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On a finite family of variational inclusions with the constraints of generalized mixed equilibrium and fixed point problems

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Abstract
In this paper, we introduce two iterative algorithms for finding common solutions of a finite family of variational inclusions for maximal monotone and inverse-strongly monotone mappings with the constraints of two problems: a generalized mixed equilibrium problem and a common fixed point problem of an infinite family of nonexpansive mappings and an asymptotically strict pseudocontractive mapping in the intermediate sense in a real Hilbert space. We prove some strong and weak convergence theorems for the proposed iterative algorithms under suitable conditions.
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1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, C be a nonempty closed convex subset of H and P_C be the metric projection of H onto C. Let $S : C \to H$ be a nonlinear mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. A mapping V is called strongly positive on H if there exists a constant $\bar{\gamma} \in (0,1]$ such that

$$\langle Vx, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

A mapping $S : C \to H$ is called L-Lipschitz-continuous if there exists a constant $L > 0$ such that

$$\|Sx - Sy\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

In particular, if $L = 1$ then S is called a nonexpansive mapping; if $L \in (0,1)$ then A is called a contraction.

Let $\varphi : C \to \mathbf{R}$ be a real-valued function, $A : H \to H$ be a nonlinear mapping and $\Theta : C \times C \to \mathbf{R}$ be a bifunction. We consider the generalized mixed equilibrium problem (GMEP)

[1] of finding $x \in C$ such that

$$\Theta(x,y) + \varphi(y) - \varphi(x) + \langle Ax, y-x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

We denote the set of solutions of GMEP (1.1) by GMEP(Θ, φ, A). The GMEP (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games and others. The GMEP is further considered and studied in, *e.g.*, [2–8].

Throughout this paper, it is assumed as in [1] that $\Theta : C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (H1)-(H4) and $\varphi : C \to \mathbf{R}$ is a lower semicontinuous and convex function with restriction (H5), where

(H1) $\Theta(x,x) = 0$ for all $x \in C$;

(H2) Θ is monotone, *i.e.*, $\Theta(x,y) + \Theta(y,x) \leq 0$ for any $x, y \in C$;

(H3) Θ is upper-hemicontinuous, *i.e.*, for each $x, y, z \in C$,

$$\limsup_{t\to 0^+} \Theta\bigl(tz + (1-t)x, y\bigr) \leq \Theta(x,y);$$

(H4) $\Theta(x,\cdot)$ is convex and lower semicontinuous for each $x \in C$;

(H5) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0.$$

Let $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ be two bifunctions, and $B_1, B_2 : C \to H$ be two nonlinear mappings. Consider the system of generalized equilibrium problems (SGEP): find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \Theta_1(x^*, x) + \langle B_1 y^*, x - x^*\rangle + \frac{1}{\mu_1}\langle x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \Theta_2(y^*, y) + \langle B_2 x^*, y - y^*\rangle + \frac{1}{\mu_2}\langle y^* - x^*, y - y^*\rangle \geq 0, & \forall y \in C, \end{cases} \tag{1.2}$$

where μ_1 and μ_2 are two positive constants.

Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0,1]$. For any $n \geq 1$, define a self-mapping W_n on H as follows:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I, \\ U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I, \\ \vdots \\ U_{n,k} = \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I, \\ U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I, \\ \vdots \\ U_{n,2} = \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I, \\ W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I. \end{cases} \tag{1.3}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

Let $f : H \to H$ be a contraction and V be a strongly positive bounded linear operator on H. Assume that $\varphi : H \to \mathbf{R}$ is a lower semicontinuous and convex functional, that $\Theta, \Theta_1, \Theta_2 : H \times H \to \mathbf{R}$ satisfy conditions (H1)-(H4), and that $A, B_1, B_2 : H \to H$ are inverse-strongly monotone. Very recently, motivated by Yao *et al.* [3], Cai and Bu [4] introduced the following hybrid extragradient-like iterative algorithm:

$$\begin{cases} z_n = S_{r_n}^{(\Theta,\varphi)}(x_n - r_n A x_n), \\ y_n = T_{\mu_1}^{\Theta_1}(I - \mu_1 B_1) T_{\mu_2}^{\Theta_2}(I - \mu_2 B_2) z_n, \\ x_{n+1} = \alpha_n(u + \gamma f(x_n)) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n(I + \mu V)) W_n y_n, \quad \forall n \geq 0, \end{cases} \tag{1.4}$$

for finding a common solution of GMEP (1.1), SGEP (1.2), and the fixed point problem of an infinite family of nonexpansive mappings $\{T_i\}_{i=1}^{\infty}$ on H, where $\{r_n\} \subset (0, \infty)$, $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$, and $x_0, u \in H$ are given. The authors proved the strong convergence of the sequence generated by the hybrid iterative algorithm (1.4) to a point $x^* \in (\bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i)) \cap \mathrm{GMEP}(\Theta, \varphi, A) \cap \mathrm{SGEP}(G)$ under some suitable conditions, where $\mathrm{SGEP}(G)$ is the fixed point set of the mapping $G := T_{\mu_1}^{\Theta_1}(I - \mu_1 B_1) T_{\mu_2}^{\Theta_2}(I - \mu_2 B_2)$. This point x^* also solves the following optimization problem:

$$\min_{x \in (\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)) \cap \mathrm{GMEP}(\Theta,\varphi,A) \cap \mathrm{SGEP}(G)} \frac{\mu}{2} \langle Vx, x \rangle + \frac{1}{2} \|x - u\|^2 - h(x), \tag{OP1}$$

where $h : H \to \mathbf{R}$ is the potential function of γf.

Let B be a single-valued mapping of C into H and R be a set-valued mapping with $D(R) = C$. Consider the following variational inclusion: find a point $x \in C$ such that

$$0 \in Bx + Rx. \tag{1.5}$$

We denote by $\mathrm{I}(B, R)$ the solution set of the variational inclusion (1.5). In particular, if $B = R = 0$, then $\mathrm{I}(B, R) = C$. If $B = 0$, then problem (1.5) becomes the inclusion problem introduced by Rockafellar [9]. It is known that problem (1.5) provides a convenient framework for the unified study of optimal solutions in many optimization related areas including mathematical programming, complementarity problems, variational inequalities, optimal control, mathematical economics, equilibria and game theory, *etc.* Let a set-valued mapping $R : D(R) \subset H \to 2^H$ be maximal monotone. We define the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$ associated with R and λ as follows:

$$J_{R,\lambda} = (I + \lambda R)^{-1}, \quad \forall x \in H,$$

where λ is a positive number.

In 1998, Huang [10] studied problem (1.5) in the case where R is maximal monotone and B is strongly monotone and Lipschitz-continuous with $D(R) = C = H$. Subsequently, Zeng *et al.* [11] further studied problem (1.5) in the case which is more general than Huang's [10]. Moreover, the authors [11] obtained the same strong convergence conclusion as in Huang's result [10]. In addition, the authors also gave the geometric convergence rate estimate for approximate solutions. Also, various types of iterative algorithms for solving variational inclusions have been further studied and developed; for more details, refer to [5, 12–17] and the references therein.

In 2011, for the case where $C = H$, Yao *et al.* [5] introduced and analyzed an iterative algorithms for finding a common element of the set of solutions of the GMEP (1.1), the set of solutions of the variational inclusion (1.5) for maximal monotone and inverse-strongly monotone mappings and the set of fixed points of a countable family of nonexpansive mappings on H.

Recently, Kim and Xu [18] introduced the concept of asymptotically κ-strict pseudocontractive mappings in a Hilbert space.

Definition 1.1 Let C be a nonempty subset of a Hilbert space H. A mapping $S : C \to C$ is said to be an asymptotically κ-strict pseudocontractive mapping with sequence $\{\gamma_n\}$ if there exist a constant $\kappa \in [0,1)$ and a sequence $\{\gamma_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\left\|S^n x - S^n y\right\|^2 \leq (1+\gamma_n)\|x-y\|^2 + \kappa \left\|x - S^n x - \left(y - S^n y\right)\right\|^2, \quad \forall n \geq 1, \forall x, y \in C.$$

Subsequently, Sahu *et al.* [19] considered the concept of asymptotically κ-strict pseudocontractive mappings in the intermediate sense, which are not necessarily Lipschitzian.

Definition 1.2 Let C be a nonempty subset of a Hilbert space H. A mapping $S : C \to C$ is said to be an asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ if there exist a constant $\kappa \in [0,1)$ and a sequence $\{\gamma_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\left\|S^n x - S^n y\right\|^2 - (1+\gamma_n)\|x-y\|^2 - \kappa \left\|x - S^n x - \left(y - S^n y\right)\right\|^2\right) \leq 0. \tag{1.6}$$

Put $c_n := \max\{0, \sup_{x,y\in C}(\|S^n x - S^n y\|^2 - (1+\gamma_n)\|x-y\|^2 - \kappa\|x - S^n x - (y - S^n y)\|^2)\}$. Then $c_n \geq 0$ ($\forall n \geq 1$), $c_n \to 0$ ($n \to \infty$), and (1.6) reduce to the relation

$$\left\|S^n x - S^n y\right\|^2 \leq (1+\gamma_n)\|x-y\|^2 + \kappa \left\|x - S^n x - \left(y - S^n y\right)\right\|^2 + c_n, \quad \forall n \geq 1, \forall x, y \in C. \tag{1.7}$$

Whenever $c_n = 0$ for all $n \geq 1$ in (1.7), then S is an asymptotically κ-strict pseudocontractive mapping with sequence $\{\gamma_n\}$. The authors [19] derived the weak and strong convergence of the modified Mann iteration processes for an asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. More precisely, they first established one weak convergence theorem for the following iterative scheme:

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrarily}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n x_n, \quad \forall n \geq 1, \end{cases}$$

where $0 < \delta \leq \alpha_n \leq 1 - \kappa - \delta$, $\sum_{n=1}^{\infty} \alpha_n c_n < \infty$, and $\sum_{n=1}^{\infty} \gamma_n < \infty$; and then obtained another strong convergence theorem for the following iterative scheme:

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ y_n = (1-\alpha_n)x_n + \alpha_n S^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \geq 1, \end{cases}$$

where $0 < \delta \leq \alpha_n \leq 1 - \kappa$, $\theta_n = c_n + \gamma_n \Delta_n$, and $\Delta_n = \sup\{\|x_n - z\|^2 : z \in \mathrm{Fix}(S)\} < \infty$.

Inspired by the above facts, we in this paper introduce two iterative algorithms for finding common solutions of a finite family of variational inclusions for maximal monotone and inverse-strongly monotone mappings with the constraints of two problems: a generalized mixed equilibrium problem and a common fixed point problem of an infinite family of nonexpansive mappings and an asymptotically strict pseudocontractive mapping in the intermediate sense in a real Hilbert space. We prove some strong and weak convergence theorems for the proposed iterative algorithms under suitable conditions. The results presented in this paper are the supplement, extension, improvement, and generalization of the previously known results in this area.

2 Preliminaries

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, *i.e.*,

$$\omega_w(x_n) := \{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

Definition 2.1 A mapping $A : C \to H$ is called

(i) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C;$$

(iii) ζ-inverse-strongly monotone if there exists a constant $\zeta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \zeta\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is easy to see that the projection P_C is 1-inverse-strongly monotone (in short, 1-ism). Inverse-strongly monotone (also referred to as co-coercive) operators have been applied widely in solving practical problems in various fields.

Definition 2.2 A differentiable function $K : H \to \mathbf{R}$ is called:

(i) convex, if

$$K(y) - K(x) \geq \langle K'(x), y - x\rangle, \quad \forall x, y \in H,$$

where $K'(x)$ is the Frechet derivative of K at x;

(ii) strongly convex, if there exists a constant $\sigma > 0$ such that

$$K(y) - K(x) - \langle K'(x), y - x\rangle \geq \frac{\sigma}{2}\|x - y\|^2, \quad \forall x, y \in H.$$

It is easy to see that if $K : H \to \mathbf{R}$ is a differentiable strongly convex function with constant $\sigma > 0$ then $K' : H \to H$ is strongly monotone with constant $\sigma > 0$.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y\in C} \|x - y\| =: d(x, C).$$

Some important properties of projections are gathered in the following proposition.

Proposition 2.1 *For given $x \in H$ and $z \in C$:*

(i) $z = P_C x \Leftrightarrow \langle x - z, y - z\rangle \le 0$, $\forall y \in C$;

(ii) $z = P_C x \Leftrightarrow \|x - z\|^2 \le \|x - y\|^2 - \|y - z\|^2$, $\forall y \in C$;

(iii) $\langle P_C x - P_C y, x - y\rangle \ge \|P_C x - P_C y\|^2$, $\forall y \in H$. (*This implies that P_C is nonexpansive and monotone.*)

By using the technique of [20], we can readily obtain the following elementary result.

Proposition 2.2 (see [6, Lemma 1 and Proposition 1]) *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying the conditions* (H1)-(H4). *Assume that*

(i) *$K : H \to \mathbf{R}$ is strongly convex with constant $\sigma > 0$ and the function $x \mapsto \langle y - x, K'(x)\rangle$ is weakly upper semicontinuous for each $y \in H$;*

(ii) *for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,*

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\big\langle K'(z) - K'(x), y_x - z\big\rangle < 0.$$

Then the following hold:

(a) *for each $x \in H$, $S_r^{(\Theta,\varphi)}(x) \ne \emptyset$;*

(b) *$S_r^{(\Theta,\varphi)}$ is single-valued;*

(c) *$S_r^{(\Theta,\varphi)}$ is nonexpansive if K' is Lipschitz-continuous with constant $\nu > 0$ and*

$$\big\langle K'(x_1) - K'(x_2), u_1 - u_2\big\rangle \ge \big\langle K'(u_1) - K'(u_2), u_1 - u_2\big\rangle, \quad \forall (x_1, x_2) \in H \times H,$$

where $u_i = S_r^{(\Theta,\varphi)}(x_i)$ for $i = 1, 2$;

(d) *for all $s, t > 0$ and $x \in H$*

$$\begin{aligned}&\big\langle K'\big(S_s^{(\Theta,\varphi)}x\big) - K'\big(S_t^{(\Theta,\varphi)}x\big), S_s^{(\Theta,\varphi)}x - S_t^{(\Theta,\varphi)}x\big\rangle \\ &\quad\le \frac{s-t}{s}\big\langle K'\big(S_s^{(\Theta,\varphi)}x\big) - K'(x), S_s^{(\Theta,\varphi)}x - S_t^{(\Theta,\varphi)}x\big\rangle;\end{aligned}$$

(e) $\mathrm{Fix}(S_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta, \varphi)$;

(f) $\mathrm{MEP}(\Theta, \varphi)$ *is closed and convex.*

In particular, whenever $\Theta : C \times C \to \mathbf{R}$ is a bifunction satisfying the conditions (H1)-(H4) *and $K(x) = \frac{1}{2}\|x\|^2$, $\forall x \in H$, then, for any $x, y \in H$,*

$$\big\|S_r^{(\Theta,\varphi)}x - S_r^{(\Theta,\varphi)}y\big\|^2 \le \big\langle S_r^{(\Theta,\varphi)}x - S_r^{(\Theta,\varphi)}y, x - y\big\rangle$$

($S_r^{(\Theta,\varphi)}$ *is firmly nonexpansive*) *and*

$$\left\|S_s^{(\Theta,\varphi)}x - S_t^{(\Theta,\varphi)}x\right\| \le \frac{|s-t|}{s}\left\|S_s^{(\Theta,\varphi)}x - x\right\|, \quad \forall s,t>0, x\in H.$$

In this case, $S_r^{(\Theta,\varphi)}$ *is rewritten as* $T_r^{(\Theta,\varphi)}$. *If, in addition*, $\varphi\equiv 0$, *then* $T_r^{(\Theta,\varphi)}$ *is rewritten as* T_r^{Θ} (*see* [21, Lemma 2.1] *for more details*).

