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Remarks on inequalities for the Casorati curvatures of slant submanifolds in quaternionic space forms

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Abstract
In this paper, we obtain an inequality for the normalized Casorati curvature of slant submanifolds in quaternionic space forms by using T Oprea's optimization method.
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1 Introduction

The Casorati curvature of an n-dimensional submanifold M of a Riemannian manifold, usually denoted by $\mathcal{C}$, is an extrinsic invariant defined as the normalized square of the length of the second fundamental form of the submanifold. In [1], Decu *et al.* introduced the normalized δ-Casorati curvatures $\delta_c(n-1)$ and $\hat{\delta}_c(n-1)$ by

$$\left[\delta_c(n-1)\right]_x = \frac{1}{2}\mathcal{C}_x + \frac{n+1}{2n(n-1)} \inf\left\{\mathcal{C}(L) \mid L \text{ a hyperplane of } T_xM\right\} \tag{1}$$

and

$$\left[\hat{\delta}_c(n-1)\right]_x = 2\mathcal{C}_x - \frac{2n-1}{2n} \sup\left\{\mathcal{C}(L) \mid L \text{ a hyperplane of } T_xM\right\},$$

where $x \in M$, and established some inequalities involving these invariants for submanifolds in real space forms. Later, Slesar *et al.* proved two inequalities relating the above normalized Casorati curvatures for a slant submanifolds in a quaternionic space form in [2]. However, it was pointed out that the coefficient $\frac{n+1}{2n(n-1)}$ in (1) is inappropriate and must be replaced by $\frac{n+1}{2n}$ [3, 4]. Following [3, 4], we define the normalized δ-Casorati curvature $\delta_C(n-1)$ by

$$\left[\delta_C(n-1)\right]_x = \frac{1}{2}\mathcal{C}_x + \frac{n+1}{2n} \inf\left\{\mathcal{C}(L) \mid L \text{ a hyperplane of } T_xM\right\}. \tag{2}$$

By using T Oprea's optimization method on Riemannian submanifolds, we establish the following inequalities in terms of $\delta_C(n-1)$ for θ-slant proper submanifolds of a quaternionic space form.

Theorem 1 *Let M^n, $n \geq 3$, be θ-slant proper submanifold of a quaternionic space form $\overline{M}^{4m}(c)$. Then the normalized δ-Casorati curvature $\delta_C(n-1)$ satisfies*

$$\rho \leq \delta_C(n-1) + \frac{c}{4}\left(1 + \frac{9}{n-1}\cos^2\theta\right), \tag{3}$$

where ρ is the normalized scalar curvature of M^n. Moreover, the equality case holds if and only if M^n is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}^{4m}(c)$, such that with respect to suitable orthonormal tangent frame $\{\xi_1, \ldots, \xi_n\}$ and normal orthonormal frame $\{\xi_{n+1}, \ldots, \xi_{4m}\}$, the shape operators $A_r = A_{e_r}$, $r \in \{n+1, \ldots, 4m\}$, take the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 & 0 \\ 0 & a & 0 & \cdots & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 0 \\ 0 & 0 & 0 & \cdots & 0 & 2a \end{pmatrix}, \qquad A_{n+2} = \cdots = A_{4m} = 0.$$

2 Preliminaries

Let (M^n, g) be an n-dimensional submanifold in an $(n+p)$-dimensional Riemannian manifold $(\overline{M}^{n+p}, \overline{g})$. The Levi-Civita connections on $\overline{M}^{n+p}$ and M^n will be denoted by $\overline{\nabla}$ and ∇, respectively. For all $X, Y \in C^\infty(TM)$, $N \in C^\infty(TM^\perp)$, the Gauss and Weingarten formulas can be expressed by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \qquad \overline{\nabla}_X N = -A_N X + \nabla_X^\perp N,$$

where h is the second fundamental form of M, $\overline{\nabla}$ is the normal connection and the shape operator A_N of M is given by

$$g(A_N X, Y) = \overline{g}\big(h(X,Y), N\big).$$

The submanifold M is said to be totally geodesic if $h = 0$. Besides, M is called invariantly quasi-umbilical if there exist p mutually orthogonal unit normal vectors $\xi_{n+1}, \ldots, \xi_{n+p}$ such that the shape operators with respect to all directions ξ_r have an eigenvalue of multiplicity $n-1$ and that for each ξ_r the distinguished eigendirection is the same [1–4].

