligned} \int_0^{s_0} (x+y)^* &= \int_{B_+} |x+y|, \\ \int_0^{s_0} (x-y)^* &= \int_{B_-} |x-y|, \\ \int_0^{s_0} \bigl(|x| + |y|\bigr)^* &= \int_{B_0} |x| + |y|. \end{aligned} \tag{45}$$

Clearly, by (44) and $s_0 \leq \gamma$, the sets B_+ and B_- are well defined.

Condition (41) implies

$$\mathrm{m}(B_+ \cap A_2) = 0 \quad \text{and} \quad \mathrm{m}(B_- \cap A_1) = 0, \tag{46}$$

since otherwise if, for example, $\mathrm{m}(B_+ \cap A_2) > 0$, then, by $\mathcal{S}(x+y) \subset A_1 \cup A_2 \cup A_3$,

$$\begin{aligned} \int_0^{s_0} (x+y)^* = \int_{B_+} |x+y| &= \int_{B_+ \cap A_2} |x+y| + \int_{B_+ \cap (A_1 \cup A_3)} |x| + |y| \\ &< \int_{B_+ \cap A_2} |x| + |y| + \int_{B_+ \cap (A_1 \cup A_3)} |x| + |y| \\ &= \int_{B_+} |x| + |y| \leq \int_0^{s_0} \bigl(|x| + |y|\bigr)^*, \end{aligned}$$

a contradiction with (41).

Case II.B.a.1. Assume $\mathrm{m}(B_+ \cap A_3) > 0$. Then $\mathrm{m}(B_+) = s_0$ implies

$$\mathrm{m}(B_+ \cap A_1) < s_0. \tag{47}$$

By (46), we have

$$\begin{aligned} \int_0^{s_0} (x+y)^* = \int_{B_+} |x+y| &= \int_{B_+ \cap A_3} |x+y| + \int_{B_+ \cap A_1} |x+y| \\ &= \int_{B_+ \cap A_3 \cap \mathcal{S}(x)} |x| + \int_{B_+ \cap A_1} |x| + \int_{B_+ \cap A_3 \cap \mathcal{S}(y)} |y| + \int_{B_+ \cap A_1} |y|. \end{aligned} \tag{48}$$

Case II.B.a.1.A. Assume $S(y) = B_+ \cap A_1$. Then $\mathrm{m}(\mathcal{S}(y)) < s_0$ by (47) and $\mathcal{S}(y) = A_1$ since $A_1 \subset \mathcal{S}(y)$. Thus

$$\begin{aligned} &A_3 \subset \mathcal{S}(x), \qquad \mathrm{m}(A_2) = 0, \\ &\int_{B_+\cap A_3\cap\mathcal{S}(y)} |y| = 0 \quad \text{and} \quad \int_{B_+\cap A_1} |y| = \sup_{\mathrm{m}(B_y)=s_0} \int_{B_y} |y|. \end{aligned} \tag{49}$$

Moreover, by (40) and $\mathcal{S}(|x| + |y|) = A_1 \cup A_3 = \mathcal{S}(x)$, we get $\mathrm{m}(\mathcal{S}(x)) > \gamma$.

We claim that

$$\int_{B_+\cap A_3\cap\mathcal{S}(x)} |x| + \int_{B_+\cap A_1} |x| < \sup_{\mathrm{m}(B_x)=s_0} \int_{B_x} |x|. \tag{50}$$

Assume for the contrary that $\int_{B_+\cap A_3\cap\mathcal{S}(x)} |x| + \int_{B_+\cap A_1} |x| = \sup_{\mathrm{m}(B_x)=s_0} \int_{B_x} |x|$. By Lemma 2.2, (48) and (49),

$$\int_0^{s_0} (x+y)^* = \sup_{\mathrm{m}(B_x)=s_0} \int_{B_x} |x| + \sup_{\mathrm{m}(B_y)=s_0} \int_{B_y} |y| = \int_0^{s_0} x^* + \int_0^{s_0} y^*. \tag{51}$$