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.1 *Let X be a real inner product space. Then we have the following inequality*:

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x,y\in X.$$

Lemma 2.2 *Let H be a real Hilbert space. Then the following hold*:

(a) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$ *for all* $x,y\in H$;

(b) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x-y\|^2$ *for all* $x,y\in H$ *and* $\lambda,\mu\in[0,1]$ *with* $\lambda+\mu=1$;

(c) *If* $\{x_n\}$ *is a sequence in H such that* $x_n \rightharpoonup x$, *it follows that*

$$\limsup_{n\to\infty}\|x_n - y\|^2 = \limsup_{n\to\infty}\|x_n - x\|^2 + \|x-y\|^2, \quad \forall y\in H.$$

Lemma 2.3 ([19, Lemma 2.5]) *Let H be a real Hilbert space. Given a nonempty closed convex subset of H and points $x,y,z\in H$ and given also a real number $a\in\mathbf{R}$, the set*

$$\left\{v\in C: \|y-v\|^2 \le \|x-v\|^2 + \langle z,v\rangle + a\right\}$$

is convex (*and closed*).

Lemma 2.4 ([19, Lemma 2.6]) *Let C be a nonempty subset of a Hilbert space H and $S: C\to C$ be an asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Then*

$$\left\|S^n x - S^n y\right\| \le \frac{1}{1-\kappa}\left(\kappa\|x-y\| + \sqrt{\bigl(1+(1-\kappa)\gamma_n\bigr)\|x-y\|^2 + (1-\kappa)c_n}\right)$$

for all $x,y\in C$ and $n\ge 1$.

Lemma 2.5 ([19, Lemma 2.7]) *Let C be a nonempty subset of a Hilbert space H and $S: C\to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Let $\{x_n\}$ be a sequence in C such that $\|x_n - x_{n+1}\|\to 0$ and $\|x_n - S^n x_n\|\to 0$ as $n\to\infty$. Then $\|x_n - Sx_n\|\to 0$ as $n\to\infty$.*

Lemma 2.6 (Demiclosedness principle [19, Proposition 3.1]) *Let C be a nonempty closed convex subset of a Hilbert space H and $S: C\to C$ be a continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Then $I-S$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in C such that $x_n\rightharpoonup x\in C$ and $\limsup_{m\to\infty}\limsup_{n\to\infty}\|x_n - S^m x_n\| = 0$, then $(I-S)x = 0$.*

Lemma 2.7 ([19, Proposition 3.2]) *Let C be a nonempty closed convex subset of a Hilbert space H and $S : C \to C$ be a continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that* $\mathrm{Fix}(S) \neq \emptyset$. *Then* $\mathrm{Fix}(S)$ *is closed and convex.*

Remark 2.1 Lemmas 2.6 and 2.7 give some basic properties of an asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Moreover, Lemma 2.6 extends the demiclosedness principles studied for certain classes of nonlinear mappings; see [19] for more details.

Lemma 2.8 ([22, p.80]) *Let $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$, and $\{\delta_n\}_{n=1}^\infty$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^\infty \delta_n < \infty$ and $\sum_{n=1}^\infty b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If, in addition, $\{a_n\}_{n=1}^\infty$ has a subsequence which converges to zero, then $\lim_{n\to\infty} a_n = 0$.

Recall that a Banach space X is said to satisfy the Opial condition [23] if, for any given sequence $\{x_n\} \subset X$ which converges weakly to an element $x \in X$, we have the inequality

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in X, y \neq x.$$

It is well known in [23] that every Hilbert space H satisfies the Opial condition.

Lemma 2.9 (see [24, Proposition 3.1]) *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\{x_n\}$ be a sequence in H. Suppose that*

$$\|x_{n+1} - p\|^2 \leq (1 + \lambda_n)\|x_n - p\|^2 + \delta_n, \quad \forall p \in C, n \geq 1,$$

where $\{\lambda_n\}$ and $\{\delta_n\}$ are sequences of nonnegative real numbers such that $\sum_{n=1}^\infty \lambda_n < \infty$ and $\sum_{n=1}^\infty \delta_n < \infty$. Then $\{P_C x_n\}$ converges strongly in C.

Lemma 2.10 (see [25]) *Let C be a closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\|, \quad \textit{for all } n,$$

then $x_n \to q$ as $n \to \infty$.

Lemma 2.11 (see [26, Lemma 3.2]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^\infty$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^\infty \mathrm{Fix}(T_n) \neq \emptyset$ and let $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Then, for every $x \in C$ and $k \geq 1$ the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Remark 2.2 (see [27, Remark 3.1]) It can be known from Lemma 2.11 that if D is a nonempty bounded subset of C, then for $\epsilon > 0$ there exists $n_0 \geq k$ such that for all $n > n_0$

$$\sup_{x\in D} \|U_{n,k}x - U_k x\| \leq \epsilon.$$

Remark 2.3 (see [27, Remark 3.2]) Utilizing Lemma 2.11, we define a mapping $W : C \to C$ as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in C.$$

Such a W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. Since W_n is nonexpansive, $W : C \to C$ is also nonexpansive. Indeed, observe that for each $x, y \in C$

$$\|Wx - Wy\| = \lim_{n\to\infty} \|W_n x - W_n y\| \leq \|x - y\|.$$

If $\{x_n\}$ is a bounded sequence in C, then we put $D = \{x_n : n \geq 1\}$. Hence, it is clear from Remark 2.2 that for an arbitrary $\epsilon > 0$ there exists $N_0 \geq 1$ such that for all $n > N_0$

$$\|W_n x_n - Wx_n\| = \|U_{n,1}x_n - U_1 x_n\| \leq \sup_{x\in D} \|U_{n,1}x - U_1 x\| \leq \epsilon.$$

This implies that

$$\lim_{n\to\infty} \|W_n x_n - Wx_n\| = 0.$$

Lemma 2.12 (see [26, Lemma 3.3]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} \operatorname{Fix}(T_n) \neq \emptyset$, and let $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Then* $\operatorname{Fix}(W) = \bigcap_{n=1}^{\infty} \operatorname{Fix}(T_n)$.

Lemma 2.13 (see [28, Theorem 10.4 (Demiclosedness Principle)]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be nonexpansive. Then $I - T$ is demiclosed on C. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y, it follows that $(I - T)x = y$. Here I is the identity operator of H.*

Recall that a set-valued mapping $R : D(R) \subset H \to 2^H$ is called monotone if, for all $x, y \in D(R)$, $f \in R(x)$, and $g \in R(y)$ imply

$$\langle f - g, x - y\rangle \geq 0.$$

A set-valued mapping R is called maximal monotone if R is monotone and $(I + \lambda R)D(R) = H$ for each $\lambda > 0$, where I is the identity mapping of H. We denote by $G(R)$ the graph of R. It is known that a monotone mapping R is maximal if and only if, for $(x, f) \in H \times H$, $\langle f - g, x - y\rangle \geq 0$ for every $(y, g) \in G(R)$, we have $f \in R(x)$. We illustrate the concept of maximal monotone mapping with the following example.

Let $A : C \to H$ be a monotone, k-Lipschitz-continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$, *i.e.*,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}.$$

Define

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $\langle Av, y - v\rangle \geq 0$ for all $y \in C$ (see [9]).

Assume that $R : D(R) \subset H \to 2^H$ is a maximal monotone mapping. Let $\lambda > 0$. In terms of Huang [10] (see also [11]), we have the following property for the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$.

Lemma 2.14 *$J_{R,\lambda}$ is single-valued and firmly nonexpansive, i.e.,*

$$\langle J_{R,\lambda}x - J_{R,\lambda}y, x - y\rangle \geq \|J_{R,\lambda}x - J_{R,\lambda}y\|^2, \quad \forall x, y \in H.$$

Consequently, $J_{R,\lambda}$ is nonexpansive and monotone.

Lemma 2.15 (see [14]) *Let R be a maximal monotone mapping with $D(R) = C$. Then for any given $\lambda > 0$, $u \in C$ is a solution of problem* (1.6) *if and only if $u \in C$ satisfies*

$$u = J_{R,\lambda}(u - \lambda Bu).$$

Lemma 2.16 (see [11]) *Let R be a maximal monotone mapping with $D(R) = C$ and let $B : C \to H$ be a strongly monotone, continuous, and single-valued mapping. Then for each $z \in H$, the equation $z \in (B + \lambda R)x$ has a unique solution x_λ for $\lambda > 0$.*

Lemma 2.17 (see [14]) *Let R be a maximal monotone mapping with $D(R) = C$ and $B : C \to H$ be a monotone, continuous and single-valued mapping. Then $(I + \lambda(R + B))C = H$ for each $\lambda > 0$. In this case, $R + B$ is maximal monotone.*

Lemma 2.18 (see [29]) *Let C be a nonempty closed convex subset of a real Hilbert space H, and $g : C \to \mathbf{R} \cup +\infty$ be a proper lower semicontinuous differentiable convex function. If x^* is a solution the minimization problem*

$$g(x^*) = \inf_{x \in C} g(x),$$

then

$$\langle g'(x), x - x^*\rangle \geq 0, \quad \forall x \in C.$$

In particular, if x^* solves (OP1), *then*

$$\langle u + (\gamma f - (I + \mu V))x^*, x - x^*\rangle \leq 0.$$

3 Strong convergence theorems

In this section, we introduce and analyze an iterative algorithm for finding common solutions of a finite family of variational inclusions for maximal monotone and inverse-strongly monotone mappings with the constraints of two problems: a generalized mixed equilibrium problem and a common fixed point problem of an infinite family of nonexpansive mappings and an asymptotically strict pseudocontractive mapping in the intermediate sense in a real Hilbert space. Under appropriate conditions imposed on the parameter sequences we will prove strong convergence of the proposed algorithm.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let N be an integer. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying* (H1)-(H4) *and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $A : H \to H$ and $B_i : C \to H$ be ζ-inverse-strongly monotone and η_i-inverse-strongly monotone, respectively, where $i \in \{1, 2, \ldots, N\}$. Let $S : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense for some $0 \leq \kappa < 1$ with sequence $\{\gamma_n\} \subset [0, \infty)$ such that $\lim_{n\to\infty} \gamma_n = 0$ and $\{c_n\} \subset [0, \infty)$ such that $\lim_{n\to\infty} c_n = 0$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let V be a $\bar{\gamma}$-strongly positive bounded linear operator and $f : H \to H$ be an l-Lipschitzian mapping with $\gamma l < (1 + \mu)\bar{\gamma}$. Assume that $\Omega := (\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)) \cap \mathrm{GMEP}(\Theta, \varphi, A) \cap (\bigcap_{i=1}^{N} \mathrm{I}(B_i, R_i)) \cap \mathrm{Fix}(S)$ is nonempty and bounded. Let W_n be the W-mapping defined by* (1.4) *and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ be three sequences in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\kappa \leq \delta_n \leq d < 1$. Assume that*:

(i) *$K : H \to \mathbf{R}$ is strongly convex with constant $\sigma > 0$ and its derivative K' is Lipschitz-continuous with constant $\nu > 0$ such that the function $x \mapsto \langle y - x, K'(x)\rangle$ is weakly upper semicontinuous for each $y \in H$;*

(ii) *for each $x \in H$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \notin D_x$,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\big\langle K'(y) - K'(x), z_x - y\big\rangle < 0;$$

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iv) $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $\forall i \in \{1, 2, \ldots, N\}$, *and* $\{r_n\} \subset [0, 2\zeta]$ *satisfies*

$$0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\zeta.$$

Pick any $x_0 \in H$ and set $C_1 = C$, $x_1 = P_{C_1} x_0$. Let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{cases} u_n = S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n, \\ z_n = J_{R_N,\lambda_{N,n}}(I - \lambda_{N,n}B_N)J_{R_{N-1},\lambda_{N-1,n}}(I - \lambda_{N-1,n}B_{N-1}) \cdots J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1)u_n, \\ k_n = \delta_n z_n + (1 - \delta_n)S^n z_n, \\ y_n = \alpha_n(u + \gamma f(x_n)) + \beta_n k_n + ((1 - \beta_n)I - \alpha_n(I + \mu V))W_n z_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases} \tag{3.1}$$

where $\theta_n = (\alpha_n + \gamma_n)\Delta_n \varrho + c_n \varrho$, $\Delta_n = \sup\{\|x_n - p\|^2 + \|u + (\gamma f - I - \mu V)p\|^2 : p \in \Omega\} < \infty$, and $\varrho = \frac{1}{1 - \sup_{n\geq 1} \alpha_n} < \infty$. If $S_r^{(\Theta,\varphi)}$ is firmly nonexpansive, then the following statements hold:

(I) $\{x_n\}$ *converges strongly to* $P_\Omega x_0$;

(II) $\{x_n\}$ *converges strongly to* $P_\Omega x_0$, *which solves the optimization problem*

$$\min_{x\in\Omega} \frac{\mu}{2}\langle Vx, x\rangle + \frac{1}{2}\|x-u\|^2 - h(x), \tag{OP2}$$

provided $\gamma_n + c_n + \|x_n - y_n\| = o(\alpha_n)$ *additionally, where* $h : H \to \mathbf{R}$ *is the potential function of* γf.

Proof Since $\lim_{n\to\infty}\alpha_n = 0$ and $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$, we may assume, without loss of generality, that $\alpha_n \le (1-\beta_n)(1+\mu\|V\|)^{-1}$. Since V is a $\bar{\gamma}$-strongly positive bounded linear operator on H, we know that

$$\|V\| = \sup\bigl\{\langle Vu, u\rangle : u \in H, \|u\| = 1\bigr\}.$$

Observe that

$$\begin{aligned}\bigl\langle\bigl((1-\beta_n)I - \alpha_n(I+\mu V)\bigr)u, u\bigr\rangle &= 1 - \beta_n - \alpha_n - \alpha_n\mu\langle Vu, u\rangle \\ &\ge 1 - \beta_n - \alpha_n - \alpha_n\mu\|V\| \\ &\ge 0,\end{aligned}$$

that is, $(1-\beta_n)I - \alpha_n(I+\mu V)$ is positive. It follows that

$$\begin{aligned}\bigl\|(1-\beta_n)I - \alpha_n(I+\mu V)\bigr\| &= \sup\bigl\{\bigl\langle\bigl((1-\beta_n)I - \alpha_n(I+\mu V)\bigr)u, u\bigr\rangle : u \in H, \|u\| = 1\bigr\} \\ &= \sup\bigl\{1 - \beta_n - \alpha_n - \alpha_n\mu\langle Vu, u\rangle : u \in H, \|u\| = 1\bigr\} \\ &\le 1 - \beta_n - \alpha_n - \alpha_n\mu\bar{\gamma}.\end{aligned}$$

Put

$$\Lambda_n^i = J_{R_i,\lambda_{i,n}}(I - \lambda_{i,n}B_i)J_{R_{i-1},\lambda_{i-1,n}}(I - \lambda_{i-1,n}B_{i-1})\cdots J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1)$$

for all $i \in \{1, 2, \ldots, N\}$ and $n \ge 1$, and $\Lambda_n^0 = I$, where I is the identity mapping on H. Then we have that $z_n = \Lambda_n^N u_n$. We divide the rest of the proof into several steps.

Step 1. We show that $\{x_n\}$ is well defined. It is obvious that C_n is closed and convex. As the defining inequality in C_n is equivalent to the inequality

$$\bigl\langle 2(x_n - z_n), z\bigr\rangle \le \|x_n\|^2 - \|z_n\|^2 + \theta_n,$$

by Lemma 2.3 we know that C_n is convex and closed for every $n \ge 1$.