In $\overline{M}^{n+p}$ we choose a local orthonormal frame $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+p}$, such that, restricting ourselves to M^n, $e_1, \ldots, e_n$ are tangent to M^n. We write $h_{ij}^r = g(h(e_i, e_j), e_r)$. Then the *mean curvature vector* H is given by

$$H = \sum_{r=n+1}^{n+p}\left(\frac{1}{n}\sum_{i=1}^{n} h_{ii}^r\right)e_r,$$

and the squared norm of h over dimension n is denoted by $\mathcal{C}$ and is called the Casorati curvature of the submanifold M. Therefore we have

$$\mathcal{C} = \frac{1}{n}\sum_{r=n+1}^{n+p}\sum_{i,j=1}^{n}\big(h_{ij}^r\big)^2.$$

Let $K(e_i \wedge e_j)$, $1 \leq i < j \leq n$, denote the *sectional curvature* of the plane section spanned by e_i and e_j. Then the *scalar curvature* of M^n is given by

$$\tau = \sum_{i<j} K(e_i \wedge e_j),$$

and the normalized scalar curvature ρ is defined by

$$\rho = \frac{2\tau}{n(n-1)}.$$

Suppose L is an l-dimensional subspace of T_xM, $x \in M$, $l \geq 2$ and $\{e_1, \ldots, e_l\}$ an orthonormal basis of L. Then the *scalar curvature* $\tau(L)$ of the l-plane L is given by

$$\tau(L) = \sum_{1 \leq \mu < \nu \leq l} K(e_\mu \wedge e_\nu),$$

and the Casorati curvature $\mathcal{C}(L)$ of the subspace L is defined as

$$\mathcal{C}(L) = \frac{1}{r} \sum_{r=n+1}^{n+p} \sum_{i,j=1}^{n} \left(h_{ij}^r\right)^2.$$

For more details of slant submanifolds in quaternionic space forms, we refer to [2, 4].

3 Optimization method on Riemannian submanifolds

Let $(N_2, \overline{g})$ be a Riemannian manifold, N_1 be a Riemannian submanifold of it, g be the metric induced on N_1 by $\overline{g}$ and $f : N_1 \to \mathbb{R}$ be a differentiable function.

Following [5–7] we considered the constrained extremum problem

$$\min_{x \in N_1} f(x), \tag{4}$$

then we have the following.

Lemma 1 ([5]) *If $x_0 \in N_1$ is the solution of the problem* (4), *then*

(i) $(\operatorname{grad} f)(x_0) \in T_{x_0}^{\perp} N_1$;

(ii) *the bilinear form*

$$\mathcal{A} : T_{x_0}N_1 \times T_{x_0}N_1 \to \mathbb{R};$$
$$\mathcal{A}(X, Y) = \operatorname{Hess}_f(X, Y) + \overline{g}\big(h(X, Y), (\operatorname{grad} f)(x_0)\big)$$

is positive semidefinite, where h is the second fundamental form of N_1 in N_2.

In [6], the above lemma was successfully applied to improve an inequality relating $\delta(2)$ obtained in [8]. Later, Chen extended the improved inequality to the general inequalities involving δ-invariants $\delta(n_1, \ldots, n_k)$ [9]. More details of δ-invariants can be found in [10–15]. Besides, the first author gave another proof of the inequalities relating the normalized δ-Casorati curvature $\hat{\delta}_c(n-1)$ for submanifolds in real space forms by using T Oprea's optimization method [16].