Moreover, by (46) and $\mathrm{m}(A_2) = 0$, we get $B_- \cap (\mathcal{S}(x-y)) \subset A_3 \subset \mathcal{S}(x)$. Furthermore, applying (41) and (45), we obtain

$$\int_0^{s_0} (x+y)^* = \int_0^{s_0} (x-y)^* = \int_{B_-} |x-y| = \int_{B_-} |x| \le \int_0^{s_0} x^* < \int_0^{s_0} x^* + \int_0^{s_0} y^*,$$

a contradiction with (51). This proves claim (50). Therefore, (48), (49) and (50) imply

$$\int_0^{s_0} (x+y)^* < \int_0^{s_0} x^* + \int_0^{s_0} y^*. \tag{52}$$

Furthermore, by the definition of s_0, for every $t > s_0$,

$$(x \pm y)^*(t) \le \big(|x| + |y|\big)^*(t) = \big(|x| + |y|\big)^*(\infty) \le x^*(\infty) + y^*(\infty) \le x^*(t) + y^*(t).$$

Thus

$$\int_{s_0}^t (x \pm y)^*(t) \le \int_{s_0}^t x^*(t) + y^*(t) \quad \text{for } t > s_0. \tag{53}$$

Finally,

$$\int_0^t (x+y)^*(t) < \int_0^t x^* + \int_0^t y^* \quad \text{for all } t \ge s_0. \tag{54}$$

Taking $t = \gamma$, we finish the proof (see Remark 2.3 and the definition of γ).

It is clear that analogous reasoning holds for the case of $S(x) = B_+ \cap A_1$.

Case II.B.a.1.B. Assume $\mathcal{S}(y) \supsetneq B_+ \cap A_1$ and $\mathcal{S}(x) \supsetneq B_+ \cap A_1$, whence

$$\mathrm{m}\big(S(x)\big) > \mathrm{m}(B_+ \cap A_1) \quad \text{and} \quad \mathrm{m}\big(S(y)\big) > \mathrm{m}(B_+ \cap A_1). \tag{55}$$

We claim that at least one of inequalities (56) or (57) holds,

$$\int_{B_+\cap A_3\cap\mathcal{S}(y)}|y| + \int_{B_+\cap A_1}|y| < \sup_{\mathrm{m}(B_y)=s_0}\int_{B_y}|y|, \tag{56}$$

$$\int_{B_+\cap A_3\cap\mathcal{S}(x)}|x| + \int_{B_+\cap A_1}|x| < \sup_{\mathrm{m}(B_x)=s_0}\int_{B_x}|x|. \tag{57}$$

If $\mathrm{m}(B_+\cap A_3\cap\mathcal{S}(y)) = 0$ or $\mathrm{m}(B_+\cap A_3\cap\mathcal{S}(x)) = 0$, then by (47) and (55) we get (56) or (57), respectively.

If $\mathrm{m}(B_+\cap A_3\cap\mathcal{S}(x)) > 0$ and $\mathrm{m}(B_+\cap A_3\cap\mathcal{S}(y)) > 0$, then

$$\mathrm{m}\big(B_+\cap\mathcal{S}(x)\big) < s_0 \quad \text{and} \quad \mathrm{m}\big(B_+\cap\mathcal{S}(y)\big) < s_0. \tag{58}$$

Assume for the contrary that (56) and (57) do not hold, *i.e.*,

$$\begin{aligned}&\int_{B_+\cap[A_1\cup(A_3\cap\mathcal{S}(y))]}|y| = \sup_{\mathrm{m}(B_y)=s_0}\int_{B_y}|y| \quad \text{and}\\ &\int_{B_+\cap[A_1\cup(A_3\cap\mathcal{S}(x))]}|x| = \sup_{\mathrm{m}(B_x)=s_0}\int_{B_x}|x|.\end{aligned} \tag{59}$$

By (46), we get

$$B_+\cap\mathcal{S}(y) = B_+\cap\big[A_1\cup\big(A_3\cap\mathcal{S}(y)\big)\big]$$

and

$$B_+\cap\mathcal{S}(x) = B_+\cap\big[A_1\cup\big(A_3\cap\mathcal{S}(x)\big)\big].$$