First of all, we show that $\Omega \subset C_n$ for all $n \ge 1$. Suppose that $\Omega \subset C_n$ for some $n \ge 1$. Take $p \in \Omega$ arbitrarily. Since $p = S_{r_n}^{(\Theta,\varphi)}(p - r_nAp)$, A is ζ-inverse strongly monotone and $0 \le r_n \le 2\zeta$, we have

$$\begin{aligned}\|u_n - p\|^2 &= \bigl\|S_{r_n}^{(\Theta,\varphi)}(I - r_nA)x_n - S_{r_n}^{(\Theta,\varphi)}(I - r_nA)p\bigr\|^2 \\ &\le \bigl\|(I - r_nA)x_n - (I - r_nA)p\bigr\|^2\end{aligned}$$

$$\begin{aligned}
&= \|(x_n - p) - r_n(Ax_n - Ap)\|^2 \\
&= \|x_n - p\|^2 - 2r_n\langle x_n - p, Ax_n - Ap\rangle + r_n^2\|Ax_n - Ap\|^2 \\
&\leq \|x_n - p\|^2 - 2r_n\zeta\|Ax_n - Ap\|^2 + r_n^2\|Ax_n - Ap\|^2 \\
&= \|x_n - p\|^2 + r_n(r_n - 2\zeta)\|Ax_n - Ap\|^2 \\
&\leq \|x_n - p\|^2.
\end{aligned} \tag{3.2}$$

Since $p = J_{R_i,\lambda_{i,n}}(I - \lambda_{i,n}B_i)p$, $\Lambda_n^i p = p$, and B_i is η_i-inverse-strongly monotone, where $\eta_i \in (0, 2\eta_i)$, $i \in \{1, 2, \ldots, N\}$, by Lemma 2.14 we deduce that

$$\begin{aligned}
\|z_n - p\|^2 &= \|J_{R_N,\lambda_{N,n}}(I - \lambda_{N,n}B_N)\Lambda_n^{N-1}u_n - J_{R_N,\lambda_{N,n}}(I - \lambda_{N,n}B_N)\Lambda_n^{N-1}p\|^2 \\
&\leq \|(I - \lambda_{N,n}B_N)\Lambda_n^{N-1}u_n - (I - \lambda_{N,n}B_N)\Lambda_n^{N-1}p\|^2 \\
&= \|(\Lambda_n^{N-1}u_n - \Lambda_n^{N-1}p) - \lambda_{N,n}(B_N\Lambda_n^{N-1}u_n - B_N\Lambda_n^{N-1}p)\|^2 \\
&\leq \|\Lambda_n^{N-1}u_n - \Lambda_n^{N-1}p\|^2 + \lambda_{N,n}(\lambda_{N,n} - 2\eta_N)\|B_N\Lambda_n^{N-1}u_n - B_N\Lambda_n^{N-1}p\|^2 \\
&\leq \|\Lambda_n^{N-1}u_n - \Lambda_n^{N-1}p\|^2 \\
&\vdots \\
&\leq \|\Lambda_n^0 u_n - \Lambda_n^0 p\|^2 \\
&= \|u_n - p\|^2.
\end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), we have

$$\|z_n - p\| \leq \|x_n - p\|. \tag{3.4}$$

By Lemma 2.2(b), we deduce from (3.1) and (3.4) that

$$\begin{aligned}
\|k_n - p\|^2 &= \|\delta_n(z_n - p) + (1 - \delta_n)(S^n z_n - p)\|^2 \\
&= \delta_n\|z_n - p\|^2 + (1 - \delta_n)\|S^n z_n - p\|^2 - \delta_n(1 - \delta_n)\|z_n - S^n z_n\|^2 \\
&\leq \delta_n\|z_n - p\|^2 + (1 - \delta_n)\big[(1 + \gamma_n)\|z_n - p\|^2 \\
&\quad + \kappa\|z_n - S^n z_n\|^2 + c_n\big] - \delta_n(1 - \delta_n)\|z_n - S^n z_n\|^2 \\
&= \big[1 + \gamma_n(1 - \delta_n)\big]\|z_n - p\|^2 + (1 - \delta_n)(\kappa - \delta_n)\|z_n - S^n z_n\|^2 + (1 - \delta_n)c_n \\
&\leq (1 + \gamma_n)\|z_n - p\|^2 + (1 - \delta_n)(\kappa - \delta_n)\|z_n - S^n z_n\|^2 + c_n \\
&\leq (1 + \gamma_n)\|z_n - p\|^2 + c_n.
\end{aligned} \tag{3.5}$$

Set $\bar{V} = I + \mu V$. Then, for $\gamma l \leq (1 + \mu)\bar{\gamma}$, by Lemma 2.1 we obtain from (3.1), (3.4), and (3.5)

$$\begin{aligned}
&\|y_n - p\|^2 \\
&\quad = \|\alpha_n(u + \gamma f(x_n)) + \beta_n k_n + ((1 - \beta_n)I - \alpha_n\bar{V})W_n z_n - p\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \|\alpha_n(u+\gamma f(x_n)-\bar{V}p)+\beta_n(k_n-p)+((1-\beta_n)I-\alpha_n\bar{V})(W_nz_n-p)\|^2 \\
&= \|\alpha_n(u+\gamma f(p)-\bar{V}p)+\alpha_n\gamma(f(x_n)-f(p)) \\
&\quad +\beta_n(k_n-p)+((1-\beta_n)I-\alpha_n\bar{V})(W_nz_n-p)\|^2 \\
&\le \|\alpha_n\gamma(f(x_n)-f(p))+\beta_n(k_n-p)+((1-\beta_n)I-\alpha_n\bar{V})(W_nz_n-p)\|^2 \\
&\quad +2\alpha_n\langle(u+\gamma f(p)-\bar{V}p),y_n-p\rangle \\
&\le [\alpha_n\gamma\|f(x_n)-f(p)\|+\beta_n\|k_n-p\|+\|((1-\beta_n)I-\alpha_n\bar{V})(W_nz_n-p)\|]^2 \\
&\quad +2\alpha_n\langle(u+\gamma f(p)-\bar{V}p),y_n-p\rangle \\
&\le [\alpha_n\gamma l\|x_n-p\|+\beta_n\|k_n-p\|+(1-\beta_n-\alpha_n-\alpha_n\mu\bar{\gamma})\|W_nz_n-p\|]^2 \\
&\quad +2\alpha_n\|u+\gamma f(p)-\bar{V}p\|\|y_n-p\| \\
&\le [\alpha_n(1+\mu)\bar{\gamma}\|x_n-p\|+\beta_n\|k_n-p\|+(1-\beta_n-\alpha_n(1+\mu)\bar{\gamma})\|z_n-p\|]^2 \\
&\quad +\alpha_n(\|u+\gamma f(p)-\bar{V}p\|^2+\|y_n-p\|^2) \\
&\le \alpha_n(1+\mu)\bar{\gamma}\|x_n-p\|^2+\beta_n\|k_n-p\|^2+(1-\beta_n-\alpha_n(1+\mu)\bar{\gamma})\|z_n-p\|^2 \\
&\quad +\alpha_n(\|u+\gamma f(p)-\bar{V}p\|^2+\|y_n-p\|^2) \\
&\le \alpha_n(1+\mu)\bar{\gamma}\|x_n-p\|^2+\beta_n((1+\gamma_n)\|z_n-p\|^2+c_n) \\
&\quad +(1-\beta_n-\alpha_n(1+\mu)\bar{\gamma})\|z_n-p\|^2+\alpha_n(\|u+\gamma f(p)-\bar{V}p\|^2+\|y_n-p\|^2) \\
&\le \alpha_n(1+\mu)\bar{\gamma}\|x_n-p\|^2+\beta_n((1+\gamma_n)\|z_n-p\|^2+c_n) \\
&\quad +(1-\beta_n-\alpha_n(1+\mu)\bar{\gamma})((1+\gamma_n)\|z_n-p\|^2+c_n) \\
&\quad +\alpha_n(\|u+\gamma f(p)-\bar{V}p\|^2+\|y_n-p\|^2) \\
&= \alpha_n(1+\mu)\bar{\gamma}\|x_n-p\|^2+(1-\alpha_n(1+\mu)\bar{\gamma})((1+\gamma_n)\|z_n-p\|^2+c_n) \\
&\quad +\alpha_n(\|u+\gamma f(p)-\bar{V}p\|^2+\|y_n-p\|^2) \\
&\le \alpha_n(1+\mu)\bar{\gamma}((1+\gamma_n)\|x_n-p\|^2+c_n) \\
&\quad +(1-\alpha_n(1+\mu)\bar{\gamma})((1+\gamma_n)\|x_n-p\|^2+c_n) \\
&\quad +\alpha_n(\|u+\gamma f(p)-\bar{V}p\|^2+\|y_n-p\|^2) \\
&= (1+\gamma_n)\|x_n-p\|^2+c_n+\alpha_n(\|u+\gamma f(p)-\bar{V}p\|^2+\|y_n-p\|^2),
\end{aligned}
$$

which hence yields

$$
\begin{aligned}
\|y_n-p\|^2 &\le \frac{1+\gamma_n}{1-\alpha_n}\|x_n-p\|^2+\frac{\alpha_n}{1-\alpha_n}\|u+\gamma f(p)-\bar{V}p\|^2+\frac{1}{1-\alpha_n}c_n \\
&= \left(1+\frac{\alpha_n+\gamma_n}{1-\alpha_n}\right)\|x_n-p\|^2+\frac{\alpha_n}{1-\alpha_n}\|u+\gamma f(p)-\bar{V}p\|^2+\frac{1}{1-\alpha_n}c_n \\
&\le \left(1+\frac{\alpha_n+\gamma_n}{1-\alpha_n}\right)\|x_n-p\|^2+\frac{\alpha_n+\gamma_n}{1-\alpha_n}\|u+\gamma f(p)-\bar{V}p\|^2+\frac{1}{1-\alpha_n}c_n \\
&= \|x_n-p\|^2+\frac{\alpha_n+\gamma_n}{1-\alpha_n}(\|x_n-p\|^2+\|u+\gamma f(p)-\bar{V}p\|^2)+\frac{1}{1-\alpha_n}c_n
\end{aligned}
$$

$$\begin{aligned}
&\le \|x_n - p\|^2 + (\alpha_n + \gamma_n)\varrho\bigl(\|x_n - p\|^2 + \bigl\|u + \gamma f(p) - \bar{V}p\bigr\|^2\bigr) + \varrho c_n \\
&\le \|x_n - p\|^2 + (\alpha_n + \gamma_n)\Delta_n\varrho + c_n\varrho \\
&= \|x_n - p\|^2 + \theta_n,
\end{aligned} \tag{3.6}$$

where $\theta_n = (\alpha_n + \gamma_n)\Delta_n\varrho + c_n\varrho$, $\Delta_n = \sup\{\|x_n - p\|^2 + \|u + \gamma f(p) - \bar{V}p\|^2 : p \in \Omega\} < \infty$, and $\varrho = \frac{1}{1-\sup_{n\ge 1}\alpha_n} < \infty$ (due to $\{\alpha_n\} \subset (0,1)$ and $\lim_{n\to\infty}\alpha_n = 0$). Hence $p \in C_{n+1}$. This implies that $\Omega \subset C_n$ for all $n \ge 1$. Therefore, $\{x_n\}$ is well defined.

Step 2. We prove that $\|x_n - k_n\| \to 0$ as $n \to \infty$.

Indeed, let $v = P_\Omega x_0$. From $x_n = P_{C_n}x_0$ and $v \in \Omega \subset C_n$, we obtain

$$\|x_n - x_0\| \le \|v - x_0\|. \tag{3.7}$$

This implies that $\{x_n\}$ is bounded and hence $\{u_n\}$, $\{z_n\}$, $\{k_n\}$, and $\{y_n\}$ are also bounded. Since $x_{n+1} \in C_{n+1} \subset C_n$ and $x_n = P_{C_n}x_0$, we have

$$\|x_n - x_0\| \le \|x_{n+1} - x_0\|, \quad \forall n \ge 1.$$

Therefore $\lim_{n\to\infty}\|x_n - x_0\|$ exists. From $x_n = P_{C_n}x_0$, $x_{n+1} \in C_{n+1} \subset C_n$, by Proposition 2.1(ii) we obtain

$$\|x_{n+1} - x_n\|^2 \le \|x_0 - x_{n+1}\|^2 - \|x_0 - x_n\|^2,$$

which implies

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.8}$$

It follows from $x_{n+1} \in C_{n+1}$ that $\|y_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \theta_n$ and hence

$$\begin{aligned}
\|x_n - y_n\|^2 &\le 2\bigl(\|x_n - x_{n+1}\|^2 + \|x_{n+1} - y_n\|^2\bigr) \\
&\le 2\bigl(\|x_n - x_{n+1}\|^2 + \|x_n - x_{n+1}\|^2 + \theta_n\bigr) \\
&= 2\bigl(2\|x_n - x_{n+1}\|^2 + \theta_n\bigr).
\end{aligned}$$

From (3.8) and $\lim_{n\to\infty}\theta_n = 0$, we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.9}$$

Also, utilizing Lemmas 2.1 and 2.2(b) we obtain from (3.1), (3.4), and (3.5)

$$\begin{aligned}
&\|y_n - p\|^2 \\
&\quad = \bigl\|\alpha_n\bigl(u + \gamma f(x_n) - \bar{V}W_n z_n\bigr) + \beta_n(k_n - p) + (1-\beta_n)(W_n z_n - p)\bigr\|^2 \\
&\quad \le \bigl\|\beta_n(k_n - p) + (1-\beta_n)(W_n z_n - p)\bigr\|^2 + 2\alpha_n\bigl\langle u + \gamma f(x_n) - \bar{V}W_n z_n, y_n - p\bigr\rangle \\
&\quad = \beta_n\|k_n - p\|^2 + (1-\beta_n)\|W_n z_n - p\|^2 - \beta_n(1-\beta_n)\|k_n - W_n z_n\|^2 \\
&\qquad + 2\alpha_n\bigl\|u + \gamma f(x_n) - \bar{V}W_n z_n\bigr\|\|y_n - p\|
\end{aligned}$$

$$\begin{aligned}
&\le \beta_n\|k_n - p\|^2 + (1-\beta_n)\|z_n - p\|^2 - \beta_n(1-\beta_n)\|k_n - W_n z_n\|^2 \\
&\quad + 2\alpha_n\bigl\|u + \gamma f(x_n) - \bar{V}W_n z_n\bigr\|\|y_n - p\| \\
&\le \beta_n\bigl((1+\gamma_n)\|z_n - p\|^2 + c_n\bigr) + (1-\beta_n)\|z_n - p\|^2 - \beta_n(1-\beta_n)\|k_n - W_n z_n\|^2 \\
&\quad + 2\alpha_n\bigl\|u + \gamma f(x_n) - \bar{V}W_n z_n\bigr\|\|y_n - p\| \\
&\le \beta_n\bigl((1+\gamma_n)\|z_n - p\|^2 + c_n\bigr) + (1-\beta_n)\bigl((1+\gamma_n)\|z_n - p\|^2 + c_n\bigr) \\
&\quad - \beta_n(1-\beta_n)\|k_n - W_n z_n\|^2 + 2\alpha_n\bigl\|u + \gamma f(x_n) - \bar{V}W_n z_n\bigr\|\|y_n - p\| \\
&= (1+\gamma_n)\|z_n - p\|^2 + c_n - \beta_n(1-\beta_n)\|k_n - W_n z_n\|^2 \\
&\quad + 2\alpha_n\bigl\|u + \gamma f(x_n) - \bar{V}W_n z_n\bigr\|\|y_n - p\| \\
&\le (1+\gamma_n)\|x_n - p\|^2 + c_n - \beta_n(1-\beta_n)\|k_n - W_n z_n\|^2 \\
&\quad + 2\alpha_n\bigl\|u + \gamma f(x_n) - \bar{V}W_n z_n\bigr\|\|y_n - p\|,
\end{aligned}$$

which leads to

$$\begin{aligned}
&\beta_n(1-\beta_n)\|k_n - W_n z_n\|^2 \\
&\quad \le \|x_n - p\|^2 - \|y_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\
&\qquad + 2\alpha_n\bigl\|u + \gamma f(x_n) - \bar{V}W_n z_n\bigr\|\|y_n - p\| \\
&\quad \le \|x_n - y_n\|\bigl(\|x_n - p\| + \|y_n - p\|\bigr) + \gamma_n\|x_n - p\|^2 + c_n \\
&\qquad + 2\alpha_n\bigl\|u + \gamma f(x_n) - \bar{V}W_n z_n\bigr\|\|y_n - p\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\gamma_n = 0$, and $\lim_{n\to\infty}c_n = 0$, it follows from (3.9) and condition (iii) that

$$\lim_{n\to\infty}\|k_n - W_n z_n\| = 0. \tag{3.10}$$

Note that

$$y_n - k_n = \alpha_n\bigl(u + \gamma f(x_n) - \bar{V}W_n z_n\bigr) + (1-\beta_n)(W_n z_n - k_n),$$

which yields

$$\begin{aligned}
\|x_n - k_n\| &\le \|x_n - y_n\| + \|y_n - k_n\| \\
&\le \|x_n - y_n\| + \bigl\|\alpha_n\bigl(u + \gamma f(x_n) - \bar{V}W_n z_n\bigr) + (1-\beta_n)(W_n z_n - k_n)\bigr\| \\
&\le \|x_n - y_n\| + \alpha_n\bigl\|u + \gamma f(x_n) - \bar{V}W_n z_n\bigr\| + (1-\beta_n)\|W_n z_n - k_n\| \\
&\le \|x_n - y_n\| + \alpha_n\bigl\|u + \gamma f(x_n) - \bar{V}W_n z_n\bigr\| + \|W_n z_n - k_n\|.
\end{aligned}$$

So, from (3.9), (3.10), and $\lim_{n\to\infty}\alpha_n = 0$, we get

$$\lim_{n\to\infty}\|x_n - k_n\| = 0. \tag{3.11}$$

Step 3. We prove that $\|x_n - u_n\| \to 0$, $\|u_n - z_n\| \to 0$, $\|z_n - Wz_n\| \to 0$, and $\|z_n - S^n z_n\| \to 0$ as $n \to \infty$.