4 Proof of Theorem 1

From the Gauss equation we can easily obtain (see (12) in [2])

$$2\tau = \frac{c}{4}\left[n(n-1) + 9n\cos^2\theta\right] + n^2\|H\|^2 - n\mathcal{C}. \tag{5}$$

We define now the following function, denoted by $\mathcal{Q}$, which is a quadratic polynomial in the components of the second fundamental form:

$$\mathcal{Q} = \frac{1}{2}n(n-1)\mathcal{C} + \frac{1}{2}(n+1)(n-1)\mathcal{C}(L) - 2\tau + \frac{c}{4}\left[n(n-1) + 9n\cos^2\theta\right]. \tag{6}$$

Without loss of generality, by assuming that L is spanned by $e_1, \dots, e_{n-1}$, one gets

$$\mathcal{Q} = \frac{n+1}{2}\sum_{\alpha=n+1}^{4m}\left[\sum_{i,j=1}^{n}\left(h_{ij}^{\alpha}\right)^2\right] + \frac{n+1}{2}\sum_{\alpha=n+1}^{4m}\left[\sum_{i,j=1}^{n-1}\left(h_{ij}^{\alpha}\right)^2\right] - \sum_{\alpha=n+1}^{4m}\left(\sum_{i=1}^{n}h_{ii}^{\alpha}\right)^2, \tag{7}$$

here we used (5) and (6).

From (7) we have

$$\begin{aligned}\mathcal{Q} &= \sum_{\alpha=n+1}^{4m}\sum_{i=1}^{n-1}\left[n\left(h_{ii}^{\alpha}\right)^2 + (n+1)\left(h_{in}^{\alpha}\right)^2\right] \\ &\quad + \sum_{\alpha=n+1}^{4m}\left[2(n+1)\sum_{1\le i<j\le n-1}\left(h_{ij}^{\alpha}\right)^2 - 2\sum_{1\le i<j\le n}h_{ii}^{\alpha}h_{jj}^{\alpha} + \frac{n-1}{2}\left(h_{nn}^{\alpha}\right)^2\right] \\ &\ge \sum_{\alpha=n+1}^{4m}\sum_{i=1}^{n-1}n\left(h_{ii}^{\alpha}\right)^2 + \sum_{\alpha=n+1}^{4m}\left[-2\sum_{1\le i<j\le n}h_{ii}^{\alpha}h_{jj}^{\alpha} + \frac{n-1}{2}\left(h_{nn}^{\alpha}\right)^2\right].\end{aligned} \tag{8}$$

For $\alpha = n+1, \dots, 4m$, let us consider the quadratic form

$$f_\alpha : \mathbb{R}^n \to \mathbb{R},$$

$$f_\alpha\left(h_{11}^{\alpha}, \dots, h_{nn}^{\alpha}\right) = \sum_{i=1}^{n-1}n\left(h_{ii}^{\alpha}\right)^2 - 2\sum_{1\le i<j\le n}h_{ii}^{\alpha}h_{jj}^{\alpha} + \frac{n-1}{2}\left(h_{nn}^{\alpha}\right)^2$$

and the constrained extremum problem

$$\min f_\alpha$$
$$\text{subject to } F : h_{11}^{\alpha} + \cdots + h_{nn}^{\alpha} = k^{\alpha},$$

where k^α is a real constant.

The partial derivatives of the function f_α are

$$\frac{\partial f_\alpha}{\partial h_{11}^{\alpha}} = 2nh_{11}^{\alpha} - 2\sum_{i=2}^{n}h_{ii}^{\alpha}, \tag{9}$$

$$\frac{\partial f_\alpha}{\partial h_{22}^{\alpha}} = 2nh_{22}^{\alpha} - 2h_{11}^{\alpha} - 2\sum_{i=3}^{n}h_{ii}^{\alpha}, \tag{10}$$

$$\dots,$$

$$\frac{\partial f_\alpha}{\partial h^\alpha_{n-1,n-1}} = 2nh^\alpha_{n-1,n-1} - 2\sum_{i=1}^{n-2} h^\alpha_{ii} - 2h^\alpha_{nn}, \tag{11}$$

$$\frac{\partial f_\alpha}{\partial h^\alpha_{nn}} = -2\sum_{i=1}^{n-1} h^\alpha_{ii} + (n-1)h^\alpha_{nn}. \tag{12}$$