Therefore, by (59) and Lemma 2.2, we have

$$\int_{B_+\cap S_y}|y| = \int_{B_+\cap[A_1\cup(A_3\cap\mathcal{S}(y))]}|y| = \sup_{\mathrm{m}(B_y)=s_0}\int_{B_y}|y| = \int_0^{s_0} y^*$$

and

$$\int_{B_+\cap S_x}|x| = \int_{B_+\cap[A_1\cup(A_3\cap\mathcal{S}(x))]}|x| = \sup_{\mathrm{m}(B_x)=s_0}\int_{B_x}|x| = \int_0^{s_0} x^*.$$

Thus, by (58), $\mathcal{S}(x)\subset B_+$ and $\mathcal{S}(y)\subset B_+$, whence $\mathcal{S}(|x|+|y|)\subset B_+$. Since $\mathrm{m}(B_+) = s_0 \le \gamma$, we get a contradiction with (40). This proves that (56) or (57) holds.

Therefore, by (48) and (56) or (57), we get $\int_0^{s_0}(x+y)^* < \int_0^{s_0} x^* + \int_0^{s_0} y^*$. Analogously as in Case II.B.a.1.A, we get (53) and then (54), which for $t = \gamma$ finishes the proof.

Considering the case of $\mathrm{m}(B_-\cap A_3) > 0$, we may follow analogously as above but with the element $(x-y)^*$.

Case II.B.a.2. Suppose $\mathrm{m}(B_+\cap A_3) = 0$ and $\mathrm{m}(B_-\cap A_3) = 0$. Then, by (46),

$$B_+\cap\mathcal{S}(x+y)\subset A_1 \quad \text{and} \quad B_-\cap\mathcal{S}(x-y)\subset A_2. \tag{60}$$

We claim $\mathrm{m}(A_1) \geq s_0$ and $\mathrm{m}(A_2) \geq s_0$. If $\mathrm{m}(A_1) < s_0$, then $\mathrm{m}(B_+ \cap \mathcal{S}(x+y)) < s_0$, whence $\mathrm{m}(\mathcal{S}(x+y)) < s_0 \leq \gamma$ by the definition of set B_+, a contradiction with (42). The case of $\mathrm{m}(A_2) < s_0$ goes analogously and proves the claim.

By (39), $\mathrm{m}(\mathcal{S}(x)) \geq 2s_0$ and $\mathrm{m}(\mathcal{S}(y)) \geq 2s_0$.

Case II.B.a.2.A. If x^* is not constant on $(0, 2s_0)$, then

$$\int_0^{2s_0} x^* < 2\int_0^{s_0} x^*.$$

Conditions (60) imply $(B_+ \cap \mathcal{S}(x+y)) \cap (B_- \cap \mathcal{S}(x-y)) = \emptyset$. Consequently,

$$\int_{B_+\cap\mathcal{S}(x+y)} |x| + \int_{B_-\cap\mathcal{S}(x-y)} |x| \leq \int_0^{2s_0} x^* < 2\int_0^{s_0} x^*.$$

Thus,

$$\int_{B_+\cap\mathcal{S}(x+y)} |x| < \int_0^{s_0} x^* \quad \text{or} \quad \int_{B_-\cap\mathcal{S}(x-y)} |x| < \int_0^{s_0} x^*.$$

Finally, by (60), one of the following holds:

$$\int_0^{s_0} (x+y)^* = \int_{B_+\cap\mathcal{S}(x+y)} |x+y| = \int_{B_+\cap\mathcal{S}(x+y)} |x| + |y| < \int_0^{s_0} x^* + \int_0^{s_0} y^*$$

or

$$\int_0^{s_0} (x-y)^* = \int_{B_-\cap\mathcal{S}(x-y)} |x-y| = \int_{B_-\cap\mathcal{S}(x-y)} |x| + |y| < \int_0^{s_0} x^* + \int_0^{s_0} y^*.$$

Analogously as in Case II.B.a.1.A, we get (53) and then (54), which for $t = \gamma$ finishes the proof.

If y^* is not constant on $(0, 2s_0)$, then we use analogous argumentation.

Case II.B.a.2.B. Assume $x^*\chi_{(0,2s_0)} = a > 0$ and $y^*\chi_{(0,2s_0)} = b > 0$. Note that this is only the case of $s_0 < \gamma$ since x^* is not constant on $(0, 2\gamma)$.