Indeed, taking into consideration that $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2\zeta$, we may assume, without loss of generality, that $\{r_n\} \subset [c,d] \subset (0,2\zeta)$. From (3.4) and (3.5) it follows that

$$\begin{aligned} \|k_n - p\|^2 &\le \bigl[1 + \gamma_n(1-\delta_n)\bigr]\|z_n - p\|^2 + (1-\delta_n)(k-\delta_n)\bigl\|z_n - S^n z_n\bigr\|^2 + (1-\delta_n)c_n \\ &\le \|z_n - p\|^2 + \gamma_n\|z_n - p\|^2 + c_n \\ &\le \|z_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n. \end{aligned} \tag{3.12}$$

Next we prove that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.13}$$

For $p \in \Omega$, we find that

$$\begin{aligned} \|u_n - p\|^2 &= \bigl\|S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n - S_{r_n}^{(\Theta,\varphi)}(I - r_n A)p\bigr\|^2 \\ &\le \bigl\|(I - r_n A)x_n - (I - r_n A)p\bigr\|^2 \\ &= \bigl\|x_n - p - r_n(Ax_n - Ap)\bigr\|^2 \\ &\le \|x_n - p\|^2 + r_n(r_n - 2\zeta)\|Ax_n - Ap\|^2. \end{aligned} \tag{3.14}$$

By (3.3), (3.12), and (3.14), we obtain

$$\begin{aligned} \|k_n - p\|^2 &\le \|z_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\ &\le \|u_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\ &\le \|x_n - p\|^2 + r_n(r_n - 2\zeta)\|Ax_n - Ap\|^2 + \gamma_n\|x_n - p\|^2 + c_n, \end{aligned}$$

which implies that

$$\begin{aligned} c(2\zeta - d)\|Ax_n - Ap\|^2 &\le r_n(2\zeta - r_n)\|Ax_n - Ap\|^2 \\ &\le \|x_n - p\|^2 - \|k_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\ &\le \|x_n - k_n\|\bigl(\|x_n - p\| + \|k_n - p\|\bigr) + \gamma_n\|x_n - p\|^2 + c_n. \end{aligned}$$

From $\lim_{n\to\infty}\gamma_n = 0$, $\lim_{n\to\infty} c_n = 0$, and (3.11), we have

$$\lim_{n\to\infty} \|Ax_n - Ap\| = 0. \tag{3.15}$$

By the firm nonexpansivity of $S_{r_n}^{(\Theta,\varphi)}$ and Lemma 2.2(a), we have

$$\begin{aligned} &\|u_n - p\|^2 \\ &\quad = \bigl\|S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n - S_{r_n}^{(\Theta,\varphi)}(I - r_n A)p\bigr\|^2 \end{aligned}$$

$$\begin{aligned}
&\le \big\langle (I-r_nA)x_n-(I-r_nA)p, u_n-p\big\rangle \\
&= \frac{1}{2}\big[\big\|(I-r_nA)x_n-(I-r_nA)p\big\|^2+\|u_n-p\|^2 \\
&\quad -\big\|(I-r_nA)x_n-(I-r_nA)p-(u_n-p)\big\|^2\big] \\
&\le \frac{1}{2}\big[\|x_n-p\|^2+\|u_n-p\|^2-\big\|x_n-u_n-r_n(Ax_n-Ap)\big\|^2\big] \\
&= \frac{1}{2}\big[\|x_n-p\|^2+\|u_n-p\|^2-\|x_n-u_n\|^2+2r_n\langle Ax_n-Ap, x_n-u_n\rangle \\
&\quad -r_n^2\|Ax_n-Ap\|^2\big],
\end{aligned}$$

which implies that

$$\|u_n-p\|^2 \le \|x_n-p\|^2-\|x_n-u_n\|^2+2r_n\|Ax_n-Ap\|\|x_n-u_n\|. \tag{3.16}$$

Combining (3.12) and (3.16), we have

$$\begin{aligned}
\|k_n-p\|^2 &\le \|z_n-p\|^2+\gamma_n\|x_n-p\|^2+c_n \\
&\le \|u_n-p\|^2+\gamma_n\|x_n-p\|^2+c_n \\
&\le \|x_n-p\|^2-\|x_n-u_n\|^2+2r_n\|Ax_n-Ap\|\|x_n-u_n\|+\gamma_n\|x_n-p\|^2+c_n,
\end{aligned}$$

which implies

$$\begin{aligned}
&\|x_n-u_n\|^2 \\
&\quad\le \|x_n-p\|^2-\|k_n-p\|^2+2r_n\|Ax_n-Ap\|\|x_n-u_n\|+\gamma_n\|x_n-p\|^2+c_n \\
&\quad\le \|x_n-k_n\|\big(\|x_n-p\|+\|k_n-p\|\big)+2r_n\|Ax_n-Ap\|\|x_n-u_n\| \\
&\qquad +\gamma_n\|x_n-p\|^2+c_n.
\end{aligned}$$

From $\lim_{n\to\infty}\gamma_n=0$, $\lim_{n\to\infty}c_n=0$, (3.11), and (3.15), we know that (3.13) holds.

Next we show that $\lim_{n\to\infty}\|B_i\Lambda_n^i u_n-B_ip\|=0$, $i=1,2,\ldots,N$. It follows from Lemma 2.14 that

$$\begin{aligned}
\big\|\Lambda_n^i u_n-p\big\|^2 &= \big\|J_{R_i,\lambda_{i,n}}(I-\lambda_{i,n}B_i)\Lambda_n^{i-1}u_n-J_{R_i,\lambda_{i,n}}(I-\lambda_{i,n}B_i)p\big\|^2 \\
&\le \big\|(I-\lambda_{i,n}B_i)\Lambda_n^{i-1}u_n-(I-\lambda_{i,n}B_i)p\big\|^2 \\
&\le \big\|\Lambda_n^{i-1}u_n-p\big\|^2+\lambda_{i,n}(\lambda_{i,n}-2\eta_i)\big\|B_i\Lambda_n^{i-1}u_n-B_ip\big\|^2 \\
&\le \|u_n-p\|^2+\lambda_{i,n}(\lambda_{i,n}-2\eta_i)\big\|B_i\Lambda_n^{i-1}u_n-B_ip\big\|^2 \\
&\le \|x_n-p\|^2+\lambda_{i,n}(\lambda_{i,n}-2\eta_i)\big\|B_i\Lambda_n^{i-1}u_n-B_ip\big\|^2.
\end{aligned} \tag{3.17}$$

Combining (3.12) and (3.17), we have

$$\begin{aligned}
\|k_n-p\|^2 &\le \|z_n-p\|^2+\gamma_n\|x_n-p\|^2+c_n \\
&\le \big\|\Lambda_n^i u_n-p\big\|^2+\gamma_n\|x_n-p\|^2+c_n
\end{aligned}$$

$$\begin{aligned}
&\le \|x_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\big\|B_i\Lambda_n^{i-1}u_n - B_ip\big\|^2 \\
&\quad + \gamma_n\|x_n - p\|^2 + c_n,
\end{aligned}$$

together with $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $i \in \{1, 2, \ldots, N\}$, implies

$$\begin{aligned}
a_i(2\eta_i - b_i)\big\|B_i\Lambda_n^{i-1}u_n - B_ip\big\|^2 &\le \lambda_{i,n}(2\eta_i - \lambda_{i,n})\big\|B_i\Lambda_n^{i-1}u_n - B_ip\big\|^2 \\
&\le \|x_n - p\|^2 - \|k_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\
&\le \|x_n - k_n\|\big(\|x_n - p\| + \|k_n - p\|\big) \\
&\quad + \gamma_n\|x_n - p\|^2 + c_n.
\end{aligned}$$

From $\lim_{n\to\infty}\gamma_n = 0$, $\lim_{n\to\infty}c_n = 0$, and (3.11), we obtain

$$\lim_{n\to\infty}\big\|B_i\Lambda_n^{i-1}u_n - B_ip\big\| = 0, \quad i = 1, 2, \ldots, N. \tag{3.18}$$

By Lemma 2.14 and Lemma 2.2(a), we obtain

$$\begin{aligned}
\big\|\Lambda_n^iu_n - p\big\|^2 &= \big\|J_{R_i,\lambda_{i,n}}(I - \lambda_{i,n}B_i)\Lambda_n^{i-1}u_n - J_{R_i,\lambda_{i,n}}(I - \lambda_{i,n}B_i)p\big\|^2 \\
&\le \big\langle (I - \lambda_{i,n}B_i)\Lambda_n^{i-1}u_n - (I - \lambda_{i,n}B_i)p, \Lambda_n^iu_n - p\big\rangle \\
&= \frac{1}{2}\Big(\big\|(I - \lambda_{i,n}B_i)\Lambda_n^{i-1}u_n - (I - \lambda_{i,n}B_i)p\big\|^2 + \big\|\Lambda_n^iu_n - p\big\|^2 \\
&\quad - \big\|(I - \lambda_{i,n}B_i)\Lambda_n^{i-1}u_n - (I - \lambda_{i,n}B_i)p - \big(\Lambda_n^iu_n - p\big)\big\|^2\Big) \\
&\le \frac{1}{2}\Big(\big\|\Lambda_n^{i-1}u_n - p\big\|^2 + \big\|\Lambda_n^iu_n - p\big\|^2 \\
&\quad - \big\|\Lambda_n^{i-1}u_n - \Lambda_n^iu_n - \lambda_{i,n}\big(B_i\Lambda_n^{i-1}u_n - B_ip\big)\big\|^2\Big) \\
&\le \frac{1}{2}\Big(\|u_n - p\|^2 + \big\|\Lambda_n^iu_n - p\big\|^2 \\
&\quad - \big\|\Lambda_n^{i-1}u_n - \Lambda_n^iu_n - \lambda_{i,n}\big(B_i\Lambda_n^{i-1}u_n - B_ip\big)\big\|^2\Big) \\
&\le \frac{1}{2}\Big(\|x_n - p\|^2 + \big\|\Lambda_n^iu_n - p\big\|^2 \\
&\quad - \big\|\Lambda_n^{i-1}u_n - \Lambda_n^iu_n - \lambda_{i,n}\big(B_i\Lambda_n^{i-1}u_n - B_ip\big)\big\|^2\Big),
\end{aligned}$$

which implies

$$\begin{aligned}
&\big\|\Lambda_n^iu_n - p\big\|^2 \\
&\quad\le \|x_n - p\|^2 - \big\|\Lambda_n^{i-1}u_n - \Lambda_n^iu_n - \lambda_{i,n}\big(B_i\Lambda_n^{i-1}u_n - B_ip\big)\big\|^2 \\
&\quad= \|x_n - p\|^2 - \big\|\Lambda_n^{i-1}u_n - \Lambda_n^iu_n\big\|^2 - \lambda_{i,n}^2\big\|B_i\Lambda_n^{i-1}u_n - B_ip\big\|^2 \\
&\qquad + 2\lambda_{i,n}\big\langle \Lambda_n^{i-1}u_n - \Lambda_n^iu_n, B_i\Lambda_n^{i-1}u_n - B_ip\big\rangle \\
&\quad\le \|x_n - p\|^2 - \big\|\Lambda_n^{i-1}u_n - \Lambda_n^iu_n\big\|^2 \\
&\qquad + 2\lambda_{i,n}\big\|\Lambda_n^{i-1}u_n - \Lambda_n^iu_n\big\|\big\|B_i\Lambda_n^{i-1}u_n - B_ip\big\|.
\end{aligned} \tag{3.19}$$

Combining (3.12) and (3.19) we get

$$\begin{aligned}\|k_n - p\|^2 &\le \|z_n - p\|^2 + \gamma_n \|x_n - p\|^2 + c_n \\ &\le \big\| \Lambda_n^i u_n - p\big\|^2 + \gamma_n \|x_n - p\|^2 + c_n \\ &\le \|x_n - p\|^2 - \big\| \Lambda_n^{i-1} u_n - \Lambda_n^i u_n\big\|^2 \\ &\quad + 2\lambda_{i,n} \big\| \Lambda_n^{i-1} u_n - \Lambda_n^i u_n\big\| \big\| B_i \Lambda_n^{i-1} u_n - B_i p\big\| \\ &\quad + \gamma_n \|x_n - p\|^2 + c_n,\end{aligned}$$

which implies

$$\begin{aligned}&\big\| \Lambda_n^{i-1} u_n - \Lambda_n^i u_n\big\|^2 \\ &\quad \le \|x_n - p\|^2 - \|k_n - p\|^2 + 2\lambda_{i,n} \big\| \Lambda_n^{i-1} u_n - \Lambda_n^i u_n\big\| \big\| B_i \Lambda_n^{i-1} u_n - B_i p\big\| \\ &\qquad + \gamma_n \|x_n - p\|^2 + c_n \\ &\quad \le \|x_n - k_n\| \big(\|x_n - p\| + \|k_n - p\|\big) + 2\lambda_{i,n} \big\| \Lambda_n^{i-1} u_n - \Lambda_n^i u_n\big\| \big\| B_i \Lambda_n^{i-1} u_n - B_i p\big\| \\ &\qquad + \gamma_n \|x_n - p\|^2 + c_n.\end{aligned}$$

From (3.11), (3.18), $\lim_{n\to\infty} \gamma_n = 0$, and $\lim_{n\to\infty} c_n = 0$, we have

$$\lim_{n\to\infty} \big\| \Lambda_n^{i-1} u_n - \Lambda_n^i u_n\big\| = 0, \quad i = 1, 2, \ldots, N. \tag{3.20}$$

From (3.20) we get

$$\begin{aligned}\|u_n - z_n\| &= \big\| \Lambda_n^0 u_n - \Lambda_n^N u_n\big\| \\ &\le \big\| \Lambda_n^0 u_n - \Lambda_n^1 u_n\big\| + \big\| \Lambda_n^1 u_n - \Lambda_n^2 u_n\big\| + \cdots + \big\| \Lambda_n^{N-1} u_n - \Lambda_n^N u_n\big\| \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{3.21}$$

By (3.13) and (3.21), we have

$$\begin{aligned}\|x_n - z_n\| &\le \|x_n - u_n\| + \|u_n - z_n\| \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{3.22}$$

From (3.8) and (3.22), we have

$$\begin{aligned}\|z_{n+1} - z_n\| &\le \|z_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - z_n\| \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{3.23}$$

By (3.11), (3.13), and (3.21), we get

$$\begin{aligned}\|k_n - z_n\| &\le \|k_n - x_n\| + \|x_n - u_n\| + \|u_n - z_n\| \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{3.24}$$

We observe that

$$k_n - z_n = (1 - \delta_n)\big(S^n z_n - z_n\big).$$

From $\delta_n \le d < 1$ and (3.24), we have

$$\lim_{n\to\infty} \big\|S^n z_n - z_n\big\| = 0. \tag{3.25}$$

We note that

$$\begin{aligned}\big\|S^n z_n - S^{n+1} z_n\big\| \le{}& \big\|S^n z_n - z_n\big\| + \|z_n - z_{n+1}\| + \big\|z_{n+1} - S^{n+1} z_{n+1}\big\| \\ &+ \big\|S^{n+1} z_{n+1} - S^{n+1} z_n\big\|.\end{aligned}$$

From (3.23), (3.25), and Lemma 2.4, we obtain

$$\lim_{n\to\infty} \big\|S^n z_n - S^{n+1} z_n\big\| = 0. \tag{3.26}$$

On the other hand, we note that

$$\|z_n - S z_n\| \le \big\|z_n - S^n z_n\big\| + \big\|S^n z_n - S^{n+1} z_n\big\| + \big\|S^{n+1} z_n - S z_n\big\|.$$

From (3.25), (3.26), and the uniform continuity of S, we have

$$\lim_{n\to\infty} \|z_n - S z_n\| = 0. \tag{3.27}$$

In addition, note that

$$\|z_n - W z_n\| \le \|z_n - k_n\| + \|k_n - W_n z_n\| + \|W_n z_n - W z_n\|.$$

So, from (3.10), (3.24), and Remark 2.3 it follows that

$$\lim_{n\to\infty} \|z_n - W z_n\| = 0. \tag{3.28}$$

Step 4. we prove that $x_n \to v = P_\Omega x_0$ as $n \to \infty$.