For an optimal solution $(h^\alpha_{11}, h^\alpha_{22}, \dots, h^\alpha_{nn})$ of the problem in question, the vector $\operatorname{grad} f_\alpha$ is normal at $\mathcal{F}$, that is, it is collinear with the vector $(1, 1, \dots, 1)$. From (9), (10), (11), and (12), it follows that a critical point of the considered problem has the form

$$\left(h^\alpha_{11}, h^\alpha_{22}, \dots, h^\alpha_{n-1,n-1}, h^\alpha_{nn}\right) = \left(t^\alpha, t^\alpha, \dots, t^\alpha, 2t^\alpha\right). \tag{13}$$

As $\sum_{i=1}^n h^\alpha_{ii} = k^\alpha$, by using (13) we have

$$h^\alpha_{11} = h^\alpha_{22} = \cdots = h^\alpha_{n-1,n-1} = \frac{1}{n+1}k^\alpha, \qquad h^\alpha_{nn} = \frac{2}{n+1}k^\alpha. \tag{14}$$

We fix an arbitrary point $x \in \mathcal{F}$. The 2-form $\mathcal{A} : T_x\mathcal{F} \times T_x\mathcal{F} \to \mathbb{R}$ has the expression

$$\mathcal{A}(X,Y) = \operatorname{Hess} f_\alpha(X,Y) + \left\langle h'(X,Y), (\operatorname{grad} f_\alpha)(x)\right\rangle,$$

where h' is the second fundamental form of $\mathcal{F}$ in $\mathbb{R}^n$ and $\langle\,,\,\rangle$ is the standard inner-product on $\mathbb{R}^n$. In the standard frame of $\mathbb{R}^n$, the Hessian of f_α has the matrix

$$\begin{pmatrix} 2n & -2 & -2 & \cdots & -2 & -2 \\ -2 & 2n & -2 & \cdots & -2 & -2 \\ -2 & -2 & 2n & \cdots & -2 & -2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & -2 & \cdots & 2n & -2 \\ -2 & -2 & -2 & \cdots & -2 & n-1 \end{pmatrix}.$$

As $\mathcal{F}$ is totally geodesic in $\mathbb{R}^n$, considering a vector X tangent to $\mathcal{F}$ at the arbitrary point x on $\mathcal{F}$, that is, verifying the relation $\sum_{i=1}^n X_i = 0$, we have

$$\begin{aligned} \mathcal{A}(X,X) &= (X_1, X_2, X_3, \dots, X_{n-1}, X_n) \begin{pmatrix} 2n & -2 & -2 & \cdots & -2 & -2 \\ -2 & 2n & -2 & \cdots & -2 & -2 \\ -2 & -2 & 2n & \cdots & -2 & -2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & -2 & \cdots & 2n & -2 \\ -2 & -2 & -2 & \cdots & -2 & n-1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_{n-1} \\ X_n \end{pmatrix} \\ &= 2(n+1)\sum_{i=1}^{n-1} X_i^2 + (n+1)X_n^2 - 2(X_1 + X_2 + \cdots + X_n)^2 \\ &= 2(n+1)\sum_{i=1}^{n-1} X_i^2 + (n+1)X_n^2 \\ &\geq 0. \end{aligned}$$

Thus the point $(h_{11}^\alpha, h_{22}^\alpha, \dots, h_{nn}^\alpha)$ given by (14) is a global minimum point, here we used Lemma 1. Inserting (14) in (8) we have

$$\mathcal{Q} \geq 0. \tag{15}$$

From (2), (6), and (15) we can derive inequality (3). The equality case of (3) holds if and only if we have the equality in all the previous inequalities. Thus

$$h_{ij}^\alpha = 0, \quad i \neq j, \forall \alpha;$$
$$h_{nn}^\alpha = 2h_{11}^\alpha = 2h_{22}^\alpha = \cdots = 2h_{n-1,n-1}^\alpha, \quad \forall \alpha.$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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