Since for a.e. $t > 0$, $|x(t)| \leq a$ and $|y(t)| \leq b$, so

$$(x \pm y)^*(t) \leq \bigl(|x| + |y|\bigr)^*(t) \leq a + b = x^*(t) + y^*(t) \tag{61}$$

for all $0 < t < 2s_0$.

If there is $t_0 \leq s_0$ such that $(|x| + |y|)^*(t_0) < x^*(t_0) + y^*(t_0)$ then, for every $t_0 < t < 2s_0$,

$$(x \pm y)^*(t) \leq \bigl(|x| + |y|\bigr)^*(t) < x^*(t) + y^*(t). \tag{62}$$

If $(|x| + |y|)^*(t) = x^*(t) + y^*(t)$ for every $t \leq s_0$, then by the definition of s_0 we get (62) for every $s_0 \leq t < 2s_0$. Thus, by (61),

$$\int_0^t (x+y)^* < \int_0^t x^* + \int_0^t y^*$$

for $s_0 < t < 2s_0$.

Analogously as in Case II.B.a.1.A, we get (53) and then (54), which for $t = \gamma > s_0$ finishes the proof.

Case II.B.b. If $s_0 > \gamma$, then we apply the argumentation in Cases II.B.a.1 to II.B.a.2.A for γ instead of s_0. □

The following corollaries have been proved directly in [20].

Corollary 2.1 *The Lorentz space* $\Gamma_{p,w}$ *is nonsquare if and only if*

(i) $\beta = 0$,

(ii) *if* $\alpha = \infty$ *then* $\int_0^\infty w = \infty$,

(iii) *if* $\alpha = 1$ *then* $\gamma > 1/2$,

where β *and* γ *are defined in* (6).

Proof Necessity. (i) Assume $\beta > 0$ and take $x = \chi_A/\|\chi_A\|$, where $0 < \mathrm{m}(A) < \beta$. Then $\|x\| = 1$. By Theorem 2.3, x is not an NSQ point.

(ii) and (iii) Suppose $\alpha = \infty$ and $\int_0^\infty w < \infty$, or $\alpha = 1$ and $\gamma \leq 1/2$. Let $x = \chi_I/\|\chi_I\|$. Then $\|x\| = 1$ and Theorem 2.2 implies that x is not an NSQ point.

Sufficiency. Let $x \in S(\Gamma_{p,w})$.

Let $\alpha = 1$. By Theorem 2.5, (i) and (iii), x is an NSQ point.

If $\alpha = \infty$ then (ii) implies $\gamma = \infty$. In view of Theorem 2.4, x is an NSQ point. □

Recall that $(\Gamma_{p,w}(I))_a \neq \Gamma_{p,w}(I)$ if and only if $I = (0,\infty)$ and $\int_0^\infty w < \infty$ (see [25]).

Corollary 2.2 *Suppose* $\alpha = \infty$ *and* $\int_0^\infty w < \infty$. *The Lorentz space* $(\Gamma_{p,w}[0,\infty))_a$ *is nonsquare if and only if*

(i) $\beta = 0$,

(ii) $\gamma = \infty$,

where β *and* γ *are defined in* (6).

Proof Necessity. (i) The proof is analogous to the proof of Corollary 2.1.

(ii) Let $\gamma < \infty$ and take $x = \chi_{(0,2\gamma)}/\|\chi_{(0,2\gamma)}\|$. Clearly, by Proposition 3.1 in [15], $x \in (\Gamma_{p,w})_a$. The definition of x and Theorem 2.2 imply that x is not an NSQ point.

Sufficiency. Let $x \in S((\Gamma_{p,w})_a)$. By Proposition 3.1 in [15], $x^*(\infty) = 0$. Thus x^* is not constant on $(0,\infty)$. Moreover, (i) implies that $\mathrm{m}(\mathcal{S}(x)) > \beta$. By Theorem 2.4, x is an NSQ point. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors have made intellectual contributions to a published study in equal parts and have written the manuscript. Authors read and approved the final manuscript.

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