Indeed, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ which converges weakly to some w. From (3.13) and (3.20)-(3.22), we see that $u_{n_i} \rightharpoonup w$, $\Lambda_{n_i}^m u_{n_i} \rightharpoonup w$, and $z_{n_i} \rightharpoonup w$, where $m \in \{1, 2, \dots, N\}$. Since S is uniformly continuous, by (3.27) we get $\lim_{n\to\infty} \|z_n - S^m z_n\| = 0$ for any $m \ge 1$. Hence from Lemma 2.6, we obtain $w \in \operatorname{Fix}(S)$. In the meantime, utilizing Lemma 2.13, we deduce from (3.28) and $z_{n_i} \rightharpoonup w$ that $w \in \operatorname{Fix}(W) = \bigcap_{n=1}^\infty \operatorname{Fix}(T_n)$ (due to Lemma 2.12). Next, we prove that $w \in \bigcap_{m=1}^N \mathrm{I}(B_m, R_m)$. As a matter of fact, since B_m is η_m-inverse-strongly monotone, B_m is a monotone and Lipschitz-continuous mapping. It follows from Lemma 2.17 that $R_m + B_m$ is maximal monotone. Let $(v, g) \in G(R_m + B_m)$, *i.e.*, $g - B_m v \in R_m v$. Again, since $\Lambda_n^m u_n = J_{R_m,\lambda_{m,n}}(I - \lambda_{m,n} B_m)\Lambda_n^{m-1} u_n$, $n \ge 1$, $m \in \{1, 2, \dots, N\}$, we have

$$\Lambda_n^{m-1} u_n - \lambda_{m,n} B_m \Lambda_n^{m-1} u_n \in (I + \lambda_{m,n} R_m)\Lambda_n^m u_n,$$

that is,

$$\frac{1}{\lambda_{m,n}}\big(\Lambda_n^{m-1}u_n - \Lambda_n^m u_n - \lambda_{m,n}B_m\Lambda_n^{m-1}u_n\big) \in R_m\Lambda_n^m u_n.$$

In terms of the monotonicity of R_m, we get

$$\left\langle v - \Lambda_n^m u_n, g - B_m v - \frac{1}{\lambda_{m,n}}\big(\Lambda_n^{m-1}u_n - \Lambda_n^m u_n - \lambda_{m,n}B_m\Lambda_n^{m-1}u_n\big)\right\rangle \ge 0$$

and hence

$$\begin{aligned}
&\langle v - \Lambda_n^m u_n, g\rangle \\
&\quad\ge \left\langle v - \Lambda_n^m u_n, B_m v + \frac{1}{\lambda_{m,n}}\big(\Lambda_n^{m-1}u_n - \Lambda_n^m u_n - \lambda_{m,n}B_m\Lambda_n^{m-1}u_n\big)\right\rangle \\
&\quad= \left\langle v - \Lambda_n^m u_n, B_m v - B_m\Lambda_n^m u_n + B_m\Lambda_n^m u_n - B_m\Lambda_n^{m-1}u_n + \frac{1}{\lambda_{m,n}}\big(\Lambda_n^{m-1}u_n - \Lambda_n^m u_n\big)\right\rangle \\
&\quad\ge \langle v - \Lambda_n^m u_n, B_m\Lambda_n^m u_n - B_m\Lambda_n^{m-1}u_n\rangle + \left\langle v - \Lambda_n^m u_n, \frac{1}{\lambda_{m,n}}\big(\Lambda_n^{m-1}u_n - \Lambda_n^m u_n\big)\right\rangle.
\end{aligned}$$

In particular,

$$\begin{aligned}
\langle v - \Lambda_{n_i}^m u_{n_i}, g\rangle &\ge \langle v - \Lambda_{n_i}^m u_{n_i}, B_m\Lambda_{n_i}^m u_{n_i} - B_m\Lambda_{n_i}^{m-1}u_{n_i}\rangle \\
&\quad + \left\langle v - \Lambda_{n_i}^m u_{n_i}, \frac{1}{\lambda_{m,n_i}}\big(\Lambda_{n_i}^{m-1}u_{n_i} - \Lambda_{n_i}^m u_{n_i}\big)\right\rangle.
\end{aligned}$$

Since $\|\Lambda_n^m u_n - \Lambda_n^{m-1}u_n\| \to 0$ (due to (3.20)) and $\|B_m\Lambda_n^m u_n - B_m\Lambda_n^{m-1}u_n\| \to 0$ (due to the Lipschitz-continuity of B_m), we conclude from $\Lambda_{n_i}^m u_{n_i} \rightharpoonup w$ and $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $i \in \{1, 2, \ldots, N\}$ that

$$\lim_{i\to\infty}\langle v - \Lambda_{n_i}^m u_{n_i}, g\rangle = \langle v - w, g\rangle \ge 0.$$

It follows from the maximal monotonicity of $B_m + R_m$ that $0 \in (R_m + B_m)w$, *i.e.*, $w \in \mathrm{I}(B_m, R_m)$. Therefore, $w \in \bigcap_{m=1}^N \mathrm{I}(B_m, R_m)$.

Next, we show that $w \in \mathrm{GMEP}(\Theta, \varphi, A)$. In fact, from $z_n = S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n$, we know that

$$\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle K'(u_n) - K'(x_n), y - u_n\rangle \ge 0, \quad \forall y \in C.$$

From (H2) it follows that

$$\varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle K'(u_n) - K'(x_n), y - u_n\rangle \ge \Theta(y, u_n), \quad \forall y \in C.$$

Replacing n by n_i, we have

$$\begin{aligned}
&\varphi(y) - \varphi(u_{n_i}) + \langle Ax_{n_i}, y - u_{n_i}\rangle + \left\langle \frac{K'(u_{n_i}) - K'(x_{n_i})}{r_{n_i}}, y - u_{n_i}\right\rangle \ge \Theta(y, u_{n_i}), \\
&\quad \forall y \in C.
\end{aligned} \tag{3.29}$$

Put $u_t = ty + (1-t)w$ for all $t \in (0,1]$ and $y \in C$. Then, from (3.29), we have

$$\begin{aligned}
&\langle u_t - u_{n_i}, Au_t\rangle \\
&\quad \geq \langle u_t - u_{n_i}, Au_t\rangle - \varphi(u_t) + \varphi(u_{n_i}) - \langle u_t - u_{n_i}, Ax_{n_i}\rangle \\
&\qquad - \left\langle \frac{K'(u_{n_i}) - K'(x_{n_i})}{r_{n_i}}, u_t - u_{n_i}\right\rangle + \Theta(u_t, u_{n_i}) \\
&\quad \geq \langle u_t - u_{n_i}, Au_t - Au_{n_i}\rangle + \langle u_t - u_{n_i}, Au_{n_i} - Ax_{n_i}\rangle - \varphi(u_t) + \varphi(u_{n_i}) \\
&\qquad - \left\langle \frac{K'(u_{n_i}) - K'(x_{n_i})}{r_{n_i}}, u_t - u_{n_i}\right\rangle + \Theta(u_t, u_{n_i}).
\end{aligned}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$ as $i \to \infty$, we deduce from the Lipschitz-continuity of A and K' that $\|Au_{n_i} - Ax_{n_i}\| \to 0$ and $\|K'(u_{n_i}) - K'(x_{n_i})\| \to 0$ as $i \to \infty$. Further, from the monotonicity of A, we have $\langle u_t - u_{n_i}, Au_t - Au_{n_i}\rangle \geq 0$. So, from (H4), we have the weakly lower semicontinuity of φ, $\frac{K'(u_{n_i})-K'(x_{n_i})}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup w$, then we have

$$\langle u_t - w, Au_t\rangle \geq -\varphi(u_t) + \varphi(w) + \Theta(u_t, w), \quad \text{as } i \to \infty. \tag{3.30}$$

From (H1), (H4), and (3.30) we also have

$$\begin{aligned}
0 &= \Theta(u_t, u_t) + \varphi(u_t) - \varphi(u_t) \\
&\leq t\Theta(u_t, y) + (1-t)\Theta(u_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(u_t) \\
&= t\big[\Theta(u_t, y) + \varphi(y) - \varphi(u_t)\big] + (1-t)\big[\Theta(u_t, w) + \varphi(w) - \varphi(w) - \varphi(u_t)\big] \\
&\leq t\big[\Theta(u_t, y) + \varphi(y) - \varphi(u_t)\big] + (1-t)\langle u_t - w, Au_t\rangle \\
&= t\big[\Theta(u_t, y) + \varphi(y) - \varphi(u_t)\big] + (1-t)t\langle y - w, Au_t\rangle,
\end{aligned}$$

and hence

$$0 \leq \Theta(u_t, y) + \varphi(y) - \varphi(u_t) + (1-t)\langle y - w, Au_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \leq \Theta(w, y) + \varphi(y) - \varphi(w) + \langle Aw, y - w\rangle.$$

This implies that $w \in \mathrm{GMEP}(\Theta, \varphi, A)$. Therefore,

$$w \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\Theta, \varphi, A) \cap \left(\bigcap_{i=1}^{N} \mathrm{I}(B_i, R_i)\right) \cap \mathrm{Fix}(S) := \Omega.$$

This shows that $\omega_w(x_n) \subset \Omega$. From (3.7) and Lemma 2.10 we infer that $x_n \to v = P_\Omega x_0$ as $n \to \infty$.

Finally, assume additionally that $\gamma_n + c_n + \|x_n - y_n\| = o(\alpha_n)$. Note that V is a $\bar{\gamma}$-strongly positive bounded linear operator and $f: H \to H$ is an l-Lipschitzian mapping with $\gamma l < (1+\mu)\bar{\gamma}$. It is clear that

$$\big\langle \big(\bar{V}x - \big(u + \gamma f(x)\big)\big) - \big(\bar{V}y - \big(u + \gamma f(y)\big)\big), x - y\big\rangle \geq \big((1+\mu)\bar{\gamma} - \gamma l\big)\|x - y\|^2, \quad \forall x, y \in H.$$

Hence we deduce that $\bar{V}x - (u + \gamma f(x))$ is $((1 + \mu)\bar{\gamma} - \gamma l)$-strongly monotone. In the meantime, it is easy to see that $\bar{V}x - (u + \gamma f(x))$ is $(\|\bar{V}\| + \gamma l)$-Lipschitzian with constant $\|\bar{V}\| + \gamma l > 0$. Thus, there exists a unique solution p in Ω to the VIP

$$\langle \bar{V}p - (u + \gamma f(p)), u - p\rangle \geq 0, \quad \forall u \in \Omega.$$

Equivalently, $p \in \Omega$ solves (OP2) (due to Lemma 2.18). Consequently, we deduce from (3.9) and $x_n \to v = P_\Omega x_0$ $(n \to \infty)$ that

$$\begin{aligned}
&\limsup_{n\to\infty} \langle (u + \gamma f(p)) - \bar{V}p, y_n - p\rangle \\
&\quad = \limsup_{n\to\infty} \big(\langle (u + \gamma f(p)) - \bar{V}p, x_n - p\rangle + \langle (u + \gamma f(p)) - \bar{V}p, y_n - x_n\rangle\big) \\
&\quad = \limsup_{n\to\infty} \langle (u + \gamma f(p)) - \bar{V}p, x_n - p\rangle \\
&\quad = \langle (u + \gamma f(p)) - \bar{V}p, v - p\rangle \leq 0.
\end{aligned} \tag{3.31}$$

Furthermore, by Lemma 2.1 we conclude from (3.1), (3.4), and (3.5) that

$$\begin{aligned}
&\|y_n - p\|^2 \\
&\quad = \big\|\alpha_n(u + \gamma f(p) - \bar{V}p) + \alpha_n\gamma\big(f(x_n) - f(p)\big) + \beta_n(k_n - p) \\
&\qquad + \big((1 - \beta_n)I - \alpha_n\bar{V}\big)(W_n z_n - p)\big\|^2 \\
&\quad \leq \big\|\alpha_n\gamma\big(f(x_n) - f(p)\big) + \beta_n(k_n - p) + \big((1 - \beta_n)I - \alpha_n\bar{V}\big)(W_n z_n - p)\big\|^2 \\
&\qquad + 2\alpha_n\langle (u + \gamma f(p) - \bar{V}p), y_n - p\rangle \\
&\quad \leq \big[\alpha_n\gamma\big\|f(x_n) - f(p)\big\| + \beta_n\|k_n - p\| + \big\|\big((1 - \beta_n)I - \alpha_n\bar{V}\big)(W_n z_n - p)\big\|\big]^2 \\
&\qquad + 2\alpha_n\langle (u + \gamma f(p) - \bar{V}p), y_n - p\rangle \\
&\quad \leq \big[\alpha_n\gamma l\|x_n - p\| + \beta_n\|k_n - p\| + (1 - \beta_n - \alpha_n - \alpha_n\mu\bar{\gamma})\|W_n z_n - p\|\big]^2 \\
&\qquad + 2\alpha_n\langle (u + \gamma f(p) - \bar{V}p), y_n - p\rangle \\
&\quad \leq \big[\alpha_n\gamma l\|x_n - p\| + \beta_n\|k_n - p\| + \big(1 - \beta_n - \alpha_n(1 + \mu)\bar{\gamma}\big)\|z_n - p\|\big]^2 \\
&\qquad + 2\alpha_n\langle (u + \gamma f(p) - \bar{V}p), y_n - p\rangle \\
&\quad = \left[\alpha_n(1 + \mu)\bar{\gamma} \cdot \frac{\gamma l}{(1 + \mu)\bar{\gamma}}\|x_n - p\| + \beta_n\|k_n - p\|\right. \\
&\qquad \left. + \big(1 - \beta_n - \alpha_n(1 + \mu)\bar{\gamma}\big)\|z_n - p\|\right]^2 \\
&\qquad + 2\alpha_n\langle (u + \gamma f(p) - \bar{V}p), y_n - p\rangle \\
&\quad \leq \alpha_n(1 + \mu)\bar{\gamma} \cdot \frac{(\gamma l)^2}{(1 + \mu)^2\bar{\gamma}^2}\|x_n - p\|^2 + \beta_n\|k_n - p\|^2 \\
&\qquad + \big(1 - \beta_n - \alpha_n(1 + \mu)\bar{\gamma}\big)\|z_n - p\|^2 + 2\alpha_n\langle (u + \gamma f(p) - \bar{V}p), y_n - p\rangle \\
&\quad \leq \alpha_n(1 + \mu)\bar{\gamma} \cdot \frac{(\gamma l)^2}{(1 + \mu)^2\bar{\gamma}^2}\|x_n - p\|^2 + \beta_n\big((1 + \gamma_n)\|z_n - p\|^2 + c_n\big)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \big(1-\beta_n-\alpha_n(1+\mu)\bar{\gamma}\big)\|z_n-p\|^2 + 2\alpha_n\big\langle\big(u+\gamma f(p)-\bar{V}p\big), y_n-p\big\rangle \\
&\le \alpha_n\frac{(\gamma l)^2}{(1+\mu)\bar{\gamma}}\|x_n-p\|^2 + \beta_n\big((1+\gamma_n)\|z_n-p\|^2+c_n\big) \\
&\quad + \big(1-\beta_n-\alpha_n(1+\mu)\bar{\gamma}\big)\big((1+\gamma_n)\|z_n-p\|^2+c_n\big) \\
&\quad + 2\alpha_n\big\langle\big(u+\gamma f(p)-\bar{V}p\big), y_n-p\big\rangle \\
&= \alpha_n\frac{(\gamma l)^2}{(1+\mu)\bar{\gamma}}\|x_n-p\|^2 + \big(1-\alpha_n(1+\mu)\bar{\gamma}\big)\big((1+\gamma_n)\|z_n-p\|^2+c_n\big) \\
&\quad + 2\alpha_n\big\langle\big(u+\gamma f(p)-\bar{V}p\big), y_n-p\big\rangle \\
&\le \alpha_n\frac{(\gamma l)^2}{(1+\mu)\bar{\gamma}}\big((1+\gamma_n)\|x_n-p\|^2+c_n\big) \\
&\quad + \big(1-\alpha_n(1+\mu)\bar{\gamma}\big)\big((1+\gamma_n)\|x_n-p\|^2+c_n\big) + 2\alpha_n\big\langle\big(u+\gamma f(p)-\bar{V}p\big), y_n-p\big\rangle \\
&= \left(1-\alpha_n\frac{(1+\mu)^2\bar{\gamma}^2-(\gamma l)^2}{(1+\mu)\bar{\gamma}}\right)\big((1+\gamma_n)\|x_n-p\|^2+c_n\big) \\
&\quad + 2\alpha_n\big\langle\big(u+\gamma f(p)-\bar{V}p\big), y_n-p\big\rangle \\
&\le \left(1-\alpha_n\frac{(1+\mu)^2\bar{\gamma}^2-(\gamma l)^2}{(1+\mu)\bar{\gamma}}\right)\|x_n-p\|^2 + \gamma_n\|x_n-p\|^2 + c_n \\
&\quad + 2\alpha_n\big\langle\big(u+\gamma f(p)-\bar{V}p\big), y_n-p\big\rangle,
\end{aligned}
$$

which hence yields

$$
\begin{aligned}
&\frac{(1+\mu)^2\bar{\gamma}^2-(\gamma l)^2}{(1+\mu)\bar{\gamma}}\|x_n-p\|^2 \\
&\le \frac{\|x_n-p\|^2-\|y_n-p\|^2}{\alpha_n} + \frac{\gamma_n\|x_n-p\|^2+c_n}{\alpha_n} + 2\big\langle\big(u+\gamma f(p)-\bar{V}p\big), y_n-p\big\rangle \\
&\le \frac{\|x_n-y_n\|}{\alpha_n}\big(\|x_n-p\|+\|y_n-p\|\big) + \frac{\gamma_n+c_n}{\alpha_n}\big(\|x_n-p\|^2+1\big) \\
&\quad + 2\big\langle\big(u+\gamma f(p)-\bar{V}p\big), y_n-p\big\rangle.
\end{aligned}
$$

Since $\gamma_n + c_n = o(\alpha_n)$, $\|x_n - y_n\| = o(\alpha_n)$, and $x_n \to v = P_\Omega x_0$, we infer from (3.31) and $0 \le \gamma l < (1+\mu)\bar{\gamma}$ that as $n \to \infty$

$$
\frac{(1+\mu)^2\bar{\gamma}^2-(\gamma l)^2}{(1+\mu)\bar{\gamma}}\|v-p\|^2 \le 0.
$$

That is, $p = v = P_\Omega x_0$. This completes the proof. □

Corollary 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying* (H1)-(H4) *and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $A : H \to H$ and $B_i : C \to H$ be ζ-inverse strongly monotone and η_i-inverse-strongly monotone, respectively, for $i = 1, 2$. Let $S : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense for some $0 \le \kappa < 1$ with sequence $\{\gamma_n\} \subset [0,\infty)$ such that $\lim_{n\to\infty} \gamma_n = 0$ and $\{c_n\} \subset [0,\infty)$ such* *that* $\lim_{n\to\infty} c_n = 0$. *Let* $\{T_n\}_{n=1}^{\infty}$ *be a sequence of nonexpansive self-mappings on* C *and* $\{\lambda_n\}$ *be a sequence in* $(0, b]$ *for some* $b \in (0, 1)$. *Let* V *be a* $\bar{\gamma}$*-strongly positive bounded linear operator and* $f : H \to H$ *be an* l*-Lipschitzian mapping with* $\gamma l < (1 + \mu)\bar{\gamma}$. *Assume that* $\Omega := (\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)) \cap \mathrm{GMEP}(\Theta, \varphi, A) \cap \mathrm{I}(B_2, R_2) \cap \mathrm{I}(B_1, R_1) \cap \mathrm{Fix}(S)$ *is nonempty and bounded. Let* W_n *be the* W*-mapping defined by* (1.4) *and* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\delta_n\}$ *be three sequences in* $(0, 1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\kappa \le \delta_n \le d < 1$. *Assume that*:

(i) $K : H \to \mathbf{R}$ *is strongly convex with constant* $\sigma > 0$ *and its derivative* K' *is Lipschitz-continuous with constant* $\nu > 0$ *such that the function* $x \mapsto \langle y - x, K'(x)\rangle$ *is weakly upper semicontinuous for each* $y \in H$;

(ii) *for each* $x \in H$, *there exist a bounded subset* $D_x \subset C$ *and* $z_x \in C$ *such that for any* $y \notin D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), z_x - y\rangle < 0;$$

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(iv) $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ *for* $i = 1, 2$, *and* $\{r_n\} \subset [0, 2\zeta]$ *satisfies*

$$0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2\zeta.$$

Pick any $x_0 \in H$ *and set* $C_1 = C$, $x_1 = P_{C_1} x_0$. *Let* $\{x_n\}$ *be a sequence generated by the following algorithm*:

$$\begin{cases} u_n = S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n, \\ z_n = J_{R_2,\lambda_{2,n}}(I - \lambda_{2,n} B_2) J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n} B_1) u_n, \\ k_n = \delta_n z_n + (1 - \delta_n) S^n z_n, \\ y_n = \alpha_n (u + \gamma f(x_n)) + \beta_n k_n + ((1 - \beta_n)I - \alpha_n (I + \mu V)) W_n z_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \ge 0, \end{cases} \tag{3.32}$$

where $\theta_n = (\alpha_n + \gamma_n)\Delta_n \varrho + c_n \varrho$, $\Delta_n = \sup\{\|x_n - p\|^2 + \|u + (\gamma f - I - \mu V)p\|^2 : p \in \Omega\} < \infty$, *and* $\varrho = \frac{1}{1 - \sup_{n\ge1} \alpha_n} < \infty$. *If* $S_r^{(\Theta,\varphi)}$ *is firmly nonexpansive, then the following statements hold*:

(I) $\{x_n\}$ *converges strongly to* $P_\Omega x_0$;

(II) $\{x_n\}$ *converges strongly to* $P_\Omega x_0$, *which solves the optimization problem*

$$\min_{x\in\Omega} \frac{\mu}{2}\langle Vx, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x), \tag{OP3}$$

provided $\gamma_n + c_n + \|x_n - y_n\| = o(\alpha_n)$ *additionally, where* $h : H \to \mathbf{R}$ *is the potential function of* γf.

Corollary 3.2 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* Θ *be a bifunction from* $C \times C$ *to* $\mathbf{R}$ *satisfying* (H1)-(H4) *and* $\varphi : C \to \mathbf{R}$ *be a lower semicontinuous and convex functional. Let* $R : C \to 2^H$ *be a maximal monotone mapping and let* $A : H \to H$ *and* $B : C \to H$ *be* ζ*-inverse strongly monotone and* ξ*-inverse-strongly monotone, respectively. Let* $S : C \to C$ *be a uniformly continuous asymptotically* κ*-strict pseudocontractive mapping in the intermediate sense for some* $0 \le \kappa < 1$ *with sequence* $\{\gamma_n\} \subset [0, \infty)$ *such that*

$\lim_{n\to\infty}\gamma_n = 0$ *and* $\{c_n\}\subset[0,\infty)$ *such that* $\lim_{n\to\infty}c_n = 0$. *Let* $\{T_n\}_{n=1}^{\infty}$ *be a sequence of nonexpansive self-mappings on* C *and* $\{\lambda_n\}$ *be a sequence in* $(0,b]$ *for some* $b\in(0,1)$. *Let* V *be a* $\bar{\gamma}$*-strongly positive bounded linear operator and* $f: H\to H$ *be an* l*-Lipschitzian mapping with* $\gamma l < (1+\mu)\bar{\gamma}$. *Assume that* $\Omega := (\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n))\cap\mathrm{GMEP}(\Theta,\varphi,A)\cap\mathrm{I}(B,R)\cap\mathrm{Fix}(S)$ *is nonempty and bounded. Let* W_n *be the* W*-mapping defined by* (1.4) *and* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\delta_n\}$ *be three sequences in* $(0,1)$ *such that* $\lim_{n\to\infty}\alpha_n = 0$ *and* $\kappa\le\delta_n\le d<1$. *Assume that*:

(i) $K: H\to\mathbf{R}$ *is strongly convex with constant* $\sigma>0$ *and its derivative* K' *is Lipschitz-continuous with constant* $\nu>0$ *such that the function* $x\mapsto\langle y-x, K'(x)\rangle$ *is weakly upper semicontinuous for each* $y\in H$;

(ii) *for each* $x\in H$, *there exist a bounded subset* $D_x\subset C$ *and* $z_x\in C$ *such that for any* $y\notin D_x$,

$$\Theta(y,z_x)+\varphi(z_x)-\varphi(y)+\frac{1}{r}\langle K'(y)-K'(x), z_x-y\rangle < 0;$$

(iii) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$;

(iv) $\{\rho_n\}\subset[a,b]\subset(0,2\xi)$, *and* $\{r_n\}\subset[0,2\zeta]$ *satisfies*

$$0<\liminf_{n\to\infty}r_n\le\limsup_{n\to\infty}r_n<2\zeta.$$

Pick any $x_0\in H$ *and set* $C_1 = C$, $x_1 = P_{C_1}x_0$. *Let* $\{x_n\}$ *be a sequence generated by the following algorithm*:

$$\begin{cases} u_n = S_{r_n}^{(\Theta,\varphi)}(I-r_nA)x_n, \\ k_n = \delta_n J_{R,\rho_n}(I-\rho_nB)u_n + (1-\delta_n)S^nJ_{R,\rho_n}(I-\rho_nB)u_n, \\ y_n = \alpha_n(u+\gamma f(x_n)) + \beta_nk_n + ((1-\beta_n)I-\alpha_n(I+\mu V))W_nJ_{R,\rho_n}(I-\rho_nB)u_n, \\ C_{n+1} = \{z\in C_n : \|y_n-z\|^2\le\|x_n-z\|^2+\theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n\ge 0, \end{cases} \tag{3.33}$$

where $\theta_n = (\alpha_n+\gamma_n)\Delta_n\varrho + c_n\varrho$, $\Delta_n = \sup\{\|x_n-p\|^2+\|u+(\gamma f-I-\mu V)p\|^2 : p\in\Omega\}<\infty$, *and* $\varrho = \frac{1}{1-\sup_{n\ge1}\alpha_n}<\infty$. *If* $S_r^{(\Theta,\varphi)}$ *is firmly nonexpansive, then the following statements hold*:

(I) $\{x_n\}$ *converges strongly to* $P_\Omega x_0$;

(II) $\{x_n\}$ *converges strongly to* $P_\Omega x_0$, *which solves the optimization problem*

$$\min_{x\in\Omega}\frac{\mu}{2}\langle Vx,x\rangle+\frac{1}{2}\|x-u\|^2-h(x), \tag{OP4}$$

provided $\gamma_n+c_n+\|x_n-y_n\| = o(\alpha_n)$ *additionally, where* $h: H\to\mathbf{R}$ *is the potential function of* γf.

4 Weak convergence theorems

In this section, we introduce and analyze another iterative algorithm for finding common solutions of a finite family of variational inclusions for maximal monotone and inverse-strongly monotone mappings with the constraints of two problems: a generalized mixed equilibrium problem and a common fixed point problem of an infinite family of nonexpansive mappings and an asymptotically strict pseudocontractive mapping in the intermediate sense in a real Hilbert space. Under mild conditions imposed on the parameter sequences we will prove weak convergence of the proposed algorithm.

Theorem 4.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let N be an integer. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying* (H1)-(H4) *and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $A : H \to H$ and $B_i : C \to H$ be ζ-inverse-strongly monotone and η_i-inverse-strongly monotone, respectively, where $i \in \{1, 2, \ldots, N\}$. Let $S : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense for some $0 \le \kappa < 1$ with sequences $\{\gamma_n\} \subset [0, \infty)$ and $\{c_n\} \subset [0, \infty)$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let V be a $\bar{\gamma}$-strongly positive bounded linear operator and $f : H \to H$ be an l-Lipschitzian mapping with $\gamma l < (1 + \mu)\bar{\gamma}$. Assume that $\Omega := (\bigcap_{n=1}^{\infty} \operatorname{Fix}(T_n)) \cap \operatorname{GMEP}(\Theta, \varphi, A) \cap (\bigcap_{i=1}^{N} \operatorname{I}(B_i, R_i)) \cap \operatorname{Fix}(S)$ is nonempty. Let W_n be the W-mapping defined by* (1.4) *and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ be three sequences in $(0, 1)$ such that $0 < \kappa + \varepsilon \le \delta_n \le d < 1$. Assume that*:

(i) *$K : H \to \mathbf{R}$ is strongly convex with constant $\sigma > 0$ and its derivative K' is Lipschitz-continuous with constant $\nu > 0$ such that the function $x \mapsto \langle y - x, K'(x)\rangle$ is weakly upper semicontinuous for each $y \in H$*;

(ii) *for each $x \in H$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \notin D_x$,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), z_x - y\rangle < 0;$$

(iii) *$\sum_{n=1}^{\infty}(\alpha_n + \gamma_n + c_n) < \infty$ and $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$*;

(iv) *$\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $\forall i \in \{1, 2, \ldots, N\}$, and $\{r_n\} \subset [0, 2\zeta]$ satisfies*

$$0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2\zeta.$$

Pick any $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{cases} u_n = S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n, \\ z_n = J_{R_N,\lambda_{N,n}}(I - \lambda_{N,n}B_N)J_{R_{N-1},\lambda_{N-1,n}}(I - \lambda_{N-1,n}B_{N-1}) \cdots J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1)u_n, \\ k_n = \delta_n z_n + (1 - \delta_n)S^n z_n, \\ x_{n+1} = \alpha_n(u + \gamma f(x_n)) + \beta_n k_n + ((1 - \beta_n)I - \alpha_n(I + \mu V))W_n z_n, \quad \forall n \ge 1. \end{cases} \tag{4.1}$$

Then $\{x_n\}$ converges weakly to $w = \lim_{n\to\infty} P_\Omega x_n$ provided $S_r^{(\Theta,\varphi)}$ is firmly nonexpansive.

Proof First, let us show that $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in \Omega$. Put

$$\Lambda_n^i = J_{R_i,\lambda_{i,n}}(I - \lambda_{i,n}B_i)J_{R_{i-1},\lambda_{i-1,n}}(I - \lambda_{i-1,n}B_{i-1}) \cdots J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1)$$

for all $i \in \{1, 2, \ldots, N\}$, $n \ge 1$, and $\Lambda_n^0 = I$, where I is the identity mapping on H. Then we see that $z_n = \Lambda_n^N u_n$. Take $p \in \Omega$ arbitrarily. Similarly to the proof of Theorem 3.1, we obtain

$$\|u_n - p\| \le \|x_n - p\|, \tag{4.2}$$

$$\|z_n - p\| \le \|u_n - p\|, \tag{4.3}$$

$$\|u_n - p\|^2 \le \|x_n - p\|^2 + r_n(r_n - 2\zeta)\|Ax_n - Ap\|^2, \tag{4.4}$$

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|Ax_n - Ap\|\|x_n - u_n\|, \tag{4.5}$$

$$\begin{aligned}&\big\|\Lambda_n^i u_n - p\big\|^2 \le \|x_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\big\|B_i\Lambda_n^{i-1}u_n - B_ip\big\|^2,\\ &\quad i \in \{1,2,\dots,N\},\end{aligned} \tag{4.6}$$

$$\begin{aligned}\big\|\Lambda_n^i u_n - p\big\|^2 \le{}& \|x_n - p\|^2 - \big\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\big\|^2\\ &+ 2\lambda_{i,n}\big\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\big\|\big\|B_i\Lambda_n^{i-1}u_n - B_ip\big\|,\\ &i \in \{1,2,\dots,N\}.\end{aligned} \tag{4.7}$$

By Lemma 2.2(b) we get

$$\begin{aligned}&\|k_n - p\|^2\\ &\quad= \big\|\delta_n(z_n - p) + (1-\delta_n)\big(S^n z_n - p\big)\big\|^2\\ &\quad= \delta_n\|z_n - p\|^2 + (1-\delta_n)\big\|S^n z_n - p\big\|^2 - \delta_n(1-\delta_n)\big\|z_n - S^n z_n\big\|^2\\ &\quad\le \delta_n\|z_n - p\|^2 + (1-\delta_n)\big[(1+\gamma_n)\|z_n - p\|^2 + \kappa\big\|z_n - S^n z_n\big\|^2 + c_n\big]\\ &\qquad - \delta_n(1-\delta_n)\big\|z_n - S^n z_n\big\|^2\\ &\quad= \big[1 + \gamma_n(1-\delta_n)\big]\|z_n - p\|^2 + (1-\delta_n)(\kappa - \delta_n)\big\|z_n - S^n z_n\big\|^2\\ &\qquad + (1-\delta_n)c_n\\ &\quad\le (1+\gamma_n)\|z_n - p\|^2 + (1-\delta_n)(\kappa - \delta_n)\big\|z_n - S^n z_n\big\|^2 + c_n\\ &\quad\le (1+\gamma_n)\|z_n - p\|^2 + c_n.\end{aligned} \tag{4.8}$$

Repeating the same arguments as in the proof of Theorem 3.1 we have

$$\big\|(1-\beta_n)I - \alpha_n(I + \mu V)\big\| \le 1 - \beta_n - \alpha_n - \alpha_n\mu\bar{\gamma}.$$

Then by Lemma 2.1 we deduce from (4.2), (4.3), (4.8), and $0 \le \gamma l \le (1+\mu)\bar{\gamma}$ that

$$\begin{aligned}&\|x_{n+1} - p\|^2\\ &\quad= \big\|\alpha_n\big(u + \gamma f(p) - \bar{V}p\big) + \alpha_n\gamma\big(f(x_n) - f(p)\big) + \beta_n(k_n - p)\\ &\qquad + \big((1-\beta_n)I - \alpha_n\bar{V}\big)(W_n z_n - p)\big\|^2\\ &\quad\le \big\|\alpha_n\gamma\big(f(x_n) - f(p)\big) + \beta_n(k_n - p) + \big((1-\beta_n)I - \alpha_n\bar{V}\big)(W_n z_n - p)\big\|^2\\ &\qquad + 2\alpha_n\big\langle\big(u + \gamma f(p) - \bar{V}p\big), x_{n+1} - p\big\rangle\\ &\quad\le \big[\alpha_n\gamma\big\|f(x_n) - f(p)\big\| + \beta_n\|k_n - p\| + \big\|\big((1-\beta_n)I - \alpha_n\bar{V}\big)(W_n z_n - p)\big\|\big]^2\\ &\qquad + 2\alpha_n\big\|u + \gamma f(p) - \bar{V}p\big\|\|x_{n+1} - p\|\\ &\quad\le \big[\alpha_n\gamma l\|x_n - p\| + \beta_n\|k_n - p\| + (1 - \beta_n - \alpha_n - \alpha_n\mu\bar{\gamma})\|W_n z_n - p\|\big]^2\\ &\qquad + 2\alpha_n\big\|u + \gamma f(p) - \bar{V}p\big\|\|x_{n+1} - p\|\\ &\quad\le \big[\alpha_n(1+\mu)\bar{\gamma}\|x_n - p\| + \beta_n\|k_n - p\| + \big(1 - \beta_n - \alpha_n(1+\mu)\bar{\gamma}\big)\|z_n - p\|\big]^2\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\alpha_n \big\| u + \gamma f(p) - \bar{V}p \big\| \|x_{n+1} - p\| \\
&\le \alpha_n(1+\mu)\bar{\gamma}\|x_n - p\|^2 + \beta_n\|k_n - p\|^2 + \big(1 - \beta_n - \alpha_n(1+\mu)\bar{\gamma}\big)\|z_n - p\|^2 \\
&\quad + 2\alpha_n \big\| u + \gamma f(p) - \bar{V}p \big\| \|x_{n+1} - p\| \\
&\le \alpha_n(1+\mu)\bar{\gamma}\|x_n - p\|^2 + \beta_n\big((1+\gamma_n)\|z_n - p\|^2 + c_n\big) \\
&\quad + \big(1 - \beta_n - \alpha_n(1+\mu)\bar{\gamma}\big)\|z_n - p\|^2 + 2\alpha_n \big\| u + \gamma f(p) - \bar{V}p \big\| \|x_{n+1} - p\| \\
&\le \alpha_n(1+\mu)\bar{\gamma}\|x_n - p\|^2 + \beta_n\big((1+\gamma_n)\|x_n - p\|^2 + c_n\big) \\
&\quad + \big(1 - \beta_n - \alpha_n(1+\mu)\bar{\gamma}\big)\|x_n - p\|^2 + 2\alpha_n \big\| u + \gamma f(p) - \bar{V}p \big\| \|x_{n+1} - p\| \\
&\le \alpha_n(1+\mu)\bar{\gamma}\big((1+\gamma_n)\|x_n - p\|^2 + c_n\big) + \beta_n\big((1+\gamma_n)\|x_n - p\|^2 + c_n\big) \\
&\quad + \big(1 - \beta_n - \alpha_n(1+\mu)\bar{\gamma}\big)\big((1+\gamma_n)\|x_n - p\|^2 + c_n\big) \\
&\quad + 2\alpha_n \big\| u + \gamma f(p) - \bar{V}p \big\| \|x_{n+1} - p\| \\
&= (1+\gamma_n)\|x_n - p\|^2 + c_n + 2\alpha_n \big\| u + \gamma f(p) - \bar{V}p \big\| \|x_{n+1} - p\| \\
&\le (1+\gamma_n)\|x_n - p\|^2 + c_n + \alpha_n\big(\big\| u + \gamma f(p) - \bar{V}p \big\|^2 + \|x_{n+1} - p\|^2\big),
\end{aligned}
$$

which hence yields

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad \le \frac{1+\gamma_n}{1-\alpha_n}\|x_n - p\|^2 + \frac{\alpha_n}{1-\alpha_n}\big\| u + \gamma f(p) - \bar{V}p \big\|^2 + \frac{1}{1-\alpha_n}c_n \\
&\quad = \left(1 + \frac{\alpha_n + \gamma_n}{1-\alpha_n}\right)\|x_n - p\|^2 + \frac{\alpha_n}{1-\alpha_n}\big\| u + \gamma f(p) - \bar{V}p \big\|^2 + \frac{1}{1-\alpha_n}c_n \\
&\quad \le \big[1 + (\alpha_n + \gamma_n)\varrho\big]\|x_n - p\|^2 + \alpha_n\varrho\big\| u + \gamma f(p) - \bar{V}p \big\|^2 + \varrho c_n,
\end{aligned}
\tag{4.9}
$$

where $\varrho = \frac{1}{1-\sup_{n\ge1}\alpha_n} < \infty$ (due to $\{\alpha_n\} \subset (0,1)$ and $\lim_{n\to\infty}\alpha_n = 0$). By Lemma 2.8, we see from $\sum_{n=1}^{\infty}(\alpha_n + \gamma_n + c_n) < \infty$ that $\lim_{n\to\infty}\|x_n - p\|$ exists. Thus $\{x_n\}$ is bounded and so are the sequences $\{u_n\}$, $\{z_n\}$, and $\{k_n\}$.

Also, utilizing Lemmas 2.1 and 2.2(b) we obtain from (4.2), (4.3), and (4.8)

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \big\| \alpha_n\big(u + \gamma f(x_n) - \bar{V}W_n z_n\big) + \beta_n(k_n - p) + (1-\beta_n)(W_n z_n - p)\big\|^2 \\
&\quad \le \big\| \beta_n(k_n - p) + (1-\beta_n)(W_n z_n - p)\big\|^2 \\
&\qquad + 2\alpha_n\big\langle u + \gamma f(x_n) - \bar{V}W_n z_n, x_{n+1} - p\big\rangle \\
&\quad = \beta_n\|k_n - p\|^2 + (1-\beta_n)\|W_n z_n - p\|^2 - \beta_n(1-\beta_n)\|k_n - W_n z_n\|^2 \\
&\qquad + 2\alpha_n\big\| u + \gamma f(x_n) - \bar{V}W_n z_n\big\| \|x_{n+1} - p\| \\
&\quad \le \beta_n\|k_n - p\|^2 + (1-\beta_n)\|z_n - p\|^2 - \beta_n(1-\beta_n)\|k_n - W_n z_n\|^2 \\
&\qquad + 2\alpha_n\big\| u + \gamma f(x_n) - \bar{V}W_n z_n\big\| \|x_{n+1} - p\| \\
&\quad \le \beta_n\big((1+\gamma_n)\|z_n - p\|^2 + c_n\big) + (1-\beta_n)\|z_n - p\|^2 \\
&\qquad - \beta_n(1-\beta_n)\|k_n - W_n z_n\|^2
\end{aligned}
$$

$$\begin{aligned}
&+ 2\alpha_n \bigl\|u + \gamma f(x_n) - \bar{V} W_n z_n\bigr\| \|x_{n+1} - p\| \\
&\le \beta_n\bigl((1+\gamma_n)\|z_n - p\|^2 + c_n\bigr) + (1-\beta_n)\bigl((1+\gamma_n)\|z_n - p\|^2 + c_n\bigr) \\
&\quad - \beta_n(1-\beta_n)\|k_n - W_n z_n\|^2 \\
&\quad + 2\alpha_n \bigl\|u + \gamma f(x_n) - \bar{V} W_n z_n\bigr\| \|x_{n+1} - p\| \\
&\le (1+\gamma_n)\|z_n - p\|^2 + c_n - \beta_n(1-\beta_n)\|k_n - W_n z_n\|^2 \\
&\quad + 2\alpha_n \bigl\|u + \gamma f(x_n) - \bar{V} W_n z_n\bigr\| \|x_{n+1} - p\| \\
&\le (1+\gamma_n)\|x_n - p\|^2 + c_n - \beta_n(1-\beta_n)\|k_n - W_n z_n\|^2 \\
&\quad + 2\alpha_n \bigl\|u + \gamma f(x_n) - \bar{V} W_n z_n\bigr\| \|x_{n+1} - p\|,
\end{aligned} \tag{4.10}$$

which leads to

$$\begin{aligned}
&\beta_n(1-\beta_n)\|k_n - W_n z_n\|^2 \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\
&\qquad + 2\alpha_n \bigl\|u + \gamma f(x_n) - \bar{V} W_n z_n\bigr\| \|x_{n+1} - p\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\gamma_n = 0$, and $\lim_{n\to\infty}c_n = 0$, it follows from the existence of $\lim_{n\to\infty}\|x_n - p\|$ and condition (iii) that

$$\lim_{n\to\infty}\|k_n - W_n z_n\| = 0. \tag{4.11}$$

Note that

$$x_{n+1} - k_n = \alpha_n\bigl(u + \gamma f(x_n) - \bar{V} W_n z_n\bigr) + (1-\beta_n)(W_n z_n - k_n),$$

which yields

$$\begin{aligned}
\|x_{n+1} - k_n\| &\le \alpha_n \bigl\|u + \gamma f(x_n) - \bar{V} W_n z_n\bigr\| + (1-\beta_n)\|W_n z_n - k_n\| \\
&\le \alpha_n \bigl\|u + \gamma f(x_n) - \bar{V} W_n z_n\bigr\| + \|W_n z_n - k_n\|.
\end{aligned}$$

So, from (4.11) and $\lim_{n\to\infty}\alpha_n = 0$, we get

$$\lim_{n\to\infty}\|x_{n+1} - k_n\| = 0. \tag{4.12}$$

In the meantime, we conclude from (4.2), (4.3), (4.8), and (4.10) that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad \le \beta_n\|k_n - p\|^2 + (1-\beta_n)\|z_n - p\|^2 - \beta_n(1-\beta_n)\|k_n - W_n z_n\|^2 \\
&\qquad + 2\alpha_n \bigl\|u + \gamma f(x_n) - \bar{V} W_n z_n\bigr\| \|x_{n+1} - p\| \\
&\quad \le \beta_n\|k_n - p\|^2 + (1-\beta_n)\|z_n - p\|^2 + 2\alpha_n \bigl\|u + \gamma f(x_n) - \bar{V} W_n z_n\bigr\| \|x_{n+1} - p\| \\
&\quad \le \beta_n\bigl[(1+\gamma_n)\|z_n - p\|^2 + (1-\delta_n)(\kappa - \delta_n)\bigl\|z_n - S^n z_n\bigr\|^2 + c_n\bigr]
\end{aligned}$$

$$
\begin{aligned}
&\quad + (1-\beta_n)\|z_n - p\|^2 + 2\alpha_n \big\|u + \gamma f(x_n) - \bar{V} W_n z_n\big\| \|x_{n+1} - p\| \\
&\leq \beta_n\big[(1+\gamma_n)\|z_n - p\|^2 + (1-\delta_n)(\kappa - \delta_n)\big\|z_n - S^n z_n\big\|^2 + c_n\big] \\
&\quad + (1-\beta_n)\big((1+\gamma_n)\|z_n - p\|^2 + c_n\big) \\
&\quad + 2\alpha_n \big\|u + \gamma f(x_n) - \bar{V} W_n z_n\big\| \|x_{n+1} - p\| \\
&\leq (1+\gamma_n)\|z_n - p\|^2 + \beta_n(1-\delta_n)(\kappa - \delta_n)\big\|z_n - S^n z_n\big\|^2 + c_n \\
&\quad + 2\alpha_n \big\|u + \gamma f(x_n) - \bar{V} W_n z_n\big\| \|x_{n+1} - p\| \\
&\leq (1+\gamma_n)\|x_n - p\|^2 + \beta_n(1-\delta_n)(\kappa - \delta_n)\big\|z_n - S^n z_n\big\|^2 + c_n \\
&\quad + 2\alpha_n \big\|u + \gamma f(x_n) - \bar{V} W_n z_n\big\| \|x_{n+1} - p\|,
\end{aligned}
$$

which, together with $0 < \kappa + \epsilon \leq \delta_n \leq d < 1$, implies that

$$
\begin{aligned}
(1-d)\epsilon\beta_n\big\|z_n - S^n z_n\big\|^2 &\leq \beta_n(1-\delta_n)(\delta_n - \kappa)\big\|z_n - S^n z_n\big\|^2 \\
&\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\
&\quad + 2\alpha_n \big\|u + \gamma f(x_n) - \bar{V} W_n z_n\big\| \|x_{n+1} - p\|.
\end{aligned}
$$

Consequently, from $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\gamma_n = 0$, $\lim_{n\to\infty} c_n = 0$, condition (iii), and the existence of $\lim_{n\to\infty}\|x_n - p\|$, we get

$$
\lim_{n\to\infty}\big\|z_n - S^n z_n\big\| = 0. \tag{4.13}
$$

Since $k_n - z_n = (1-\delta_n)(S^n z_n - z_n)$, from (4.13) we have

$$
\lim_{n\to\infty}\|k_n - z_n\| = 0. \tag{4.14}
$$

Combining (4.4), (4.8), and (4.10), we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \beta_n\big((1+\gamma_n)\|z_n - p\|^2 + c_n\big) + (1-\beta_n)\|z_n - p\|^2 - \beta_n(1-\beta_n)\|k_n - W_n z_n\|^2 \\
&\quad + 2\alpha_n \big\|u + \gamma f(x_n) - \bar{V} W_n z_n\big\| \|x_{n+1} - p\| \\
&\leq \beta_n\big((1+\gamma_n)\|z_n - p\|^2 + c_n\big) + (1-\beta_n)\big((1+\gamma_n)\|z_n - p\|^2 + c_n\big) \\
&\quad + 2\alpha_n \big\|u + \gamma f(x_n) - \bar{V} W_n z_n\big\| \|x_{n+1} - p\| \\
&= (1+\gamma_n)\|z_n - p\|^2 + c_n + 2\alpha_n \big\|u + \gamma f(x_n) - \bar{V} W_n z_n\big\| \|x_{n+1} - p\| \\
&\leq \|u_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n + 2\alpha_n \big\|u + \gamma f(x_n) - \bar{V} W_n z_n\big\| \|x_{n+1} - p\| \\
&\leq \|x_n - p\|^2 + r_n(r_n - 2\zeta)\|Ax_n - Ap\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\
&\quad + 2\alpha_n \big\|u + \gamma f(x_n) - \bar{V} W_n z_n\big\| \|x_{n+1} - p\|,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&r_n(2\zeta - r_n)\|Ax_n - Ap\|^2 \\
&\quad\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\
&\qquad + 2\alpha_n \big\|u + \gamma f(x_n) - \bar{V} W_n z_n\big\| \|x_{n+1} - p\|.
\end{aligned}
$$

From condition (iv), $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\gamma_n = 0$, $\lim_{n\to\infty}c_n = 0$, and the existence of $\lim_{n\to\infty}\|x_n - p\|$, we get

$$\lim_{n\to\infty}\|Ax_n - Ap\| = 0. \tag{4.15}$$

Combining (4.5), (4.8), and (4.10), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \|z_n - p\|^2 + \gamma_n\|z_n - p\|^2 + c_n \\
&\quad + 2\alpha_n\big\|u + \gamma f(x_n) - \bar{V}W_n z_n\big\|\|x_{n+1} - p\| \\
&\le \|u_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\
&\quad + 2\alpha_n\big\|u + \gamma f(x_n) - \bar{V}W_n z_n\big\|\|x_{n+1} - p\| \\
&\le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|Ax_n - Ap\|\|x_n - u_n\| \\
&\quad + \gamma_n\|x_n - p\|^2 + c_n + 2\alpha_n\big\|u + \gamma f(x_n) - \bar{V}W_n z_n\big\|\|x_{n+1} - p\|,
\end{aligned}$$

which implies

$$\begin{aligned}
\|x_n - u_n\|^2 &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2r_n\|Ax_n - Ap\|\|x_n - u_n\| \\
&\quad + \gamma_n\|x_n - p\|^2 + c_n + 2\alpha_n\big\|u + \gamma f(x_n) - \bar{V}W_n z_n\big\|\|x_{n+1} - p\|.
\end{aligned}$$

From (4.15), $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\gamma_n = 0$, $\lim_{n\to\infty}c_n = 0$, and the existence of $\lim_{n\to\infty}\|x_n - p\|$, we obtain

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{4.16}$$

Combining (4.6), (4.8), and (4.10), we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad\le \|z_n - p\|^2 + \gamma_n\|z_n - p\|^2 + c_n + 2\alpha_n\big\|u + \gamma f(x_n) - \bar{V}W_n z_n\big\|\|x_{n+1} - p\| \\
&\quad\le \big\|\Lambda_n^i u_n - p\big\|^2 + \gamma_n\|z_n - p\|^2 + c_n + 2\alpha_n\big\|u + \gamma f(x_n) - \bar{V}W_n z_n\big\|\|x_{n+1} - p\| \\
&\quad\le \|x_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\big\|B_i\Lambda_n^{i-1}u_n - B_i p\big\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\
&\qquad + 2\alpha_n\big\|u + \gamma f(x_n) - \bar{V}W_n z_n\big\|\|x_{n+1} - p\|,
\end{aligned}$$

which implies

$$\begin{aligned}
&\lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\big\|B_i\Lambda_n^{i-1}u_n - B_i p\big\|^2 \\
&\quad\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\
&\qquad + 2\alpha_n\big\|u + \gamma f(x_n) - \bar{V}W_n z_n\big\|\|x_{n+1} - p\|.
\end{aligned}$$

From $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $i \in \{1, 2, \dots, N\}$, $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\gamma_n = 0$, $\lim_{n\to\infty}c_n = 0$, and the existence of $\lim_{n\to\infty}\|x_n - p\|$, we obtain

$$\lim_{n\to\infty}\big\|B_i\Lambda_n^{i-1}u_n - B_i p\big\| = 0, \quad i \in \{1, 2, \dots, N\}. \tag{4.17}$$

Combining (4.7), (4.8), and (4.10), we get

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad\le \|z_n-p\|^2+\gamma_n\|z_n-p\|^2+c_n+2\alpha_n\big\|u+\gamma f(x_n)-\bar{V}W_nz_n\big\|\|x_{n+1}-p\| \\
&\quad\le \big\|\Lambda_n^iu_n-p\big\|^2+\gamma_n\|x_n-p\|^2+c_n+2\alpha_n\big\|u+\gamma f(x_n)-\bar{V}W_nz_n\big\|\|x_{n+1}-p\| \\
&\quad\le \|x_n-p\|^2-\big\|\Lambda_n^{i-1}u_n-\Lambda_n^iu_n\big\|^2+2\lambda_{i,n}\big\|\Lambda_n^{i-1}u_n-\Lambda_n^iu_n\big\|\big\|B_i\Lambda_n^{i-1}u_n-B_ip\big\| \\
&\qquad+\gamma_n\|x_n-p\|^2+c_n+2\alpha_n\big\|u+\gamma f(x_n)-\bar{V}W_nz_n\big\|\|x_{n+1}-p\|,
\end{aligned}$$

which implies

$$\begin{aligned}
&\big\|\Lambda_n^{i-1}u_n-\Lambda_n^iu_n\big\|^2 \\
&\quad\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\lambda_{i,n}\big\|\Lambda_n^{i-1}u_n-\Lambda_n^iu_n\big\|\big\|B_i\Lambda_n^{i-1}u_n-B_ip\big\| \\
&\qquad+\gamma_n\|x_n-p\|^2+c_n+2\alpha_n\big\|u+\gamma f(x_n)-\bar{V}W_nz_n\big\|\|x_{n+1}-p\|.
\end{aligned}$$

From (4.17), $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\gamma_n=0$, $\lim_{n\to\infty}c_n=0$, and the existence of $\lim_{n\to\infty}\|x_n-p\|$, we obtain

$$\lim_{n\to\infty}\big\|\Lambda_n^{i-1}u_n-\Lambda_n^iu_n\big\|=0,\quad i\in\{1,2,\ldots,N\}. \tag{4.18}$$

By (4.18), we have

$$\begin{aligned}
\|u_n-z_n\|&=\big\|\Lambda_n^0u_n-\Lambda_n^Nu_n\big\| \\
&\le \big\|\Lambda_n^0u_n-\Lambda_n^1u_n\big\|+\big\|\Lambda_n^1u_n-\Lambda_n^2u_n\big\|+\cdots+\big\|\Lambda_n^{N-1}u_n-\Lambda_n^Nu_n\big\| \\
&\to 0\quad\text{as } n\to\infty.
\end{aligned} \tag{4.19}$$

From (4.16) and (4.19), we have

$$\begin{aligned}
\|x_n-z_n\|&\le\|x_n-u_n\|+\|u_n-z_n\| \\
&\to 0\quad\text{as } n\to\infty.
\end{aligned} \tag{4.20}$$

By (4.14) and (4.20), we obtain

$$\begin{aligned}
\|k_n-x_n\|&\le\|k_n-z_n\|+\|z_n-x_n\| \\
&\to 0\quad\text{as } n\to\infty,
\end{aligned} \tag{4.21}$$

which, together with (4.12) and (4.21), implies that

$$\begin{aligned}
\|x_{n+1}-x_n\|&\le\|x_{n+1}-k_n\|+\|k_n-x_n\| \\
&\to 0\quad\text{as } n\to\infty.
\end{aligned} \tag{4.22}$$

On the other hand, we observe that

$$\|z_{n+1}-z_n\|\le\|z_{n+1}-x_{n+1}\|+\|x_{n+1}-x_n\|+\|x_n-z_n\|.$$

By (4.20) and (4.22), we have

$$\lim_{n\to\infty} \|z_{n+1} - z_n\| = 0. \tag{4.23}$$

We note that

$$\begin{aligned}\|z_n - Sz_n\| &\le \|z_n - z_{n+1}\| + \left\|z_{n+1} - S^{n+1}z_{n+1}\right\| + \left\|S^{n+1}z_{n+1} - S^{n+1}z_n\right\| \\ &\quad + \left\|S^{n+1}z_n - Sz_n\right\|.\end{aligned}$$

From (4.13), (4.23), Lemma 2.4, and the uniform continuity of S, we obtain

$$\lim_{n\to\infty} \|z_n - Sz_n\| = 0. \tag{4.24}$$

In addition, note that

$$\|z_n - Wz_n\| \le \|z_n - k_n\| + \|k_n - W_n z_n\| + \|W_n z_n - Wz_n\|$$

So, from (4.11), (4.14), and Remark 2.3 it follows that

$$\lim_{n\to\infty} \|z_n - Wz_n\| = 0. \tag{4.25}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to w. From (4.20) and (4.21), we have $z_{n_i} \rightharpoonup w$ and $k_{n_i} \rightharpoonup w$. From (4.24) and the uniform continuity of S, we have $\lim_{n\to\infty} \|z_n - S^m z_n\| = 0$ for any $m \ge 1$. So, from Lemma 2.6, we have $w \in \mathrm{Fix}(S)$. In the meantime, by (4.25) and Lemma 2.13, we get $w \in \mathrm{Fix}(W) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ (due to Lemma 2.12). Utilizing similar arguments to those in the proof of Theorem 3.1, we can derive $w \in \mathrm{GMEP}(\Theta, \varphi, A) \cap (\bigcap_{i=1}^{N} \mathrm{I}(B_i, R_i))$. Consequently, $w \in \Omega$. This shows that $\omega_w(x_n) \subset \Omega$.

Next let us show that $\omega_w(x_n)$ is a single-point set. As a matter of fact, let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup w'$. Then we get $w' \in \Omega$. If $w \neq w'$, from the Opial condition, we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - w\| &= \lim_{i\to\infty} \|x_{n_i} - w\| \\ &< \lim_{i\to\infty} \left\|x_{n_i} - w'\right\| \\ &= \lim_{n\to\infty} \left\|x_n - w'\right\| \\ &= \lim_{j\to\infty} \left\|x_{n_j} - w'\right\| \\ &< \lim_{j\to\infty} \|x_{n_j} - w\| \\ &= \lim_{n\to\infty} \|x_n - w\|.\end{aligned}$$

This attains a contradiction. So we have $w = w'$. Put $v_n = P_\Omega x_n$. Since $w \in \Omega$, we have $\langle x_n - v_n, v_n - w\rangle \ge 0$. By Lemma 2.9, we see that $\{v_n\}$ converges strongly to some $w_0 \in \Omega$.

Since $\{x_n\}$ converges weakly to w, we have

$$\langle w - w_0, w_0 - w\rangle \geq 0.$$

Therefore we obtain $w = w_0 = \lim_{n\to\infty} P_\Omega x_n$. This completes the proof. □

Corollary 4.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying* (H1)-(H4) *and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $A : H \to H$ and $B_i : C \to H$ be ζ-inverse-strongly monotone and η_i-inverse-strongly monotone, respectively, for $i = 1, 2$. Let $S : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense for some $0 \leq \kappa < 1$ with sequences $\{\gamma_n\} \subset [0, \infty)$ and $\{c_n\} \subset [0, \infty)$. Let $\{T_n\}_{n=1}^\infty$ be a sequence of nonexpansive self-mappings on C and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let V be a $\bar{\gamma}$-strongly positive bounded linear operator and $f : H \to H$ be an l-Lipschitzian mapping with $\gamma l < (1 + \mu)\bar{\gamma}$. Assume that $\Omega := (\bigcap_{n=1}^\infty \mathrm{Fix}(T_n)) \cap \mathrm{GMEP}(\Theta, \varphi, A) \cap \mathrm{I}(B_2, R_2) \cap \mathrm{I}(B_1, R_1) \cap \mathrm{Fix}(S)$ is nonempty. Let W_n be the W-mapping defined by* (1.4) *and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$ be three sequences in $(0, 1)$ such that $0 < \kappa + \varepsilon \leq \delta_n \leq d < 1$. Assume that*:

(i) *$K : H \to \mathbf{R}$ is strongly convex with constant $\sigma > 0$ and its derivative K' is Lipschitz-continuous with constant $\nu > 0$ such that the function $x \mapsto \langle y - x, K'(x)\rangle$ is weakly upper semicontinuous for each $y \in H$;*

(ii) *for each $x \in H$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \notin D_x$,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\big\langle K'(y) - K'(x), z_x - y\big\rangle < 0;$$

(iii) $\sum_{n=1}^\infty (\alpha_n + \gamma_n + c_n) < \infty$ *and* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iv) $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ *for $i = 1, 2$, and $\{r_n\} \subset [0, 2\zeta]$ satisfies*

$$0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\zeta.$$

Pick any $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{cases} u_n = S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n, \\ z_n = J_{R_2,\lambda_{2,n}}(I - \lambda_{2,n}B_2)J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1)u_n, \\ k_n = \delta_n z_n + (1 - \delta_n)S^n z_n, \\ x_{n+1} = \alpha_n(u + \gamma f(x_n)) + \beta_n k_n + ((1 - \beta_n)I - \alpha_n(I + \mu V))W_n z_n, \quad \forall n \geq 1. \end{cases} \tag{4.26}$$

Then $\{x_n\}$ converges weakly to $w = \lim_{n\to\infty} P_\Omega x_n$ provided $S_r^{(\Theta,\varphi)}$ is firmly nonexpansive.

Corollary 4.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying* (H1)-(H4) *and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $R : C \to 2^H$ be a maximal monotone mapping and let $A : H \to H$ and $B : C \to H$ be ζ-inverse-strongly monotone and ξ-inverse-strongly monotone, respectively. Let $S : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense for some $0 \leq \kappa < 1$ with sequences $\{\gamma_n\} \subset [0, \infty)$ and*

$\{c_n\} \subset [0,\infty)$. *Let* $\{T_n\}_{n=1}^{\infty}$ *be a sequence of nonexpansive self-mappings on* C *and* $\{\lambda_n\}$ *be a sequence in* $(0,b]$ *for some* $b \in (0,1)$. *Let* V *be a* $\bar{\gamma}$*-strongly positive bounded linear operator and* $f : H \to H$ *be an* l*-Lipschitzian mapping with* $\gamma l < (1+\mu)\bar{\gamma}$. *Assume that* $\Omega := (\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)) \cap \mathrm{GMEP}(\Theta,\varphi,A) \cap \mathrm{I}(B,R) \cap \mathrm{Fix}(S)$ *is nonempty. Let* W_n *be the* W*-mapping defined by* (1.4) *and* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\delta_n\}$ *be three sequences in* $(0,1)$ *such that* $0 < \kappa + \varepsilon \le \delta_n \le d < 1$. *Assume that*:

(i) $K : H \to \mathbf{R}$ *is strongly convex with constant* $\sigma > 0$ *and its derivative* K' *is Lipschitz-continuous with constant* $\nu > 0$ *such that the function* $x \mapsto \langle y - x, K'(x)\rangle$ *is weakly upper semicontinuous for each* $y \in H$;

(ii) *for each* $x \in H$, *there exist a bounded subset* $D_x \subset C$ *and* $z_x \in C$ *such that for any* $y \notin D_x$,

$$\Theta(y,z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), z_x - y\rangle < 0;$$

(iii) $\sum_{n=1}^{\infty}(\alpha_n + \gamma_n + c_n) < \infty$ *and* $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;

(iv) $\{\rho_n\} \subset [a,b] \subset (0,2\xi)$, *and* $\{r_n\} \subset [0,2\zeta]$ *satisfies*

$$0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2\zeta.$$

Pick any $x_1 \in H$ *and let* $\{x_n\}$ *be a sequence generated by the following algorithm*:

$$\begin{cases} u_n = S_{r_n}^{(\Theta,\varphi)}(I - r_nA)x_n, \\ k_n = \delta_n J_{R,\rho_n}(I - \rho_nB)u_n + (1-\delta_n)S^nJ_{R,\rho_n}(I-\rho_nB)u_n, \\ x_{n+1} = \alpha_n(u + \gamma f(x_n)) + \beta_n k_n + ((1-\beta_n)I \\ \qquad - \alpha_n(I + \mu V))W_nJ_{R,\rho_n}(I - \rho_nB)u_n, \quad \forall n \ge 1. \end{cases} \tag{4.27}$$

Then $\{x_n\}$ *converges weakly to* $w = \lim_{n\to\infty} P_\Omega x_n$ *provided* $S_r^{(\Theta,\varphi)}$ *is firmly nonexpansive*.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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